Prof. Dr. Roman DMYTRYSHYN

CURRENT TRENDS IN ANALYSIS APPROXIMATION THEORY AND THEIR APPLICATIONS

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Addres: Selçuk Üniversitesi Yayınları, Alaeddin Keykubat Yerleşkesi, Akademi mah. Yeni İstanbul Cad. No:369
Posta Kodu: 42130, Selçuklu-Konya/Türkiye
Web: yayinevi.selcuk.edu.tr
E-posta: yayinevi@selcuk.edu.tr
Phone: 0 (332) 241 00 41

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Introduction

The book brings together a collection of contemporary results in analysis, approximation theory, and their applications. It extensively discusses known and current results, as well as problematic questions and open problems. The book comprises ten chapters, representing five directions of mathematical analysis, unified by the theory of functions.

Splines and their generalizations play an important role in many questions of approximation theory and its applications. The goal of Chapter 1 is to generalize known in the literature theorems about existence and uniqueness of interpolating splines in Hilbert and Banach spaces to the case of metric groups. This allows, in particular, to include into consideration interpolating splines in linear metric spaces, F-spaces and other situations. We also adduce some applications of the obtained results, mainly focusing on applications to problems of optimal recovery.

The second and the third chapters is representation of recent results of the Lviv school of complex analysis. The notion of directionally bounded L-index for various classes of analytic functions is very important to study local and asymptotic properties of analytic solutions of ordinary and partial differential equations, and their systems. Use arbitrary positive continuous function L in the notion allows to consider analytic functions with bounded multiplicities of zero points. The authors selected such subclass of the functions as slice holomorphic functions in the unit polydisc. They are functions, which are analytic on the intersection of unit polydisc with every slice of the form $z^0 + t\mathbf{b}$, where $t \in \mathbb{C}$ is the complex parameter, $b \in \mathbb{C}^n$ is a fixed direction, z^0 is any point from the polydisc. Other resuts in the third chapter concern asymtotic relations of Wiman-Valiron's theory for analytic functions in domains admitting geometric exhaustion directional strips.

The third direction (Chapters 4 and 7) is devoted to symmetric analytic functions on infinite-dimensional Banach spaces. Algebras of symmetric polynomials and analytic functions on Banach spaces play an important role in nonlinear functional analysis and applications. In Chapter 4, the authors considered the problem of the description of the spectra of algebras of bounded type symmetric analytic functions on ℓ_p. This problem is completely solved for the case $p = 1$ using a representation of the spectrum by analytic functions of exponential type. Chapter 7 is devoted to block-symmetric polynomials on Cartesian products of Banach spaces and gathers results concerning algebraic bases in algebras of such polynomials.

The fourth direction is devoted to continued fractions and branched continued fractions, some of the most intriguing approximation tools of mathematical analysis. Chapter 5 discusses the representation of Appel's hypergeometric functions by branched continued fractions. Here, the main problems are the construction of expansions of ratios of Appel's hypergeometric functions into branched continued fractions, the establishment of their domains of convergence, which are also domains of analytic continuation of these functions, and the approximation of some special functions by branched continued fractions. Chapter 6 is devoted to continued fractions. The stability to perturbations of specific classes of continued fractions with complex elements (formulas for relative errors, sufficient conditions for stability to perturbations, and sets of stability to perturbations) and their application to the approximation of special functions, including Bessel functions and ratios of Horn's confluent hypergeometric functions, are considered.

In Chapters 8-10, approximation errors for linear and nonlinear methods on important functional classes in various metrics are considered using approximation aggregates constructed on different systems, in particular, on trigonometric and on a system defined by integer functions of exponential type. Namely, the chapters cover best approximations, approximations by step hyperbolic Fourier sums, best orthogonal and best m-term trigonometric approximations, greedy approximations, widths and some their analogs from the point of view of finding exact order values.

In the eighth chapter, the results are presented on the investigation of the Sobolev and Nikol'skii–Besov classes of periodic functions with dominating mixed smoothness. The approximation error of respective characteristics is estimated in the important Lebesgue subspaces. Special attention is paid to some features compared to the approximation in classical Lebesgue spaces.

The ninth section is devoted to the approximation of Wiener-type functional classes with generalized weights given by certain positive functions. Here, the dependence is discussed of the respective approximation errors on the asymptotic behavior of such weight functions. The errors are obtained in Wiener-Stepanets spaces and Lebesgue spaces.

The tenth chapter is devoted to the approximation properties of the Nikol'skii–Besov classes of non-periodic functions with dominating mixed smoothness and the error is estimated in the norm of the Lebesgue space.

Finally, we are grateful to Tuncer Acar for inviting us to contribute to the book Current Trends in Analysis, Approximation Theory, and Their Applications and to the staff of the Şelcuk University Press for their for their professional support during the preparation of this book.

Vladyslav BABENKO, Yuliya BABENKO, Andriy BANDURA, Iryna CHERNEGA, Roman DMYTRYSHYN, Volodymyr HLADUN, Oleh KOVALENKO, Victoria KRAVTSIV, Andriy KURYLIAK, Nataliia PARFINOVYCH, Kateryna POZHARSKA, Anatolii ROMANYUK, Tetyana SALO, Andrii SHYDLICH, Oleh SKASKIV, Taras VASYLYSHYN, Sergii YANCHENKO, Andrii ZAGORODNYUK

Vladyslav BABENKO, Yuliya BABENKO, Oleh KOVALENKO,
and Nataliia PARFINOVYCH

1 Interpolating splines in metric groups and some of their applications

1.1 Introduction. Interpolating splines in Hilbert and Banach spaces

Let (X, ρ_X) be a metric space, and simultaneously X be a commutative group with an operation $+$ and a neutral element θ_X. We assume that the group operation and the metric in X agree in the following sense:

$$\forall x, y, u \in X, \rho_X(x+u, y+u) = \rho_X(x, y). \tag{1.1}$$

Under these conditions the set X will be called a *metric group*. In order to somewhat shorten notations, for $x \in X$ we sometimes write $\|x\|_X$ instead of $\rho_X(x, \theta_X)$. In view of (1.1), one has $\|x - y\|_X = \rho_X(x, y)$; these notations agree with the relation between the norm and the corresponding metric in a normed space.

Let Y be some group (the group operation will also be denoted by $+$, everywhere below we assume that all considered groups are commutative) and $T\colon X \to Y$ be some homomorphism such that its kernel $\mathcal{N}(T) := \{x \in X\colon Tx = \theta_Y\}$ is closed. Such a homomorphism will be called an *information homomorphism*. If there is a Hausdorff topology in Y such that the mapping T is continuous, then the set $\mathcal{N}(T)$ is automatically closed.

Splines and their generalizations, in particular splines in Hilbert (see [2, 13]) and Banach (see [15]) spaces play an important role in many questions of approximation theory and its applications. We refer the reader to works [7, 8, 18] and references therein for further information and results that involve such splines and their further generalizations.

Generalizing the definition of interpolating splines in Hilbert spaces, we con-

Vladyslav BABENKO, Oles Honchar Dnipro National University, 72 Nauky avenue, 49045, Dnipro, Ukraine, e-mail: babenko.vladislav@gmail.com

Yuliya BABENKO, Kennesaw State University, Marietta, Kennesaw, GA 30060, United States, e-mail: ybabenko@kennesaw.edu

Oleh KOVALENKO, Oles Honchar Dnipro National University, 72 Nauky avenue, 49045, Dnipro, Ukraine, e-mail: olegkovalenko90@gmail.com

Nataliia PARFINOVYCH, Oles Honchar Dnipro National University, 72 Nauky avenue, 49045, Dnipro, Ukraine, e-mail: parfinovich@mmf.dnu.edu.ua

sider the following diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ {\scriptstyle R}\big\downarrow & & \\ W & & \end{array}$$

where X and W are metric groups, Y is a group, $R\colon X \to W$ and $T\colon X \to Y$ are homomorphisms.

For a given $x \in X$ consider the following extremal problem

$$\|Rs\|_W \to \inf \text{ over } s \in X \text{ such that } Ts = Tx. \tag{1.2}$$

Definition 1.1. *If there exists an element $s_x = s(x; R, T) \in X$ that delivers the infimum in* (1.2)*, then it will be called a T-interpolating R-spline for x.*

For Hilbert spaces W, X, Y extremal problem (1.2) has been studied by Atteia [2], Golomb [13] and Laurent [18]. If X, Y are Hilbert or Banach spaces, then by $\mathcal{L}(X,W)$ we, as usually, denote the space of all linear bounded operators $A\colon X \to W$.

Theorem 1.1 (Atteia [2])**.** *Assume that W, X, Y are Hilbert spaces, operators $R \in \mathcal{L}(X, W)$ and $T \in \mathcal{L}(X, Y)$ are surjections such that*

$$\mathcal{N}(R) + \mathcal{N}(T) \text{ is closed in } X, \tag{1.3}$$

$$\mathcal{N}(R) \cap \mathcal{N}(T) = \{\theta_X\}. \tag{1.4}$$

Then for each $x \in X$ there exists a unique T-interpolating R-spline.

The following generalization of the preceding theorem by Utreras [22] allows to consider constrained interpolation problems.

Consider the following diagram

$$\begin{array}{ccc} C \subset X & \xrightarrow{T} & Y \\ {\scriptstyle R}\big\downarrow & & \\ W & & \end{array} \tag{1.5}$$

where X, Y, W are Hilbert spaces, R and T are linear continuous operators and C is a closed convex set. For an element $x \in X$ such that

$$T^{-1}(Tx) \cap C \neq \emptyset \tag{1.6}$$

we consider the following extremal problem

$$\|Rs\|_W \to \inf \text{ over } s \in C \text{ such that } Ts = Tx. \tag{1.7}$$

Definition 1.2. *If there exists an element $s_x = s(x; C, R, T) \in C$ that delivers the infimum in* (1.7)*, then it will be called a C-constrained T-interpolating R-spline for x.*

Theorem 1.2 (Utreras [22])**.** *Assume that W, X, Y are Hilbert spaces, operators $R \in \mathcal{L}(X, W)$ and $T \in \mathcal{L}(X, Y)$ are surjections, $C \subset X$ is a closed convex subset. If condition* (1.4) *holds and for $x \in X$*

$$\mathcal{N}(R) + (\mathcal{N}(T) \cap (C - x)) \text{ is closed in } X, \tag{1.8}$$

then there exists a unique C-constrained T-interpolating R-spline for x.

Let (X, ρ_X) be a metric space and $F \subset X$ be a closed set. We need the following well-known definitions.

Definition 1.3. *The quantity*

$$E(x; F; X) := \inf_{u \in F} \rho_X(x, u) = \inf_{u \in F} \|x - u\|_X$$

is called the best approximation of an element $x \in X$ by the set F in the space X.

Definition 1.4. *The set*

$$P_F x = \{u \in F \colon \rho_X(x, u) = E(x; F; X)\}$$

is called the metric projection of the element x into the set F. If for each $x \in X$ the set $P_F x$ is a singleton (and in this case we write $P_F x = y$ instead of $P_F x = \{y\}$), then the set F is said to be a Chebyshev set in X.

Chebyshev sets, as well as metric projections play an important role in approximation theory. Many examples of Chebyshev sets can be found e.g., in [9, 21]. A discussion of metric projections can be found in [21].

Observe that one does not expect to find an element of the best approximation in a set F, if F is not closed. The following statement (see e.g., [15, Theorem 1.2]) explains the relation between condition (1.3) in Theorem 1.1 or condition (1.8) in Theorem 1.2 and the requirement for some set to be Chebyshev (see e.g., (1.10) in Theorem 1.4): if W, X, Y are Banach spaces, $R \in \mathcal{L}(X, W)$ and $T \in \mathcal{L}(X, Y)$ are such that the range $\mathcal{R}(R)$ of the operator R is closed, then the set $R(\mathcal{N}(T))$ is closed if and only if the set $\mathcal{N}(R) + \mathcal{N}(T)$ is closed. Since (see e.g., [12, Corollary 2.51]), a closed subspace of a strictly convex reflexive Banach space is a Chebyshev set, in a Hilbert space condition (1.3) implies that the set $R(\mathcal{N}(T))$ is Chebyshev and we arrive at condition (1.10) in Theorem 1.4.

Holmes [15, Theorem 1.3] proved the following generalization of the theorems on existence and uniqueness of the splines to the case of Banach spaces. We formulate the theorem using similar to the above notations.

Theorem 1.3 (Holmes [15]). *Assume that W, X, Y are Banach spaces, and $R \in \mathcal{L}(X, W)$, $T \in \mathcal{L}(X, Y)$. There is a unique T-interpolating R-spline for $x \in X$ if and only if* (1.4) *is true and the following condition hold:*

$$P_{R(\mathcal{N}(T))}(Rx) \text{ is a singleton.}$$

The goal of this article is to generalize known in the literature theorems about existence and uniqueness of interpolating splines in Hilbert [2] and Banach [15] spaces to the case of metric groups. This allows, in particular, to include into consideration interpolating splines in linear metric spaces (see e.g., [20]), F-spaces (see e.g., [11, Chapter II.1] or [24, Chapter I.9]) and other situations. We also adduce some applications of the obtained results, mainly focusing on applications to problems of optimal recovery.

The chapter is organized as follows. In Section 1.2 we give sufficient conditions that guarantee existence and uniqueness of T-interpolating R-splines in the case when X and W are metric groups, and Y is a group; this result generalizes the Holmes theorem. In Section 1.3 we give a generalization of the Utreras theorem. In Section 1.4 we show some applications of T-interpolating R-splines to the problems of optimal recovery. Finally, in Section 1.5 we adduce some illustrative examples.

1.2 Existence and uniqueness of interpolating splines in metric groups. Generalization of the Holmes Theorem

Remind that a theorem about existence and uniqueness of a T-interpolating R-spline in normed spaces was proved by Holmes [15]. Our goal is to generalize it to the case of metric groups.

Before formulating the theorem, we note that for homomorphisms $R\colon X \to W$ and $T\colon X \to Y$ between groups X, Y, W and each $x \in X$ one has

$$\{u \in X\colon Tu = Tx\} = T^{-1}(Tx) = x + \mathcal{N}(T), \tag{1.9}$$

and

$$R(T^{-1}(Tx)) = Rx + R(\mathcal{N}(T)).$$

Thus, taking into account that the set $R(\mathcal{N}(T))$ is θ_W-symmetric, we can reformulate the definition of a T-interpolating R-spline as follows.

Definition 1.5. *For a given $x \in X$ an element $s_x \in X$ is called a T-interpolating R-spline, if*

$$\|Rs_x\|_W = E(\theta_X, R(T^{-1}(Tx)), W) = E(Rx, R(\mathcal{N}(T)), W).$$

Theorem 1.4. *Let W be a metric group and X, Y be groups. Assume that $R\colon X \to W$ and $T\colon X \to Y$ are homomorphisms such that condition* (1.4) *holds and*

$$R(\mathcal{N}(T)) \text{ is a Chebyshev set in } W. \tag{1.10}$$

Then for each $x \in X$ there exists a unique T-interpolating R-spline $s_x = s(x;R,T) \in X$. Moreover, the restriction $R|_{\mathcal{N}(T)}\colon \mathcal{N}(T) \to R(\mathcal{N}(T))$ has an inverse

$$\left(R|_{\mathcal{N}(T)}\right)^{-1} : R(\mathcal{N}(T)) \to \mathcal{N}(T)$$

and the spline s_x can be represented as

$$s_x = x - \left(R|_{\mathcal{N}(T)}\right)^{-1} P_{R(\mathcal{N}(T))} Rx.$$

Remark 1.1. *Under the conditions of Theorem 1.4 set*

$$S(R,T) = \{s_x\colon x \in X\} = \left\{s_x = x - \left(R|_{\mathcal{N}(T)}\right)^{-1} P_{R(\mathcal{N}(T))} Rx\colon x \in X\right\}.$$

The mapping

$$T|_{S(R,T)}\colon S(R,T) \to \mathcal{R}(T)$$

is a bijection and hence has an inverse

$$\left(T|_{S(R,T)}\right)^{-1} : \mathcal{R}(T) \to S(R,T).$$

For each $x \in X$ the T-interpolating R-spline s_x can be represented as

$$s_x = \left(T|_{S(R,T)}\right)^{-1} (Tx).$$

Proof. First of all we show that condition (1.4) implies that the restriction

$$R|_{\mathcal{N}(T)}\colon \mathcal{N}(T) \to R(\mathcal{N}(T))$$

has an inverse

$$\left(R|_{\mathcal{N}(T)}\right)^{-1} : R(\mathcal{N}(T)) \to \mathcal{N}(T).$$

From the definition it follows that $R|_{\mathcal{N}(T)}$ is a surjection, so it is enough to show that $R|_{\mathcal{N}(T)}$ is an injection. Let $x, y \in \mathcal{N}(T)$ be such that $Rx = Ry$. Then $x - y \in \mathcal{N}(R)$. Moreover, $x, y \in \mathcal{N}(T)$ implies that $x - y \in \mathcal{N}(T)$. Thus

$$x - y \in \mathcal{N}(R) \cap \mathcal{N}(T),$$

which together with condition (1.4) implies $x = y$.

Since $R(\mathcal{N}(T))$ is a Chebyshev set, for the element Rx there exists a unique element of the best approximation $P_{R(\mathcal{N}(T))}(Rx)$. Consider an element s_x defined by the formula

$$s_x = x - \left(R|_{\mathcal{N}(T)}\right)^{-1} \left(P_{R(\mathcal{N}(T))}(Rx)\right).$$

From the definition it follows that

$$Rs_x = Rx - P_{R(\mathcal{N}(T))}(Rx). \tag{1.11}$$

Hence, the condition from Definition 1.5 holds for the element s_x. Thus, s_x is the T-interpolating R-spline for x.

Next we show that a T-interpolating R-spline for x is unique. Let $s' \in X$ be such that $Ts' = Tx = Ts_x$ and

$$\|Rs'\|_W = \|Rs_x\|_W. \tag{1.12}$$

The first condition implies that $s' - s_x \in \mathcal{N}(T)$ i.e., $s' = s_x + w$ with some

$$w \in \mathcal{N}(T). \tag{1.13}$$

Hence $Rs' = Rs_x + Rw$, and $Rw \in R(\mathcal{N}(T))$. Taking into account (1.11), we obtain

$$Rs' = Rx - (P_{R(\mathcal{N}(T))}(Rx) - Rw).$$

If $Rw \neq \theta_W$, then condition (1.10) implies

$$\|Rs'\|_W > E(Rx, R(\mathcal{N}(T)), W) = \|Rs_x\|_W,$$

which contradicts to (1.12). Thus, $Rw = \theta_W$ i.e., $w \in \mathcal{N}(R)$. Together with (1.13) and condition (1.4) this implies $w = \theta_X$, and hence $s' = s_x + w = s_x$. This finishes the proof of uniqueness of a T-interpolating R-spline for x. □

From formula (1.11) we obtain the following statement.

Corollary 1.1. *Under the assumptions of Theorem 1.4, for arbitrary $x \in X$*

$$\|Rs_x\|_X \leq \|Rx\|_X.$$

Consider an important partial case of Theorem 1.4. Let $W = X$ and $R = I_X$, where $I_X\colon X \to X$ is the identity mapping. Instead of $S(I_X, T)$ we write $S(T)$, so that $S(T) = \{s_x = x - P_{\mathcal{N}(T)}x\colon x \in H\}$. The elements of the set $S(T)$ will be called T-interpolating splines.

Corollary 1.2. *Let X be a metric group, Y be a group and $T\colon X \to Y$ be a homomorphism. If condition* (1.10) *holds (with R being the identity operator), then for each $x \in X$ there exists a unique spline $s_x \in S(T)$ such that $Ts_x = Tx$.*

Moreover, the restriction $T|_{S(T)}\colon S(T) \to \mathcal{R}(T)$ is a bijection, and for arbitrary $x \in X$ the T-interpolating spline s_x can be represented as

$$s_x = \left(T|_{S(T)}\right)^{-1}(Tx).$$

1.3 Constrained interpolating splines in metric groups. Generalization of the Utreras Theorem

Consider diagram (1.5), but now we assume that W is a metric group, X, Y are groups, $R\colon X \to W$ and $T\colon X \to Y$ are homomorphisms, and $C \subset X$. The following theorem is a generalization of the Utreras theorem.

Theorem 1.5. *Assume that W is a metric group and X, Y are groups, $C \subset X$, condition* (1.4) *holds, and for a given $x \in X$ such that condition* (1.6) *is true,*

$$R((C - x) \cap \mathcal{N}(T)) \text{ is a Chebyshev set in } W. \tag{1.14}$$

Then there exists a unique C-constrained T-interpolating R-spline $s_x = s(x; C, R, T)$.

Proof. Taking into account equality (1.9), we obtain that

$$T^{-1}(Tx) \cap C = x + (C - x) \cap \mathcal{N}(T).$$

Since R is a homomorphism, condition (1.14) implies that $R(T^{-1}(Tx) \cap C)$ is a Chebyshev set. Extremal problem (1.7) can be rewritten as a problem to find

$$\inf_{x \in T^{-1}(Tx) \cap C} \|Rx\|_W = \inf_{w \in R(T^{-1}(Tx) \cap C)} \|w\|_W,$$

hence there exists an element $s_x \in T^{-1}(Tx) \cap C$ such that

$$Rs_x = P_{R(T^{-1}(Tx) \cap C)}(\theta_W)$$

is the unique solution to this problem.

Next we prove that condition (1.4) actually implies that the element s_x is unique i.e., if

$$Rs = Rs_x, \quad s \in T^{-1}(Tx) \cap C,$$

then $s = s_x$. Indeed, first of all, since $Ts = Ts_x = Tx$, we obtain that $s - s_x \in \mathcal{N}(T)$. On the other hand, $s - s_x \in \mathcal{N}(R)$, which together with (1.4) implies $s = s_x$. □

Remark 1.2. *We note that requirement* (1.14) *is actually overly restrictive. It suffices to assume only that the set $R(x + (C - x) \cap \mathcal{N}(T))$ contains a unique element of the best approximation for θ_W.*

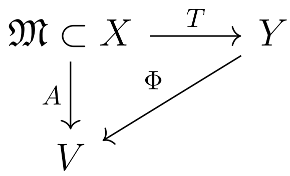
Remark 1.3. *If $C = X$, then Theorem 1.5 implies the statement on existence and uniqueness from Theorem 1.4. Moreover, an analogous to Remark 1.1 representation of the spline s_x in terms of the element Tx holds.*

1.4 Optimal recovery problems and interpolating splines in metric groups

Let a metric space V and sets X and Y be given. For a mapping $A\colon X \to V$ and an information mapping $T\colon X \to Y$, arbitrary mapping $\Phi\colon Y \to V$ is called a method of recovery of the mapping A based on the information provided by T. If $\mathfrak{M} \subset X$ is given, then the quantity

$$\mathcal{E}(A, \mathfrak{M}; T; \Phi; V) := \sup_{x \in \mathfrak{M}} \|A(x) - \Phi(Tx)\|_V$$

is called the error of the method Φ on the class $\mathfrak{M}$. This is illustrated by the following diagram.



Definition 1.6. *The problem of optimal recovery of a mapping A on the class $\mathfrak{M}$ based on the information about $x \in \mathfrak{M}$ that is given with the help of an operator T, consists of computing the value*

$$\mathcal{E}(A, \mathfrak{M}; T; V) := \inf_{\Phi} \mathcal{E}(A, \mathfrak{M}; T; \Phi; V),$$

and finding a mapping (called an optimal method of recovery) $\overline{\Phi}$ that delivers the infimum on the right hand side of this equality, if such an operator exists.

In the case, when $V = X$ and $A = I_X$ (the identity operator), we obtain the problem of optimal recovery of elements of the set $\mathfrak{M}$. In this case we write $\mathcal{E}(\mathfrak{M}; T; X)$ instead of $\mathcal{E}(I_X, \mathfrak{M}; T; V)$.

The set $\mathfrak{M}$ is very often chosen as follows. Assume that an operator $R\colon X \to W$ and a set $\mathfrak{N} \subset W$ is given. Then set

$$\mathfrak{M} = R^{-1}(\mathfrak{N}). \tag{1.15}$$

Below we assume that X, V, W are metric groups, and Y is a group. Let also the information operator $T\colon X \to Y$ and the operator $R\colon X \to W$ be homomorphisms, a centrally symmetric set $\mathfrak{N} \subset W$ be given, and the set $\mathfrak{M}$ is defined by formula (1.15). Under the assumptions of Theorem 1.4, for every $y \in \mathcal{R}(T)$ we set

$$\overline{\Phi}(y) = A((T|_{S(R,T)})^{-1} y), \tag{1.16}$$

so that $\overline{\Phi}(Tx) = As_x$.

We describe an approach to the solution of an optimal recovery problem; although, this kind of results is well known in the literature in various contexts, we give the proofs for completeness. We start with a theorem that gives an estimate from below for the quantity $\mathcal{E}(A, \mathfrak{M}; T; V)$ and is well-known in many situations, cf. e.g., [19, Theorem 1].

Theorem 1.6. *If the mappings A and R are odd and for arbitrary $v \in V$ one has*

$$\|v + v\|_V = 2\|v\|_V, \tag{1.17}$$

then for arbitrary method of recovery Φ one has

$$\mathcal{E}(A, \mathfrak{M}; \Phi; T; V) \geq \sup_{x \in \mathfrak{M} \cap \mathcal{N}(T)} \|Ax\|_V,$$

and therefore

$$\mathcal{E}(A, \mathfrak{M}; T; V) \geq \sup_{x \in \mathfrak{M} \cap \mathcal{N}(T)} \|Ax\|_V. \tag{1.18}$$

Proof. Since A is odd, for an arbitrary element $v \in V$, using (1.1) and (1.17) we obtain

$$\max\{\rho_V(Ax, v), \rho_V(A(-x), v)\} \geq \frac{1}{2}(\rho_V(Ax, v) + \rho_V(-Ax, v))$$
$$\geq \frac{1}{2}\rho_V(Ax, -Ax) = \frac{1}{2}\rho_V(Ax + Ax, \theta_V) = \rho_V(Ax, \theta_V).$$

Moreover, since $\mathfrak{N}$ is centrally symmetric and R is odd, $x \in \mathfrak{M} \iff (-x) \in \mathfrak{M}$. Thus we obtain for an arbitrary method of recovery Φ

$$\mathcal{E}(A, \mathfrak{M}; T; \Phi; V) = \sup_{x \in \mathfrak{M}} \|Ax - \Phi(Tx)\|_V \geq \sup_{x \in \mathfrak{M} \cap \mathcal{N}(T)} \|Ax - \Phi(\theta_Y)\|_V =$$
$$= \sup_{x \in \mathfrak{M} \cap \mathcal{N}(T)} \max\{\|Ax - \Phi(\theta_Y)\|_V, \|A(-x) - \Phi(\theta_Y)\|_V\} \geq \sup_{x \in \mathfrak{M} \cap \mathcal{N}(T)} \|Ax\|_V.$$

The theorem is proved. □

A rather general result for an estimate from above is contained in the following theorem. We formulate it in a form convenient to illustrate the application of abstract splines to the problems of optimal recovery.

Theorem 1.7. *Let a mapping $A \colon X \to V$ be such that for arbitrary $x, y \in X$ one has*

$$\|Ax - Ay\|_V \leq \|A(x - y)\|_V. \tag{1.19}$$

Assume that one of the following properties hold:

1. *There exists $x_0 \in \mathfrak{M} \cap \mathcal{N}(T)$ such that*

$$\sup_{x \in \mathfrak{M}} \|Ax - As_x\|_V = \|Ax_0 - As_{x_0}\|_V; \tag{1.20}$$

2. *$\mathfrak{M} = R^{-1}(\mathfrak{N})$, with $\mathfrak{N} \subset W$ that satisfies condition*

$$w \in \mathfrak{N} \implies P_{R(\mathcal{N}(T))} w \in \mathfrak{N}. \tag{1.21}$$

Then

$$\mathcal{E}(A,\mathfrak{M};T;V)\le\sup_{x\in\mathfrak{M}}\|Ax-As_x\|_V\le\sup_{x\in\mathfrak{M}\cap\mathcal{N}(T)}\|Ax\|_V.$$

Proof. We start with the proof of the theorem in the case when condition (1.20) holds. Since $s_{x_0}=\theta_X$,

$$\sup_{x\in\mathfrak{M}}\|Ax-As_x\|_V\le\|Ax_0-As_{x_0}\|_V\le\|A(x_0-s_{x_0})\|_V=$$
$$=\|Ax_0\|_V\le\sup_{x\in\mathfrak{M}\cap\mathcal{N}(T)}\|Ax\|_V$$

as required.

Next assume that condition (1.21) holds. Using assumption (1.19), and the fact that for all $w\in W$, $T((R|_{\mathcal{N}(T)})^{-1}P_{R(\mathcal{N}(T))}w)=\theta_Y$, we have

$$\mathcal{E}(A,\mathfrak{M};T;V)\le\sup_{x\in\mathfrak{M}}\|Ax-\overline{\Phi}(Tx)\|_V=\sup_{x\in\mathfrak{M}}\|Ax-As_x\|_V\le$$
$$\le\sup_{x\in\mathfrak{M}}\|A(x-s_x)\|_V=\sup_{x\in\mathfrak{M}}\left\|A\left((R|_{\mathcal{N}(T)})^{-1}P_{R(\mathcal{N}(T))}Rx\right)\right\|_V=$$
$$=\sup_{w\in\mathfrak{N}}\left\|A\left((R|_{\mathcal{N}(T)})^{-1}P_{R(\mathcal{N}(T))}w\right)\right\|_V\le\sup_{x\in\mathfrak{M}\cap\mathcal{N}(T)}\|Ax\|_V.$$

□

Let F be a subset of a metric space (V,ρ_V). Recall that the quantity

$$r(F)=\inf_{v\in V}\sup_{f\in F}\rho_V(v,f)$$

is called the *Chebyshev radius* of the set F, and the set

$$c(F):=\{v\in V:\ \sup_{f\in F}\rho_V(v,f)=r(F)\}$$

is called the *Chebyshev center* of the set F (the set $c(F)$ might be empty). It is well known (see e.g., [19, Theorem 4]) that under mild conditions

$$\mathcal{E}(A,\mathfrak{M};T;V)=\sup_{y\in T(\mathfrak{M})}r(A(T^{-1}(y))),$$

and if each set $A(T^{-1}(y))$ has a Chebyshev center, $y\in T(\mathfrak{M})$, then any method

$$y\mapsto\overline{\Phi}(y)\in c(y)$$

is optimal (such methods are often called *central methods of recovery*). In this context, quantity on the right-hand side of estimate (1.18) is nothing else but the Chebyshev radius of the set $A(T^{-1}(\theta_Y))$. Condition (1.20) is intimately related to the requirement

$$\sup_{y\in T(\mathfrak{M})}r(A(T^{-1}(y)))=r(A(T^{-1}(\theta_Y))).$$

Finally, condition (1.21) guarantees that for all $x \in \mathfrak{M}$

$$x - s_x = x - (T|_{S(R,T)})^{-1}(Tx) \in \mathfrak{M} \cap \mathcal{N}(T),$$

which in turn under mild conditions implies that method (1.16) is an optimal one, cf. [19, Theorem 3].

If X and Y are Hilbert spaces, $T \in \mathcal{L}(X, Y)$ is a surjection and $\mathfrak{M}$ is the unit ball in X, then (see [19, Theorem 5]) one can explicitly specify a central method of recovery:

$$\overline{\Phi}(y) = AT^*(TT^*)^{-1}y, y \in T(\mathfrak{M}),$$

where T^* is the adjoint to T operator. It looks like abstract splines allow to build optimal method of recovery in some situations when the set $\mathfrak{M}$ is given by (1.15).

Applications of abstract splines to problems of optimal recovery are also contained in [4, 6].

1.5 Examples

1. If $W = L_2[a, b]$ is the space of square integrable on $[a, b]$ functions with the usual norm, X is the space $L_2^2[a, b]$ of continuously differentiable functions $x\colon [a, b] \to \mathbb{R}$ such that x' is absolutely continuous and $Rx := x'' \in W$, $Y = \mathbb{R}^n$, $n \geq 2$, $a \leq t_1 < \ldots < t_n \leq b$ and $T(x) = (x(t_1), \ldots, x(t_n))$, then according to the result by Holladay [14, Theorem 1], the abstract spline s_x from Corollary 1.1 is actually the natural (i.e., $s_x''(t_1) = s_x''(t_n) = 0$) cubic spline that interpolates x at the points $t_1, \ldots, t_n$.

Here $\mathcal{N}(R)$ is the space of linear on $[a, b]$ functions, and $\mathcal{N}(T)$ is the space of functions that vanish at points $t_1, \ldots, t_n$. Thus condition (1.4) holds. Moreover, the set $R(\mathcal{N}(T))$ is a (closed) subspace of a strictly convex reflexive Banach space $L_2[a, b]$, hence is a Chebyshev set, see e.g., [12, Corollary 2.51]. Thus condition (1.10) also holds.

A generalization of Holladay's result to the case of the class $L_2^n[a, b]$, $n > 2$ was obtained in [10], and corresponding results in the periodic case are contained in [1, 23].

2. Let H be a separable Hilbert space with an orthonormal basis $\{e_k\}_{k=1}^{\infty}$, $Y = \mathbb{R}^n$, $R\colon H \to H$ be the identity map, and the information homomorphism be

$$Tx = ((x, e_1), \ldots, (x, e_n)).$$

Then $\mathcal{N}(T) = \{x \in H\colon x \perp e_1, \ldots, e_n\}$,

$$P_{\mathcal{N}(T)}x = \sum_{k=n+1}^{\infty} (x, e_k)e_k \text{ and } s_x = x - P_{\mathcal{N}(T)}x = \sum_{k=1}^{n}(x, e_k)e_k.$$

Observe that for arbitrary $x \in H$, $\|P_{\mathcal{N}(T)}x\| \le \|x\|$, and hence if $\mathfrak{N}$ is the unit ball $\{x \in H\colon \|x\| \le 1\}$, then condition (1.21) holds. A related example is contained in [19, Example 1.1].

3. Let $W = L_p[0,1], 1 < p < \infty$, be the space of integrable in a power p functions with the usual norm, $Y = \mathbb{R}^n$, $n \in \mathbb{N}$. Define an information homomorphism by the formula

$$Tx = ((Tx)_1, (Tx)_2, \ldots, (Tx)_n) =$$
$$= \left(n\int_0^{1/n} x(u)du,\ n\int_{1/n}^{2/n} x(u)du,\ \ldots,\ n\int_{(n-1)/n}^{1} x(u)du \right).$$

The kernel of this homomorphism is the space

$$\mathcal{N}(T) = \left\{ x \in L_p[0,1]\colon\ \int_{(k-1)/n}^{k/n} x(u)du = 0, k = 1,\ \ldots, n \right\}.$$

It is again a closed subspace of a strictly convex reflexive Banach space $L_p[0,1]$, hence is a Chebyshev set. For each $x \in L_p[0,1]$ define a function $s_x\colon [0,1] \to \mathbb{R}$ by the formula

$$s_x(t) = n\int_{(k-1)/n}^{k/n} x(u)du, \quad \text{if} \quad t \in \left(\frac{k-1}{n}, \frac{k}{n}\right), \quad k = 1, \ldots, n. \tag{1.22}$$

Then $y := x - s_x \in \mathcal{N}(T)$ and for arbitrary $\varphi \in \mathcal{N}(T)$, one has

$$\int_0^1 \varphi(t)\cdot|x(t)-y(t)|^{p-1}\text{sgn}\,(x(t)-y(t))dt =$$
$$= \sum_{k=1}^{n} |(Tx)_k|^{p-1}\text{sgn}\,(Tx)_k \int_{(k-1)/n}^{k/n} \varphi(t)dt = 0.$$

Thus by the criterion of the element of the best approximation in $L_p[0,1]$ (see e.g., [17, Theorem 1.4.5]) $P_{\mathcal{N}(T)}x = y$. Hence, the function s_x defined by (1.22) is the T-interpolating spline.

Observe that in the case $p = 2$ for the space (χ_A denotes the characteristic function of a set A)

$$F = \text{span}\left\{\chi_{[0,1/n]}, \chi_{[1/n,2/n]}, \ldots, \chi_{[(n-1)/n,1]}\right\}$$

again using the criterion of the element of the best approximation in $L_2[0,1]$, one has

$$E(x;F;L_2[0,1]) = \inf_{u\in F}\|x-u\|_{L_2[0,1]} = \|x-s_x\|_{L_2[0,1]} = \|P_{\mathcal{N}(T)}x\|_{L_2[0,1]}.$$

Thus, for $p = 2$ one has

$$\|P_{\mathcal{N}(T)}x\|_{L_2[0,1]} \le \|x\|_{L_2[0,1]},$$

and hence condition (1.21) holds, if $\mathfrak{N}$ is the unit ball in $L_2[0,1]$.

On the other hand, suppose that $1 < p < 2$, and for $0 < \varepsilon \le n^{-1}$ consider the function

$$x(t) = \begin{cases} \varepsilon^{-1/p}, & t \in [0,\varepsilon], \\ 0, & t \in (\varepsilon, 1]. \end{cases}$$

Obviously, $\|x\|_{L_p[0,1]} = 1$, and (below $1/p + 1/p' = 1$)

$$P_{\mathcal{N}(T)}x = \begin{cases} \varepsilon^{-1/p} - n\varepsilon^{1/p'}, & t \in (0,\varepsilon), \\ -n\varepsilon^{1/p'}, & t \in (\varepsilon, n^{-1}), \\ 0, \quad t > n^{-1}. \end{cases}$$

Moreover,

$$\|P_{\mathcal{N}(T)}x\|^p_{L_p[0,1]} = \left(\varepsilon^{-1/p} - n\varepsilon^{1/p'}\right)^p \varepsilon + (n^{-1} - \varepsilon)n^p\varepsilon^{p/p'} = (1-n\varepsilon)^p + (1-n\varepsilon)(n\varepsilon)^{p-1}.$$

For the function

$$f(t) = (1-t)(t^{p-1} + (1-t)^{p-1})$$

it is easy to see that $f(0) = 1$ and

$$\lim_{t\to+0} f'(t) = +\infty,$$

thus for small enough $\varepsilon > 0$ one has $\|P_{\mathcal{N}(T)}x\|_{L_p[0,1]} > 1$. This means that (1.21) does not hold for the unit ball $\mathfrak{N}$ in $L_p[0,1]$, $p \in (1,2)$.

In order to finish discussion of this example, consider the following application of the defined splines to the problem of optimal recovery, when the estimate from above can be obtained with the help of Theorem 1.7, although condition (1.21) does not hold (while condition (1.20) does hold).

Let $\omega\colon [0,\infty) \to \mathbb{R}$ be a modulus of continuity (i.e., a continuous non-decreasing semi-additive function that vanishes at 0) and denote by $H^\omega[0,1]$ the class of continuous functions $x\colon [0,1] \to \mathbb{R}$ such that

$$\forall s,t \in [0,1], |x(s) - x(t)| \le \omega(|s-t|).$$

Note that $H^\omega[0,1] \subset L_p[0,1]$ for all $1 \le p \le \infty$. Moreover, the T-interpolating spline s_x defined in (1.22) also belongs to $L_p[0,1]$ for all $1 \le p \le \infty$.

Using a known result on sharp estimates for the deviation between a function $x \in H^\omega[0,1]$ and the piecewise-constant function that interpolates the average values of x (see e.g., [17, Theorem 7.3.2]), we obtain that for a concave modulus of continuity ω and all $p \in [1,3]$

$$\mathcal{E}(H^\omega[0,1], T, L_p[0,1]) \le \sup_{x \in H^\omega[0,1]} \|x - s_x\|_{L_p[0,1]} = \|x_{n,\omega}\|_{L_p[0,1]},$$

where the function $x_{n,\omega}$ belongs to $H^\omega[0,1] \cap \mathcal{N}(T)$ and is defined as follows:

$$x_{n,\omega}(t) = \begin{cases} \frac{1}{2}\omega\left(\frac{1}{n} - 2t\right), & 0 \le t \le \frac{1}{2n}, \\ \frac{1}{2}\omega\left(2t - \frac{1}{n}\right), & \frac{1}{2n} \le t \le \frac{1}{n}, \end{cases}$$

and

$$x_{n,\omega}(t) = (-1)^k x_{n,\omega}\left(t - \frac{k}{n}\right)$$

for

$$\frac{k}{n} \le t \le \frac{k+1}{n}, \quad k = 0, 1, \ldots, n-1.$$

This implies the estimate

$$\mathcal{E}(H^\omega[0,1], T, L_p[0,1]) \le \sup_{x \in H^\omega[0,1] \cap \mathcal{N}(T)} \|x\|_{L_p[0,1]} = \|x_{n,\omega}\|_{L_p[0,1]},$$

On the other hand, the estimate

$$\mathcal{E}(H^\omega[0,1], T, L_p[0,1]) \ge \sup_{x \in H^\omega[0,1] \cap \mathcal{N}(T)} \|x\|_{L_p[0,1]} = \|x_{n,\omega}\|_{L_p[0,1]},$$

follows from Theorem 1.6.

In the case $p = \infty$ it is easy to obtain (see e.g., [16, p.158]) that for an arbitrary modulus of continuity ω

$$\sup_{x \in H^\omega[0,1]} \|x - s_x\|_{L_\infty[0,1]} = n \int_0^{1/n} \omega(u)du = \|\overline{x}_{n,\omega}\|_{L_\infty[0,1]},$$

where the function $\overline{x}_{n,\omega}$ belongs to $H^\omega[0,1] \cap \mathcal{N}(T)$ and is defined by the following formula:

$$\overline{x}_{n,\omega}(t) = \left| \min_{0 \le k \le \left[\frac{n}{2}\right]} \omega\left(\left|t - \frac{2k}{n}\right|\right) - n\int_0^{1/n} \omega(u)du \right|, \quad t \in [0,1].$$

Hence, we obtain

$$\mathcal{E}(H^\omega[0,1], T, L_\infty[0,1]) \le \sup_{x \in H^\omega[0,1] \cap \mathcal{N}(T)} \|x\|_{L_\infty[0,1]} = \|\overline{x}_{n,\omega}\|_{L_\infty[0,1]}.$$

As above, the estimate

$$\mathcal{E}(H^\omega[0,1], T, L_\infty[0,1]) \geq \sup_{x \in H^\omega[0,1] \cap \mathcal{N}(T)} \|x\|_{L_\infty[0,1]} = \|\overline{x}_{n,\omega}\|_{L_\infty[0,1]}$$

from below follows from Theorem 1.6.

4. Let Ω be a metric compact with a metric ρ_Ω and $B(\Omega)$ be the space of bounded functions $x\colon \Omega \to \mathbb{R}$ with the norm

$$\|x\|_{B(\Omega)} = \sup_{t \in \Omega} |x(t)|.$$

Assume also that points $t_1, \ldots, t_n \in \Omega$ be given. Let

$$\Omega_i' := \left\{ t \in \Omega\colon \quad \min_{j=\overline{1,n}} \{\rho_\Omega(t, t_j)\} = \rho_\Omega(t, t_i) \right\}, \quad i = 1, \ldots, n,$$

and

$$\Omega_1 = \Omega_1', \quad \Omega_k = \Omega_k' \setminus \bigcup_{i=1}^{k-1} \Omega_i', \quad i = 2, \ldots, n.$$

As the set W we consider the subspace of $B(\Omega)$ consisting of the functions $x \in B(\Omega)$ such that $x(t_k) = 0$ for all $k = 1, \ldots, n$.

A linear bounded operator $R\colon X \to W$ is defined by the formula

$$(Rx)(t) = x(t) - x(t_k), \quad \text{if} \quad t \in \Omega_k, \quad k = 1, \ldots, n.$$

Let also $Y = \mathbb{R}^n$ and $T\colon X \to Y$ be defined by the formula

$$Tx = (x(t_1). \ldots, x(t_n)).$$

It is clear that $\mathcal{N}(T) = W$ and $R(\mathcal{N}(T)) = W$.

Obviously, the set $R(\mathcal{N}(T)) = W$ is a Chebyshev set in W and for arbitrary $x \in X$

$$P_{R(\mathcal{N}(T))}(Rx) = Rx.$$

Thus, for the spline s_x we obtain $Rs_x \equiv 0$, that is s_x is a piecewise-constant function

$$s_x(t) = x(t_k) \text{ if } t \in \Omega_k, \; k = 1, \ldots, n.$$

Solution of some optimal recovery problems for integral operators on the class $H^\omega(T)$ was obtained in [5]; this result was generalized in [3] to the case of functions that take values in semi-linear metric spaces.

5. Let $\mathbb{Z}$ be the group of integer numbers with the usual operation $+$ and the metric $\rho(v,w)=|v-w|$. Assume that $n\in\mathbb{N}$, $n>1$, and denote by $\mathbb{Z}_n$ the group of residues modulo n. Let $W=X=\mathbb{Z}$, $Y=\mathbb{Z}_n$, R be the identity map, and $T\colon\mathbb{Z}\to\mathbb{Z}_n$ be the remainder of w divided by n. It is easy to see that $\mathcal{N}(T)=n\mathbb{Z}$; $n\mathbb{Z}$ is a Chebyshev set in $\mathbb{Z}$ if and only if n is odd. In the case of an odd n it is also easy to get an explicit formula for the projection operator, namely

$$P_{n\mathbb{Z}}w=\begin{cases} w-Tw, & Tw<\frac{n}{2},\\ w-Tw+n, & Tw>\frac{n}{2},\end{cases}$$

and

$$s_w=w-P_{n\mathbb{Z}}w=\begin{cases} Tw, & Tw<\frac{n}{2},\\ Tw-n, & Tw>\frac{n}{2},\end{cases}$$

i.e., s_w is the residue in the least absolute residue system

$$\left\{-\frac{n-1}{2},\ \dots,\ \frac{n-1}{2}\right\}.$$

6. Let

$$X=\left\{x\colon\mathbb{Z}\to\mathbb{Z}\colon\ \sum_{k\in\mathbb{Z}}|x(k)|<\infty\right\}$$

with a metric

$$\rho(x,y)=\sum_{k\in\mathbb{Z}}|x(k)-y(k)|,$$

$Y=n\mathbb{Z}$, with some natural n. Define $T\colon X\to Y$,

$$(Tx)(nk)=x(nk),\quad k\in\mathbb{Z}.$$

Obviously,

$$\mathcal{N}(T)=\{x\in X\colon\ x(nk)=0\ \forall\,k\in\mathbb{Z}\}.$$

The set $\mathcal{N}(T)$ is Chebyshev in X, for all $x\in X$

$$(P_{\mathcal{N}(T)}x)(k)=\begin{cases} 0, & \text{if}\quad k\in n\mathbb{Z},\\ x(nk), & \text{if}\quad k\notin n\mathbb{Z},\end{cases}$$

and the T-interpolating spline s_x is defined by

$$(s_x)(k)=\begin{cases} x(k), & \text{if}\quad k\in n\mathbb{Z},\\ 0, & \text{if}\quad k\notin n\mathbb{Z}.\end{cases}$$

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Andriy BANDURA, Tetyana SALO, and Oleh SKASKIV

2 Slice holomorphic functions in the polydics: directionally bounded index

2.1 Holomorphy in a fixed direction: motivation, definition and denotations

In recent years, analytic functions of several variables with bounded index have been intensively investigated. The main objects of investigations are such function classes: entire functions of several variables [12, 19, 37], functions analytic in a polydisc [4, 13], in a ball [1, 16] or in the Cartesian product of the complex plane and the unit disc [15], slice entire functions [9, 10] and slice analytic functions in the unit ball [7, 30].

For entire functions, analytic functions in a unit ball or in a unit polydisc there were proposed two approaches to introduce a concept of index boundedness in a multidimensional complex space. They generate two function classes: functions of bounded $\mathbf{L}$-index in joint variables and functions of bounded L-index in a direction, where $\mathbf{L}$ and L are some positive continuous functions. They are defined in some multidimensional complex domains and satisfy certain growth conditions at the boundary of these domains. The first approach requires joint holomorphy for multivariate functions of complex variables. But the second approach needs only holomorphy in one fixed direction and joint continuity in all variables. To present more details, we will introduce some notations and definitions from [2].

Let $\mathbf{0} = (0, \ldots, 0)$, $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{R}_+^* = [0, +\infty)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direction, $\mathbb{D}^n = \{z \in \mathbb{C}^n : |z_j| < 1\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $L : \mathbb{D}^n \to \mathbb{R}_+$ be a continuous function such that for all $z \in \mathbb{D}^n$

$$L(z) > \beta \max_{1 \le j \le n} \frac{|b_j|}{1 - |z_j|}, \ \beta = \text{const} > 1. \tag{2.1}$$

Remark 2.1. *Notice that if* $\eta \in [0, \beta]$, $z \in \mathbb{D}^n$ *and* $|t| \le \frac{\eta}{L(z)}$ *then* $z + t\mathbf{b} \in \mathbb{D}^n$.

Andriy BANDURA, Ivano-Frankivsk National Technical University of Oil and Gas, 15 Karpatska street, 76019, Ivano-Frankivsk, Ukraine, e-mail: andriy.bandura@nung.edu.ua

Tetyana SALO, Lviv Polytechnic National University, 12 Stepana Bandery street, 79000, Lviv, Ukraine, e-mail: tetiana.m.salo@lpnu.ua

Oleh SKASKIV, Ivan Franko National University of Lviv, 1 Universytetska street, 79000, Lviv, Ukraine, e-mail: oleh.skaskiv@lnu.edu.ua

For a given $z \in \mathbb{D}^n$ we denote $S_z = \{t \in \mathbb{C} : z + t\mathbf{b} \in \mathbb{D}^n\}$. The slice functions on S_z for fixed $z^0 \in \mathbb{D}^n$ we will denote as $g_{z^0}(t) = F(z^0 + t\mathbf{b})$ and $l_{z^0}(t) = L(z^0 + t\mathbf{b})$ for $t \in S_z$.

Definition 2.1 ([2]). *An analytic function $F : \mathbb{D}^n \to \mathbb{C}$ is called a function of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and for all $z \in \mathbb{D}^n$ one has*

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m! L^m(z)} \le \max_{0 \le k \le m_0} \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)}, \tag{2.2}$$

where $\partial_{\mathbf{b}}^0 F(z) = F(z), \partial_{\mathbf{b}} F(z) = \sum_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j$, $\partial_{\mathbf{b}}^k F(z) = \partial_{\mathbf{b}}\left(\partial_{\mathbf{b}}^{k-1} F(z)\right)$, $k \ge 2$.

The least such integer number m_0, obeying (2.2), is called the L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ of the function F and is denoted by $N_{\mathbf{b}}(F, L)$. If such m_0 does not exist, then we put $N_{\mathbf{b}}(F, L) = \infty$, and the function F is said to be of unbounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ in this case. Let $l : \mathbb{D} \to \mathbb{R}_+$ be a continuous function such that $l(z) > \frac{\beta}{1-|z|}$. For $n = 1$, $\mathbf{b} = 1$, $L(z) \equiv l(z)$ $(z \in \mathbb{D})$ inequality (2.2) defines an analytic function in the unit disc of bounded l-index with the l-index $N(F, l) \equiv N_1(F, l)$. The definition was firstly introduced for entire functions by B. Lepson [22] (see also [23]). Let $N_{\mathbf{b}}(F, L, z^0)$ stands for the L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ of the function F at the point z^0, i.e., it is the least integer m_0, for which inequality (2.2) is satisfied at this point $z = z^0$. By analogy, the notation $N(f, l, z^0)$ is defined if $n = 1$, i.e., in the case of analytic functions in the unit disc ((2.2) is true for $F = f$, $L = l$, $\mathbf{b} = 1$).

Methods of investigation of properties of functions with bounded L-index in a direction often use the restriction of the function to the slices $\{z^0 + t\mathbf{b} \colon\ t \in \mathbb{C}\}$. For fixed $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ and $z^0 \in \mathbb{C}^n$, using considerations from the one-dimensional case, mathematicians obtain the estimates which are uniform in $z^0 \in \mathbb{C}^n$. This is a short description of main idea. In view of this, Prof S. Yu. Favorov (2015) posed the following **problem** in a conversation with Prof. O. B. Skaskiv.

Problem 2.1 ([36]). *Let $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direction, $L : \mathbb{C}^n \to \mathbb{R}_+$ be a continuous function. Is it possible to replace the condition "F is holomorphic in $\mathbb{C}^n$" by the condition "F is holomorphic on all slices $z^0 + t\mathbf{b}$" and to deduce all known properties of analytic functions of bounded L-index in direction for this function class?*

There was presented a negative answer to Favorov's question [36]. This relaxation of restrictions by the function F does not allow the proving of some theorems. Here by $\overline{D}$ we denote a closure of domain D. There was proved the following proposition.

Proposition 2.1 ([36], Theorem 5). *For every direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *there exists a function* $F(z)$ *and a bounded domain* $D \subset \mathbb{C}^n$ *with following properties:*

1. F *is holomorphic function of bounded index on every slice* $\{z^0 + t\mathbf{b} : t \in \mathbb{C}\}$ *for each fixed* $z^0 \in \mathbb{C}^n$;

2. F *is not analytic function in* $\mathbb{C}^n$;

3. F *does not satisfy* (2.2) *in* $\overline{D}$, *i.e., for any* $p \in \mathbb{Z}_+$ *there exists* $m \in \mathbb{Z}_+$ *and* $z_p \in \overline{D}$
$$\frac{|\partial_{\mathbf{b}}^m F(z_p)|}{m!} > \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z_p)|}{k!} : 0 \le k \le p\right\}.$$

Let D be a bounded domain in $\mathbb{C}^n$. If inequality (2.2) holds for all $z \in D$ instead $\mathbb{C}^n$, then F is called *function of bounded* L*-index in the direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *in the domain* D. The least such integer m_0 is called the L*-index in the direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *in the domain* D and is denoted by $N_{\mathbf{b}}(F, L, D) = m_0$.

Proposition 2.2 ([36], Theorem 2). *Let* D *be a bounded domain in* $\mathbb{D}^n$, $\mathbf{b} \in \mathbb{D}^n \setminus \{\mathbf{0}\}$ *be arbitrary direction. If* $L\colon \mathbb{D}^n \to \mathbb{R}_+$ *is continuous function and* $F(z)$ *is an analytic function such that* $(\forall z^0 \in \overline{D})\colon \quad F(z^0 + t\mathbf{b}) \not\equiv 0$, *then* $N_{\mathbf{b}}(F, L, D) < \infty$.

Hence, if we replace holomorphy in $\mathbb{C}^n$ by holomorphy on the slices $\mathbb{C}^n \cap \{z^0 + t\mathbf{b} : t \in \mathbb{C}\}$, then conclusion of Proposition 2.2 is not valid. Thus, Proposition 2.1 shows impossibility to replace joint holomorphy by slice holomorphy without additional hypothesis. The proof of Proposition 2.2 uses continuity in joint variables (see [36], Equation (6)). In view of this, there was constructed theory of slice entire functions [10, 11] and slice holomorphic functions in the unit ball [7, 30]. Since the unit polydisc is not biholomorphic equivalent to unit ball, it leads to the following question

Problem 2.2. *What are additional conditions providing validity of Proposition 2.2 for slice holomorphic functions in the unit polydisc?*

Please note that the positivity and continuity of the function L are weak restrictions to deduce constructive results. Thus, we assume additional restrictions by the function L from [2, 6].

Let us denote
$$\lambda_{\mathbf{b}}(\eta) = \sup_{z\in\mathbb{D}^n} \sup_{t_1,t_2\in S_z} \left\{\frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})} \colon |t_1 - t_2| \le \frac{\eta}{\min\{L(z+t_1\mathbf{b}), L(z+t_2\mathbf{b})\}}\right\}.$$

By $Q_{\mathbf{b}}(\mathbb{D}^n)$, as in [3], we denote a class of positive continuous functions $L : \mathbb{D}^n \to \mathbb{R}_+$, satisfying (2.1) and the condition
$$\forall \eta \in [0; \beta] : \ \lambda_{\mathbf{b}}(\eta) < +\infty.$$

For a positive continuous function $l(t)$, $t \in \mathbb{D}$, and $\eta \in [0;\beta]$ we define $\lambda(\eta) \equiv \lambda_{\mathbf{b}}(\eta)$ in the cases, when $\mathbf{b} = 1$, $n = 1$, $L \equiv l$. As in [25], let $Q(\mathbb{D}) \equiv Q_1(\mathbb{D})$ be a class of positive continuous functions $l(t)$, $t \in \mathbb{D}$, obeying the condition $0 < \lambda(\eta) < +\infty$ for all $\eta > 0$. Besides, we denote by $\langle a, c\rangle = \sum_{j=1}^{n} a_j\overline{c_j}$, where $a, c \in \mathbb{D}^n$.

Let $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ be a class of functions which are holomorphic on every slices $\{z^0 + t\mathbf{b} : t \in S_{z^0}\}$ for each $z^0 \in \mathbb{D}^n$ and let $\mathcal{H}_{\mathbf{b}}(\mathbb{D}^n)$ be a class of functions from $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ which are joint continuous. The notation $\partial_{\mathbf{b}}F(z)$ stands for the derivative of the function $g_z(t)$ at the point 0, i.e., for every $p \in \mathbb{N}$ $\partial_{\mathbf{b}}^p F(z) = g_z^{(p)}(0)$, where $g_z(t) = F(z + t\mathbf{b})$ is analytic function of complex variable $t \in \mathbb{C}$ for given $z \in \mathbb{D}^n$. In this research, we will often call this derivative as directional derivative because if F is analytic function in $\mathbb{D}^n$ then the derivatives of the function $g_z(t)$ matches with directional derivatives of the function F.

Please note that if $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{D}^n)$ then for every $p \in \mathbb{N}$ $\partial_{\mathbf{b}}F \in \mathcal{H}_{\mathbf{b}}(\mathbb{D}^n)$. It can be proved by using of Cauchy's formula.

Together the hypothesis on joint continuity and the hypothesis on holomorphy in one direction **do not imply** holomorphy in whole n-dimensional complex unit polydisc. We give some examples to demonstrate it. For $n = 2$ let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function, $g : \mathbb{D} \to \mathbb{C}$ be a continuous function. Then $f(z_1)g(z_2)$, $f(z_1) \pm g(z_2)$ are functions which are holomorphic in the direction $(1, 0)$ and are joint continuous in $\mathbb{D}^2$. Moreover, the function $f(z_1 \cdot g(z_2))$ has the same properties if $|g(z)| = 1$. If, in addition, we have performed an affine transformation

$$\begin{cases} z_1 = b_2 z_1' + b_1 z_2', \\ z_2 = b_2 z_1' - b_1 z_2' \end{cases}$$

then the new functions are also holomorphic in the direction (b_1, b_2) and are joint continuous in $\mathbb{D}^2$, where $|b_1 b_2| = 1/2$. This conditions concerns the jacobian of the transofrmation and provides that unit polydisc will be mapped on a unit polydisc.

A function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ is said to be of *bounded L-index in the direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for all $m \in \mathbb{Z}_+$ and each $z \in \mathbb{D}^n$ inequality (2.2) is true. All notations, introduced above for analytic functions of bounded L-index in direction, keep for functions from $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$.

2.2 Sufficient Sets

Now we prove several assertions that establish a connection between functions of bounded L-index in direction and functions of bounded l-index of one variable. The similar results for analytic functions in the unit ball were obtained in [8], for slice holomorphic functions in $\mathbb{C}^n$ [10], for entire multivariate functions [38]. The next proofs use ideas from the mentioned papers.

Proposition 2.3. *If a function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ then for every $z \in \mathbb{D}^n$ the analytic function $g_z(t)$ is of bounded l_z-index and $N(g_z, l_z) \le N_{\mathbf{b}}(F, L)$.*

Proof. Let $z \in \mathbb{D}^n$, $g(t) \equiv g_z(t)$, $l(t) \equiv l_z(t)$. As for all $p \in \mathbb{N}$

$$g^{(p)}(t) = \partial_{\mathbf{b}}^p F(z + t\mathbf{b}), \tag{2.3}$$

then by the definition of boundedness of L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ for all $t \in S_z$ and $p \in \mathbb{Z}_+$ we obtain

$$\frac{|g^{(p)}(t)|}{p! l^p(t)} = \frac{|\partial_{\mathbf{b}}^p F(z^0 + t\mathbf{b})|}{p! L^p(z^0 + t\mathbf{b})} \le \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z^0 + t\mathbf{b})|}{k! L^k(z^0 + t\mathbf{b})} : 0 \le k \le N_{\mathbf{b}}(F, L)\right\} =$$

$$= \max\left\{\frac{|g^{(k)}(t)|}{k! l^k(t)} : 0 \le k \le N_{\mathbf{b}}(F, L)\right\}.$$

Hence, we obtain that $g(t)$ is of bounded l-index and $N(g, l) \le N_{\mathbf{b}}(F, L)$. Proposition 2.3 is proved. □

Equality (2.3) implies that the proposition holds.

Proposition 2.4. *If a function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ then*

$$N_{\mathbf{b}}(F, L) = \max\{N(g_z, l_z) : z \in \mathbb{D}^n\}.$$

Theorem 2.1. *A function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if and only if there exists a number $M > 0$ such that for all $z \in \mathbb{D}^n$ the function $g_z(t)$ is of bounded l_z-index with $N(g_z, l_z) \le M < +\infty$, as a function of variable $t \in \mathbb{C}$. Thus, $N_{\mathbf{b}}(F, L) = \max\{N(g_z, l_z) : z \in \mathbb{D}^n\}$.*

Proof. The necessity follows from Proposition 2.3.

Sufficiency. Since $N(g_z, l_z) \le M$, there exists $\max\{N(g_z, l_z) : z \in \mathbb{D}^n\}$. We denote $N_{\mathbf{b}}(F, L) = \max\{N(g_z, l_z) : z \in \mathbb{D}^n\} < +\infty$. Suppose that $N_{\mathbf{b}}(F, L)$ is not the L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ of the function $F(z)$. It means that there exists $n^* > N_{\mathbf{b}}(F, L)$ and $z^* \in \mathbb{D}^n$ such that

$$\frac{|\partial_{\mathbf{b}}^{n^*} F(z^*)|}{n^*! L^{n^*}(z^*)} > \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z^*)|}{k! L^k(z^*)} : 0 \le k \le N_{\mathbf{b}}(F, L)\right\}. \tag{2.4}$$

Since for $g_z(t) = F(z + t\mathbf{b})$ we have $g_z^{(p)}(t) = \partial_{\mathbf{b}}^p F(z + t\mathbf{b})$, inequality (2.4) can be rewritten as $\frac{\left|g_{z^*}^{(n^*)}(0)\right|}{n^*! l_{z^*}^{n^*}(0)} > \max\left\{\frac{|g_{z^*}^{(k)}(0)|}{k! l_{z^*}^k(0)} : 0 \le k \le N_{\mathbf{b}}(F, L)\right\}$, but it is impossible (it contradicts that all l_z-indices $N(g_{z_0}, l_z)$ are not greater than $N_{\mathbf{b}}(F, L)$). Thus, $N_{\mathbf{b}}(F, L)$ is the L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ of the function $F(z)$. Theorem 2.1 is proved. □

However, maximum can be calculated on a set A with a property $\bigcup_{z^0 \in A}\{z^0 + t\mathbf{b} : t \in S_{z^0}\} = \mathbb{D}^n$. Thus, the following assertion is valid.

Lemma 2.1. *If a function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ has bounded L-index in the direction $\mathbf{b}$, then*

$$N_{\mathbf{b}}(F, L) = \begin{cases} \max\{N(g_{z^0}, l_{z^0}) : z^0 \in \mathbb{D}^n, z^0_{j_0} = 0\}, & \text{if } j_0 \text{ is such that } b_{j_0} \neq 0, \\ \max\{N(g_{z^0}, l_{z^0}) : z^0 \in \mathbb{D}^n, \sum\limits_{j=1}^{n} z^0_j = 0\}, & \text{if } \sum_{j=1}^n b_j \neq 0. \end{cases}.$$

Proof. We prove that for every $z \in \mathbb{D}^n$ there exist $z^0 \in \mathbb{C}^n$ and $t \in S_{z^0}$ with $z = z^0 + t\mathbf{b}$ and $z^0_{j_0} = 0$. Put $t = z_{j_0}/b_{j_0}$, $z^0_j = z_j - t\mathbf{b}_j$, $j \in \{1, 2, \ldots, n\}$. Clearly, $z^0_{j_0} = 0$ for this choice.

However, the point z^0 may not be contained in $\mathbb{D}^n$. But there exists $t \in \mathbb{C}$ that $z^0 + t\mathbf{b} \in \mathbb{D}^n$. Let $z^0 \notin \mathbb{D}^n$ and $|z| = R_1 < 1$. Therefore, $|z^0 + t\mathbf{b}| = |z - \frac{z_{j_0}}{b_{j_0}}\mathbf{b} + t\mathbf{b}| = |z + (t - \frac{z_{j_0}}{b_{j_0}})\mathbf{b}| \leq |z| + |t - \frac{z_{j_0}}{b_{j_0}}| \cdot |\mathbf{b}| \leq R_1 + |t - \frac{z_{j_0}}{b_{j_0}}| \cdot |\mathbf{b}| < 1$. Thus, $|t - \frac{z_{j_0}}{b_{j_0}}| < \frac{1-R_1}{|\mathbf{b}|}$.

In the second part we prove that for every $z \in \mathbb{D}^n$ there exist $z^0 \in \mathbb{C}^n$ and $t \in S_{z^0}$ such that $z = z^0 + t\mathbf{b}$ and $\sum_{j=1}^n z^0_j = 0$. Put $t = \frac{\sum_{j=1}^n z_j}{\sum_{j=1}^n b_j}$ and $z^0_j = z_j - t\mathbf{b}_j$, $1 \leq j \leq n$. Thus, the following equality is valid $\sum_{j=1}^n z^0_j = \sum_{j=1}^n (z_j - tb_j) = \sum_{j=1}^n z_j - \sum_{j=1}^n b_j t = 0$.

Lemma 2.1 is proved. □

Note that for a given $z \in \mathbb{D}^n$ we can pick uniquely $z^0 \in \mathbb{C}^n$ and $t \in S_{z^0}$ such that $\sum_{j=1}^n z^0_j = 0$ and $z = z^0 + t\mathbf{b}$.

Remark 2.2. *If for some $z^0 \in \mathbb{C}^n$ one has $\{z^0 + t\mathbf{b} : t \in \mathbb{C}\} \bigcap \mathbb{D}^n = \emptyset$ then we put $N(g_{z^0}, l_{z^0}) = 0$.*

Theorem 2.2. *Let $A_0 \subset \mathbb{C}^n$ be such that $\bigcup_{z \in A_0}\{z + t\mathbf{b} : t \in S_z\} = \mathbb{D}^n$. A function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if and only if there exists $M > 0$ such that for all $z^0 \in A_0$ the function $g_{z^0}(t)$ is of bounded l_{z^0}-index with $N(g_{z^0}, l_{z^0}) \leq M < +\infty$, as a function of variable $t \in S_{z^0}$ and $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A_0\}$.*

Proof. By Theorem 2.1 the analytic function F is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if and only if there exists number $M > 0$ such that for every $z^0 \in \mathbb{D}^n$ the function $g_{z^0}(t)$ is of bounded l_{z^0}-index $N(g_{z^0}, l_{z^0}) \leq M < +\infty$, as a function of variable $t \in S_{z^0}$. But in view of property of the set A_0 for every $z^0 + t\mathbf{b}$ there exist $\widetilde{z}^0 \in A_0$ and $\widetilde{t} \in \mathbb{D}_{\widetilde{z}^0}$ such that $z^0 + t\mathbf{b} = \widetilde{z}^0 + \widetilde{t}\mathbf{b}$. In other words, for all

$p \in \mathbb{Z}_+$ $(g_{z_0}(t))^{(p)} = (g_{\widetilde{z}_0}(\widetilde{t}))^{(p)}$. But $\widetilde{t}$ depends on t. Thus, the condition that $g_{z^0}(t)$ is of bounded l_{z^0}-index for all $z^0 \in \mathbb{D}^n$ is equivalent to the condition $g_{\widetilde{z}^0}(t)$ is of bounded $l_{\widetilde{z}^0}$-index for all $\widetilde{z}^0 \in A_0$. □

Remark 2.3. *An intersection $H \cap \mathbb{D}^n_{\mathbf{b}}$ of an arbitrary hyperplane $H = \{z \in \mathbb{C}^n : \langle z, c\rangle = 1\}$ and the set $\mathbb{D}^n_{\mathbf{b}} = \{z + \frac{1-\langle z,\mathbf{c}\rangle}{\langle \mathbf{b},c\rangle}\mathbf{b} \colon z \in \mathbb{D}^n\}$, where $\langle \mathbf{b}, c\rangle \neq 0$, satisfies conditions of Theorem 2.2.*

We prove that for every $w \in \mathbb{D}^n$ there exist $z \in H \bigcap \mathbb{D}^n_{\mathbf{b}}$ and $t \in \mathbb{C}$ such that $w = z + t\mathbf{b}$.

Choosing $z = w + \frac{1-\langle w,c\rangle}{\langle \mathbf{b},c\rangle}\mathbf{b} \in H \bigcap \mathbb{D}^n_{\mathbf{b}}$, $t = \dfrac{\langle w, c\rangle - 1}{\langle \mathbf{b}, c\rangle}$, we obtain

$$z + t\mathbf{b} = w + \frac{1 - \langle w, c\rangle}{\langle \mathbf{b}, c\rangle}\mathbf{b} + \frac{\langle w, c\rangle - 1}{\langle \mathbf{b}, c\rangle}\mathbf{b} = w.$$

Theorem 2.3 requires replacement of the space $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ by the space $\mathcal{H}_{\mathbf{b}}(\mathbb{D}^n)$. In other words, we use joint continuity in its proof.

Theorem 2.3. *Let $\overline{A} = \mathbb{D}^n$, i.e., A be an everywhere dense set in $\mathbb{D}^n$ and let a function $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{D}^n)$. The function F is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if and only if there exists $M > 0$ such that for all $z^0 \in A$ a function $g_{z^0}(t)$ is of bounded l_{z^0}-index $N(g_{z^0}, l_{z^0}) \le M < +\infty$ and $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A\}$.*

Proof. The necessity follows from Theorem 2.1.

Sufficiency. Since $\overline{A} = \mathbb{D}^n$, then for every $z^0 \in \mathbb{D}^n$ there exists a sequence $z^{(m)}$, that $z^{(m)} \to z^0$ as $m \to +\infty$ and $z^{(m)} \in A$ for all $m \in \mathbb{N}$. However, $F(z + t\mathbf{b})$ is of bounded l_z-index for all $z \in A$ as a function of variable t. That is why in view the definition of bounded l_z-index there exists $M > 0$ that for all $z \in A$, $t \in \mathbb{C}$, $p \in \mathbb{Z}_+$ $\frac{|g_z^{(p)}(t)|}{p! l^p(t)} \le \max\left\{\frac{|g_z^{(k)}(t)|}{k! l_z^k(t)} : 0 \le k \le M\right\}$.

Substituting instead of z a sequence $z^{(m)} \in A$, $z^{(m)} \to z^0$, we obtain that for every $m \in \mathbb{N}$

$$\frac{|\partial^p_{\mathbf{b}} F(z^{(m)} + t\mathbf{b})|}{p! L^p(z^{(m)} + t\mathbf{b})} \le \max\left\{\frac{|\partial^k_{\mathbf{b}} F(z^{(m)} + t\mathbf{b})|}{k! L^k(z^{(m)} + t\mathbf{b})} : 0 \le k \le M\right\}.$$

Since F is joint continuous, we can write $\partial^p_{\mathbf{b}} F(z) = \frac{p!}{2\pi i}\int_{|t|=r} \frac{F(z+t\mathbf{b})}{t^{p+1}} dt$. Hence, F and $\partial^p_{\mathbf{b}} F$ are continuous in $\mathbb{D}^n$ for all $p \in \mathbb{N}$ and L is a positive continuous function. Thus, in the obtained expression the limiting transition is possible as $m \to +\infty$ $(z^{(m)} \to z^0)$. Evaluating the limit as $m \to +\infty$ we obtain that for all $z^0 \in \mathbb{D}^n$, $t \in \mathbb{C}$, $m \in \mathbb{Z}_+$

$$\frac{|\partial^p_{\mathbf{b}} F(z^0 + t\mathbf{b})|}{p! L^p(z^0 + t\mathbf{b})} \le \max\left\{\frac{|\partial^k_{\mathbf{b}} F(z^0 + t\mathbf{b})|}{k! L^k(z^0 + t\mathbf{b})} : 0 \le k \le M\right\}.$$

This inequality implies that $F(z+t\mathbf{b})$ is of bounded $L(z+t\mathbf{b})$-index as a function of variable t for every given $z \in \mathbb{D}^n$. Applying Theorem 2.1 we obtain the desired conclusion. Theorem 2.3 is proved. $\square$

Remark 2.3 and Theorem 2.3 yield the following corollary.

Corollary 2.1. *Let A_0 be such that its closure is $\overline{A}_0 = \{z \in \mathbb{C}^n : \langle z, c\rangle = 1\} \bigcap \mathbb{D}^n_{\mathbf{b}}$, where $\langle c, \mathbf{b}\rangle \neq 0$, $\mathbb{D}^n_{\mathbf{b}} = \{z + \frac{1-\langle z,\mathbf{c}\rangle}{\langle \mathbf{b},c\rangle}\mathbf{b} \colon z \in \mathbb{D}^n\}$. A function $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{D}^n)$ is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if and only if there exists number $M > 0$ such that for all $z^0 \in A_0$ the function $g_{z^0}(t)$ is of bounded l_{z^0}-index with $N(g_{z^0}, l_{z^0}) \leq M < +\infty$, as a function of variable $t \in S_{z^0}$. And $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A_0\}$.*

Proof. In view of Remark 2.3 in Theorem 2.2 we can take $B_0 = \{z \in \mathbb{C}^n : \langle z, c\rangle = 1\} \cap \{z + \frac{1-\langle z,\mathbf{c}\rangle}{\langle \mathbf{b},c\rangle}\mathbf{b} \colon z \in \mathbb{D}^n\}$, where $\langle c, \mathbf{b}\rangle \neq 0$. Let A_0 be a dense set in B_0, $\overline{A_0} = B_0$. Repeating considerations of Theorem 2.3, we obtain the desired conclusion.

Indeed, the necessity follows from Theorem 2.3 (in this theorem same condition is satisfied for all $z^0 \in \mathbb{C}^n$, and we need this condition for all $z^0 \in A_0$).

To prove the sufficiency, we use the density of the set A_0. Obviously, for every $z^0 \in B_0$ there exists a sequence $z^{(m)} \to z^0$ and $z^{(m)} \in A_0$. But $g_z(t)$ is of bounded l_z-index for all $z \in A_0$. Taking the conditions of Corollary 2.1 into account, for some $M > 0$ and for all $z \in A_0$, $t \in \mathbb{C}$, $p \in \mathbb{Z}_+$ the following inequality holds $\frac{g_z^{(p)}(t)}{p! l_z^p(t)} \leq \max\left\{\frac{|g_z^{(k)}(t)|}{k! l_z^k(t)} : 0 \leq k \leq M\right\}$.

Substituting an arbitrary sequence $z^{(m)} \in A$, $z^{(m)} \to z^0$ instead of $z \in A^0$, we have $\frac{|g^{(p)}_{z^{(m)}}(t)|}{p! l^p_{z^{(m)}}(t)} \leq \max\left\{\frac{|g^{(k)}_{z^{(m)}}(t)|}{k! l^k_{z^{(m)}}(t)} : 0 \leq k \leq M\right\}$, that is

$$\frac{\left|\partial^p_{\mathbf{b}} F(z^{(m)}+t\mathbf{b})\right|}{L^p(z^{(m)}+t\mathbf{b})} \leq \max_{0\leq k\leq M} \frac{\left|\partial^k_{\mathbf{b}} F(z^{(m)}+t\mathbf{b})\right|}{k! L^k(z^{(m)}+t\mathbf{b})}.$$

However, F is an analytic function in $\mathbb{D}^n$, L is a positive continuous. So we calculate a limit as $m \to +\infty$ ($z^m \to z$). For all $z^0 \in B_0$, $t \in S_{z^0}$, $m \in \mathbb{Z}_+$ we have

$$\frac{\left|\partial^p_{\mathbf{b}} F(z^0+t\mathbf{b})\right|}{L^p(z^0+t\mathbf{b})} \leq \max_{0\leq k\leq M} \frac{\left|\partial^k_{\mathbf{b}} F(z^0+t\mathbf{b})\right|}{k! L^k(z^0+t\mathbf{b})}.$$

Therefore, $F(z^0+t\mathbf{b})$ is of bounded $L(z^0+t\mathbf{b})$-index as a function of t at each $z^0 \in \mathbb{D}^n$. By Theorem 2.3 and Remark 2.3 F is of bounded L-index in the direction $\mathbf{b}$. $\square$

Proposition 2.5. *Let (r_p) be a positive sequence such that $r_p \to 1$ as $p \to \infty$, $D_p = \{z \in \mathbb{C}^n : |z| = r_p\}$, A_p be a dense set in D_p (i.e. $\overline{A}_p = D_p$) and $A = \bigcup\limits_{p=1}^{\infty} A_p$.*

A function $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{D}^n)$ *is of bounded* L*-index in the direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *if and only if there exists number* $M > 0$ *such that for all* $z^0 \in A$ *the function* $g_{z^0}(t)$ *is of bounded* l_{z^0}*-index* $N(g_{z^0}, l_{z^0}) \le M < +\infty$*, as a function of variable* $t \in S_{z^0}$*. And* $N_{\mathbf{b}}(F, L) = \max\{N(g_{z^0}, l_{z^0}) : z^0 \in A\}$.

Proof. Theorem 2.1 implies the necessity of this theorem.

Sufficiency. It is easy to prove $\{z + t\mathbf{b} : t \in S_z,\ z \in A\} = \mathbb{D}^n$. Further, we repeat arguments with the proof of sufficiency in Theorem 2.3 and obtain the desired conclusion. □

2.3 Local behavior of directional derivative

The following proposition is important in theory of functions of bounded index. It initializes series of propositions which are necessary to prove logarithmic criterion of index boundedness. It was first obtained by G. H. Fricke [42] for entire functions of bounded index. Later the proposition was generalized for entire functions of bounded l-index [31], analytic functions of bounded l-index [39], entire functions of bounded L-index in direction [28], functions analytic in a polydisc [13] or in a ball [16] with bounded $\mathbf{L}$-index in joint variables, for slice holomorphic functions in $\mathbb{C}^n$ [10].

Theorem 2.4. *Let* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$*. A function* $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ *is of bounded* L*-index in the direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *if and only if for each* $\eta \in (0; \beta]$ *there exist* $n_0 = n_0(\eta) \in \mathbb{Z}_+$ *and* $P_1 = P_1(\eta) \ge 1$ *such that for every* $z \in \mathbb{D}^n$ *there exists* $k_0 = k_0(z) \in \mathbb{Z}_+$, $0 \le k_0 \le n_0$, *and*

$$\max\left\{\left|\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})\right| : |t| \le \frac{\eta}{L(z)}\right\} \le P_1 \left|\partial_{\mathbf{b}}^{k_0} F(z)\right|. \tag{2.5}$$

Proof. Our proof is based on the proof of appropriate theorem for analytic functions in the unit polydisc having bounded L-index in direction [1] and for slice holomorphic functions in $\mathbb{C}^n$ [10].

Necessity. Let $N_{\mathbf{b}}(F; L) \equiv N < +\infty$, and $[a]$, $a \in \mathbb{R}$, stands for the integer part of the number a in this proof. We denote

$$q(\eta) = [2\eta(N + 1)(\lambda_{\mathbf{b}}(\eta))^{2N+1}] + 1.$$

For $z \in \mathbb{D}^n$ and $p \in \{0, 1, \ldots, q(\eta)\}$ we put

$$R_p^{\mathbf{b}}(z, \eta) = \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z + t\mathbf{b})|}{k! L^k(z + t\mathbf{b})} : |t| \le \frac{p\eta}{q(\eta)L(z)}, 0 \le k \le N\right\},$$

$$\widetilde{R}_p^{\mathbf{b}}(z, \eta) = \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z + t\mathbf{b})|}{k! L^k(z)} : |t| \le \frac{p\eta}{q(\eta)L(z)}, 0 \le k \le N\right\}.$$

However, $|t| \le \frac{p\eta}{q(\eta)L(z)} \le \frac{\eta}{L(z)}$, then $\lambda_{\mathbf{b}}\left(\frac{p\eta}{q(\eta)}\right) \le \lambda_{\mathbf{b}}(\eta)$. It is clear that $R_p^{\mathbf{b}}(z,\eta)$, $\widetilde{R}_p^{\mathbf{b}}(z,\eta)$ are well-defined. Moreover,

$$\begin{gathered} R_p^{\mathbf{b}}(z,\eta) = \\ = \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z)} \left(\frac{L(z)}{L(z+t\mathbf{b})}\right)^k : 0 \le k \le N, |t| \le \frac{p\eta}{q(\eta)L(z)} \right\} \le \\ \le \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z)} \left(\lambda_{\mathbf{b}}\left(\frac{p\eta}{q(\eta)}\right)\right)^k : |t| \le \frac{p\eta}{q(\eta)L(z)}, 0 \le k \le N \right\} \le \\ \le \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z)} (\lambda_{\mathbf{b}}(\eta))^k : |t| \le \frac{p\eta}{q(\eta)L(z)}, 0 \le k \le N \right\} \le \\ \le (\lambda_{\mathbf{b}}(\eta))^N \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z)} : |t| \le \frac{p\eta}{q(\eta)L(z)}, 0 \le k \le N \right\} = \\ = \widetilde{R}_p^{\mathbf{b}}(z,\eta)(\lambda_{\mathbf{b}}(\eta))^N, \end{gathered} \tag{2.6}$$

$$\begin{gathered} \widetilde{R}_p^{\mathbf{b}}(z,\eta) = \\ = \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z+t\mathbf{b})} \left(\frac{L(z+t\mathbf{b})}{L(z)}\right)^k : |t| \le \frac{p\eta}{q(\eta)L(z)}, 0 \le k \le N \right\} \le \\ \le \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z+t\mathbf{b})} \left(\lambda_{\mathbf{b}}\left(\frac{p\eta}{q(\eta)}\right)\right)^k : |t| \le \frac{p\eta}{q(\eta)L(z)}, 0 \le k \le N \right\} \le \\ \le \max\left\{ (\lambda_{\mathbf{b}}(\eta))^k \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z+t\mathbf{b})} : |t| \le \frac{p\eta}{q(\eta)L(z)}, 0 \le k \le N \right\} \le \\ \le (\lambda_{\mathbf{b}}(\eta))^N \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z+t\mathbf{b})|}{k!L^k(z+t\mathbf{b})} : |t| \le \frac{p\eta}{q(\eta)L(z)}, \ 0 \le k \le N \right\} = \\ = R_p^{\mathbf{b}}(z,\eta)(\lambda_{\mathbf{b}}(\eta))^N. \end{gathered} \tag{2.7}$$

Let $k_p^z \in \mathbb{Z}$, $0 \le k_p^z \le N$, and $t_p^z \in S_z$, $|t_p^z| \le \frac{p\eta}{q(\eta)L(z)}$, be such that

$$\widetilde{R}_p^{\mathbf{b}}(z,\eta) = \frac{|\partial_{\mathbf{b}}^{k_p^z} F(z+t_p^z\mathbf{b})|}{k_p^z! L^{k_p^z}(z)}. \tag{2.8}$$

However, for every given $z \in \mathbb{D}^n$ the function $g_z(t) = F(z+t\mathbf{b})$ and its derivatives are analytic as functions of variable t. Then by the maximum modulus principle, equality (2.8) holds for t_p^z such that $|t_p^z| = \frac{p\eta}{q(\eta)L(z)}$. We set $\widetilde{t_p^z} = \frac{p-1}{p} t_p^z$. Then

$$|\widetilde{t_p^z}| = \frac{(p-1)\eta}{q(\eta)L(z)}, \tag{2.9}$$

$$|\widetilde{t_p^z} - t_p^z| = \frac{|t_p^z|}{p} = \frac{\eta}{q(\eta)L(z)}. \tag{2.10}$$

It follows from (2.9) and the definition of $\widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta)$ that

$$\widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta) \ge \frac{|\partial_{\mathbf{b}}^{k_p^z} F(z+\widetilde{t_p^z}\mathbf{b})|}{k_p^z! L^{k_p^z}(z)}.$$

Therefore,

$$0 \le \widetilde{R}_p^{\mathbf{b}}(z,\eta) - \widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta) \le \frac{\left|\partial_{\mathbf{b}}^{k_p^z} F(z+t_p^z\mathbf{b})\right| - \left|\partial_{\mathbf{b}}^{k_p^z} F(z+\widetilde{t}_p^z\mathbf{b})\right|}{k_p^z! L^{k_p^z}(z)} = \\ = \frac{1}{k_p^z! L^{k_p^z}(z)} \int_0^1 \frac{d}{ds} \left|\partial_{\mathbf{b}}^{k_p^z} F(z+(\widetilde{t}_p^z + s(t_p^z - \widetilde{t}_p^z))\mathbf{b})\right| ds. \tag{2.11}$$

For every analytic complex-valued function of real variable $\varphi(s)$, $s \in \mathbb{R}$, the inequality $\frac{d}{ds}|\varphi(s)| \le \left|\frac{d}{ds}\varphi(s)\right|$ holds, where $\varphi(s) \neq 0$. Applying this inequality to (2.11) and using the mean value theorem we obtain

$$\widetilde{R}_p^{\mathbf{b}}(z,t_0,\eta) - \widetilde{R}_{p-1}^{\mathbf{b}}(z,t_0,\eta) \le \\ \le \frac{|t_p^z - \widetilde{t}_p^z|}{k_p^z! L^{k_p^z}(z)} \int_0^1 \left|\partial_{\mathbf{b}}^{k_p^z+1} F(z+(\widetilde{t}_p^z + s(t_p^z - \widetilde{t}_p^z))\mathbf{b})\right| ds = \\ = \frac{|t_p^z - \widetilde{t}_p^z|}{k_p^z! L^{k_p^z}(z)} \left|\partial_{\mathbf{b}}^{k_p^z+1} F(z+(\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z))\mathbf{b})\right| = \\ = L(z)(k_p^z+1)|t_p^z - \widetilde{t}_p^z| \frac{|\partial_{\mathbf{b}}^{k_p^z+1} F(z+(\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z))\mathbf{b})|}{(k_p^z+1)! L^{k_p^z+1}(z)},$$

where $s^* \in [0,1]$. The point $\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z)$ belongs to the set

$$\left\{ t \in \mathbb{C} : |t| \le \frac{p\eta}{q(\eta)L(z)} \le \frac{\eta}{L(z)} \right\}.$$

Using the definition of boundedness of L-index in direction, the definition of $q(\eta)$, inequalities (2.6) and (2.10), for $k_p^z \le N$ we have

$$\widetilde{R}_p^{\mathbf{b}}(z,\eta) - \widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta) \le \frac{|\partial_{\mathbf{b}}^{k_p^z+1} F(z+(\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z))\mathbf{b})|}{(k_p^z+1)! L^{k_p^z+1}(z+(\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z))\mathbf{b})} \times \\ \times \left(\frac{L(z+(\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z))\mathbf{b})}{L(z)}\right)^{k_p^z+1} L(z)(k_p^z+1)|t_p^z - \widetilde{t}_p^z| \le \\ \le \eta \frac{N+1}{q(\eta)} (\lambda_{\mathbf{b}}(\eta))^{N+1} \max\left\{ \frac{|\partial_{\mathbf{b}}^{k} F(z+(\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z))\mathbf{b})|}{k! L^{k}(z+(\widetilde{t}_p^z + s^*(t_p^z - \widetilde{t}_p^z))\mathbf{b})} : 0 \le k \le N \right\} \le \\ \le \eta \frac{N+1}{q(\eta)} (\lambda_{\mathbf{b}}(\eta))^{N+1} R_p^{\mathbf{b}}(z,\eta) \le \\ \le \frac{\eta(N+1)(\lambda_{\mathbf{b}}(\eta))^{2N+1}}{[2\eta(N+1)(\lambda_{\mathbf{b}}(\eta))^{2N+1}]+1} \widetilde{R}_p^{\mathbf{b}}(z,\eta) \le \frac{1}{2} \widetilde{R}_p^{\mathbf{b}}(z,\eta)$$

It follows that $\widetilde{R}_p^{\mathbf{b}}(z,\eta) \le 2\widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta)$. Using inequalities (2.6) and (2.7), we obtain for $R_p^{\mathbf{b}}(z,\eta)$

$$R_p^{\mathbf{b}}(z,\eta) \le 2(\lambda_{\mathbf{b}}(\eta))^N \widetilde{R}_{p-1}^{\mathbf{b}}(z,\eta) \le 2(\lambda_{\mathbf{b}}(\eta))^{2N} R_{p-1}^{\mathbf{b}}(z,\eta).$$

Hence,

$$\begin{gathered}\max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z+t\mathbf{b})|}{k!L^{k}(z+t\mathbf{b})}:|t|\le\frac{\eta}{L(z)},0\le k\le N\right\}=R^{\mathbf{b}}_{q(\eta)}(z,\eta)\le\\ \le 2(\lambda_{\mathbf{b}}(\eta))^{2N}R^{\mathbf{b}}_{q(\eta)-1}(z,\eta)\le(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{2}R^{\mathbf{b}}_{q(\eta)-2}(z,\eta)\le\\ \le\cdots\le(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)}R^{\mathbf{b}}_{0}(z,\eta)=\\ =(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)}\max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{k!L^{k}(z)}:0\le k\le N\right\}.\end{gathered}\tag{2.12}$$

Let $k_z\in\mathbb{Z}$, $0\le k_z\le N$, and $\widetilde{t_z}\in\mathbb{C}$, $|\widetilde{t_z}|=\frac{\eta}{L(z)}$ be such that

$$\frac{|\partial_{\mathbf{b}}^{k_z}F(z)|}{k_z!L^{k_z}(z)}=\max_{0\le k\le N}\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{k!L^{k}(z)},$$

and

$$|\partial_{\mathbf{b}}^{k_z}F(z+\widetilde{t_z}\mathbf{b})|=\max\{|\partial_{\mathbf{b}}^{k_z}F(z+t\mathbf{b})|:|t|\le\eta/L(z)\}.$$

Inequality (2.12) implies

$$\begin{gathered}\frac{|\partial_{\mathbf{b}}^{k_z}F(z+\widetilde{t_z}\mathbf{b})|}{k_z!L^{k_z}(z+\widetilde{t_z}\mathbf{b})}\le\max\left\{\frac{|\partial_{\mathbf{b}}^{k_z}F(z+t\mathbf{b})|}{k_z!L^{k_z}(z+t\mathbf{b})}:|t|=\frac{\eta}{L(z)}\right\}\le\\ \le\max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z+t\mathbf{b})|}{k!L^{k}(z+t\mathbf{b})}:|t|=\frac{\eta}{L(z)},0\le k\le N\right\}\le\\ \le(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)}\frac{|\partial_{\mathbf{b}}^{k_z}F(z)|}{k_z!L^{k_z}(z)}.\end{gathered}$$

Hence,

$$\begin{gathered}\max\left\{|\partial_{\mathbf{b}}^{k_z}F(z+t\mathbf{b})|:|t|\le\eta/L(z)\right\}\le\\ \le(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)}\frac{L^{k_z}(z+\widetilde{t_z}\mathbf{b})}{L^{k_z}(z)}|\partial_{\mathbf{b}}^{k_z}F(z)|\le\\ \le(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)}(\lambda_{\mathbf{b}}(\eta))^{N}|\partial_{\mathbf{b}}^{k_z}F(z)|\le\\ \le(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)}(\lambda_{\mathbf{b}}(\eta))^{N}|\partial_{\mathbf{b}}^{k_z}F(z)|.\end{gathered}$$

Thus, we obtain (2.5) with $n_0=N_{\mathbf{b}}(F,L)$ and

$$P_1(\eta)=(2(\lambda_{\mathbf{b}}(\eta))^{2N})^{q(\eta)}(\lambda_{\mathbf{b}}(\eta))^{N}>1.$$

Sufficiency. Suppose that for each $\eta\in(0;\beta]$ there exist $n_0=n_0(\eta)\in\mathbb{Z}_+$ and $P_1=P_1(\eta)\ge 1$ such that for every $z\in\mathbb{D}^n$ there exists $k_0=k_0(z)\in\mathbb{Z}_+$, $0\le k_0\le n_0$, for which inequality (2.5) holds. We choose $\eta>1$ and $j_0\in\mathbb{N}$ such that $P_1 \le \eta^{j_0}$. For given $z \in \mathbb{D}^n$, $k_0 = k_0(z)$ and $j \ge j_0$ by Cauchy's formula for $F(z + t\mathbf{b})$ as a function of one variable t

$$\partial_{\mathbf{b}}^{k_0+j} F(z) = \frac{j!}{2\pi i} \int_{|t|=\eta/L(z)} \frac{\partial_{\mathbf{b}}^{k_0} F(z+t\mathbf{b})}{t^{j+1}} dt.$$

Therefore, in view of (2.5) we have

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{j!} \le \frac{L^j(z)}{\eta^j} \max\left\{|\partial_{\mathbf{b}}^{k_0} F(z+t\mathbf{b})| : |t| = \frac{\eta}{L(z)}\right\} \le P_1 \frac{L^j(z)}{\eta^j} |\partial_{\mathbf{b}}^{k_0} F(z)|,$$

Hence, for all $j \ge j_0$, $z \in \mathbb{D}^n$

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{(k_0+j)! L^{k_0+j}(z)} \le \frac{j! k_0!}{(j+k_0)!} \frac{P_1}{\eta^j} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z)} \le \eta^{j_0 - j} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z)} \le \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z)}.$$

Since $k_0 \le n_0$, the numbers $n_0 = n_0(\eta)$ and $j_0 = j_0(\eta)$ are independent of z and t_0, this inequality means that a function F has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ and $N_{\mathbf{b}}(F, L) \le n_0 + j_0$. The proof of Theorem 2.4 is complete. □

Theorem 2.4 implies the next proposition that describes the boundedness of L-index in direction for an equivalent function to L.

Theorem 2.5. *Let $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$, $\frac{1}{\beta} < \theta_1 \le \theta_2 < +\infty$, $\theta_1 L(z) \le L^*(z) \le \theta_2 L(z)$. A function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ is of bounded L^*-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if and only if F is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$.*

Proof. Obviously, if $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$ and $\theta_1 L(z) \le L^*(z) \le \theta_2 L(z)$, then $L^* \in Q_{\mathbf{b}}(\mathbb{D}^n)$ with $\beta^* \in [\theta_1\beta; \theta_2\beta]$ and $\beta^* > 1$ instead $\beta > 1$. Let $N_{\mathbf{b}}(F, L^*) < +\infty$. Therefore, by Theorem 2.4 for each η^*, $0 < \eta^* < \beta\theta_2$, there exist $n_0(\eta^*) \in \mathbb{Z}_+$ and $P_1(\eta^*) \ge 1$ such that for every $z \in \mathbb{D}^n$, $t_0 \in S_z$ and some k_0, $0 \le k_0 \le n_0$, inequality (2.5) is valid with L^* and η^* instead of L and η. Taking $\eta* = \theta_2\eta$ we obtain

$$\begin{aligned} P_1 |\partial_{\mathbf{b}}^{k_0} F(z)| \ge \max\left\{|\partial_{\mathbf{b}}^{k_0} F(z+t\mathbf{b})| : |t| \le \eta^*/L^*(z)\right\} \ge \\ \ge \max\left\{|\partial_{\mathbf{b}}^{k_0} F(z+t\mathbf{b})| : \ |t| \le \eta/L(z)\right\}. \end{aligned}$$

Therefore, by Theorem 2.4 the function $F(z)$ is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$. The converse assertion is obtained by replacing L on L^*. □

Theorem 2.6. *Let $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$, $m \in \mathbb{C} \setminus \{0\}$. A function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n$ if and only if $F(z)$ is of bounded L-index in the direction $m\mathbf{b}$.*

Proof. Let a function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ be of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$. By Theorem 2.4 $(\forall \eta > 0)$ $(\exists n_0(\eta) \in \mathbb{Z}_+)$ $(\exists P_1(\eta) \geq 1)$ $(\forall z \in \mathbb{D}^n)$ $(\exists k_0 = k_0(z) \in \mathbb{Z}_+, 0 \leq k_0 \leq n_0)$, and the following inequality is valid

$$\max\left\{|\partial_{\mathbf{b}}^{k_0} F(z+t\mathbf{b})| : |t| \leq \eta/L(z)\right\} \leq P_1|\partial_{\mathbf{b}}^{k_0} F(z)|. \tag{2.13}$$

Since $\frac{\partial^k F}{\partial (m\mathbf{b})^k} = (m)^k \frac{\partial^k F}{\partial \mathbf{b}^k}$, inequality (2.13) is equivalent to the inequality

$$\max\left\{|m|^{k_0}|\partial_{\mathbf{b}}^{k_0} F(z+t\mathbf{b})| : |t| \leq \eta/L(z)\right\} \leq P_1|m|^{k_0}|\partial_{\mathbf{b}}^{k_0} F(z)|$$

as well as to the inequality

$$\max\left\{\left|\partial_{m\mathbf{b}}^{k_0} F\big(z+\frac{t}{m}m\mathbf{b}\big)\right| : |t/m| \leq \eta/(|m|L(z))\right\} \leq P_1|\partial_{m\mathbf{b}}^{k_0} F(z)|.$$

Denoting $t^* = \frac{t}{m}$, $\eta^* = \frac{\eta}{|m|}$, we obtain

$$\max\left\{|\partial_{m\mathbf{b}}^{k_0} F(z+t^* m\mathbf{b})| : |t^*| \leq \eta^*/L(z)\right\} \leq P_1|\partial_{\mathbf{b}}^{k_0} F(z)|.$$

By Theorem 2.4 the function $F(z)$ is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$. The converse assertion can be proved similarly. □

Please note that Proposition 2.5 can be slightly refined. The following proposition is easy deduced from (2.2).

Proposition 2.6. *Let $L_1(z)$, $L_2(z)$ be positive continuous functions, $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ be a function of bounded L_1-index in the direction $\mathbf{b}$, for all $z \in \mathbb{D}^n$ the inequality $L_1(z) \leq L_2(z)$ holds. Then $N_{\mathbf{b}}(L_2, F) \leq N_{\mathbf{b}}(L_1, F)$.*

Using Fricke's idea [40], we deduce a modification of Theorem 2.4. Our proof is similar to proof in [58].

Theorem 2.7. *Let $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$, $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$. If there exist $\eta \in (0, \beta]$, $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that for any $z \in \mathbb{D}^n$ there exists $k_0 = k_0(z) \in \mathbb{Z}_+$, $0 \leq k_0 \leq n_0$, and*

$$\max\{|\partial_{\mathbf{b}}^{k_0} F(z+t\mathbf{b})| : |t| \leq \eta/L(z)\} \leq P_1|\partial_{\mathbf{b}}^{k_0} F(z)|,$$

then the function F has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$.

Proof. Besides the mentioned paper of Fricke [40], our proof is similar to proofs in [58] (analytic functions in the unit ball of bounded L-index in direction), [31] (entire functions of bounded l-index) and [10] (slice holomorphic functions in $\mathbb{C}^n$).

Assume that there exist $\eta \in (0, \beta]$, $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that for any $z \in \mathbb{D}^n$ there exists $k_0 = k_0(z) \in \mathbb{Z}_+$, $0 \leq k_0 \leq n_0$, and

$$\max\{|\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : |t| \leq \frac{\eta}{L(z)}\} \leq P_1 |\partial_{\mathbf{b}}^{k_0} F(z)|. \tag{2.14}$$

If $\eta \in (1, \beta]$, then we choose $j_0 \in \mathbb{N}$ such that $P_1 \leq \eta^{j_0}$. And for $\eta \in (0; 1]$ we choose $j_0 \in \mathbb{N}$ such that $\frac{j_0! k_0!}{(j_0+k_0)!} P_1 < 1$. The j_0 is well-defined because

$$\frac{j_0! k_0!}{(j_0 + k_0)!} P_1 = \frac{k_0!}{(j_0 + 1)(j_0 + 2) \cdot \ldots \cdot (j_0 + k_0)} P_1 \to 0, \ j_0 \to \infty.$$

Applying integral Cauchy's formula to the function $g_z(t) = F(z + t\mathbf{b})$ as analytic function of one complex variable t for $j \geq j_0$ we obtain that for every $z \in \mathbb{D}^n$ there exists $k_0 = k_0(z)$, $0 \leq k_0 \leq n_0$, and

$$\partial_{\mathbf{b}}^{k_0+j} F(z) = \frac{j!}{2\pi i} \int\limits_{|t| = \frac{\eta}{L(z)}} \frac{\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})}{t^{j+1}} dt.$$

Taking into account (2.14), we deduce

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{j!} \leq \frac{L^j(z)}{\eta^j} \max \left\{ |\partial_{\mathbf{b}}^{k_0} F(z + t\mathbf{b})| : \ |t| = \frac{\eta}{L(z)} \right\} \leq P_1 \frac{L^j(z)}{\eta^j} |\partial_{\mathbf{b}}^{k_0} F(z)|. \tag{2.15}$$

In view of choice j_0 with $\eta \in (1, \beta]$, for all $j \geq j_0$ one has

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{(k_0 + j)! L^{k_0+j}(z)} \leq \frac{j! k_0!}{(j + k_0)!} \frac{P_1}{\eta^j} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z + t_0\mathbf{b})} \leq \eta^{j_0 - j} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z)} \leq \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0! L^{k_0}(z)}.$$

Since $k_0 \leq n_0$, the numbers $n_0 = n_0(\eta)$ and $j_0 = j_0(\eta)$ do not depend of z, and $z \in \mathbb{D}^n$ is arbitrary, the last inequality is equivalent to the assertion that F has boudned L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ and $N_{\mathbf{b}}(F, L) \leq n_0 + j_0$.

If $\eta \in (0, 1)$, then from (2.15) it follows that for all $j \geq j_0$

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{(k_0 + j)! L^{k_0+j}(z)} \leq \frac{j! k_0! P_1}{(j + k_0)!} \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{\eta^j k_0! L^{k_0}(z)} \leq \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{\eta^j k_0! L^{k_0}(z)}$$

or in view of choice j_0

$$\frac{|\partial_{\mathbf{b}}^{k_0+j} F(z)|}{(k_0 + j)!} \frac{\eta^{k_0+j}}{L^{k_0+j}(z)} \leq \frac{|\partial_{\mathbf{b}}^{k_0} F(z)|}{k_0!} \frac{\eta^{k_0}}{L^{k_0}(z)}.$$

Thus, the function F is of bounded $\tilde{L}$-index in the direction $\mathbf{b}$, where $\tilde{L}(z) = \frac{L(z)}{\eta}$. Then by Lemma 2.5 the function F has bounded L-index in the direction

$\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, if $\eta\beta > 1$. When $\eta \le \frac{1}{\beta}$, we choose arbitrary $\gamma > \frac{1}{\eta\beta}$. By Lemma 2.5 the function F is of bounded L_1-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, where $L_1(z) = \eta\gamma\tilde{L}(z)$. Then be Lemma 2.6 the function F has bounded L_1-index in the direction $\gamma\mathbf{b}$. Since $\partial^k_{\gamma\mathbf{b}} F = \gamma^k \partial^k_{\mathbf{b}} F$ and $L^k_1(z) = \gamma^k L^k(z)$, in inequality (2.2) with the definition of L-index boundedness in direction the corresponding multiplier γ is reduced. Hence, the function F is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$. Theorem is proved. □

2.4 L-index in direction in a domain compactly embedded in the polydisc

Let D be an arbitrary bounded domain in $\mathbb{D}^n$ such that $\operatorname{dist}(D, \mathbb{B}^n) > 0$. If inequality (2.2) holds for all $z \in D$ instead $\mathbb{D}^n$, then the function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ is called *a function of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ in the domain D*. The least such integer m_0 is called the *L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ in domain D* and is denoted by $N_{\mathbf{b}}(F, L, D) = m_0$. The notation $\overline{D}$ stands for a closure of the domain D.

Theorem 2.8. *Let D be a bounded domain in $\mathbb{D}^n$ such that $d = \operatorname{dist}(D, \mathbb{B}^n) = \inf_{z \in D}(1 - |z|) > 0$, $\beta > 1$, $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direction. If $L\colon \mathbb{D}^n \to \mathbb{R}_+$ is continuous function such that $L(z) \ge \frac{\beta|\mathbf{b}|}{d}$, and a function $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{D}^n)$ be such that $(\forall z^0 \in \overline{D})\colon \quad F(z^0 + t\mathbf{b}) \not\equiv 0$, then $N_{\mathbf{b}}(F, L, D) < \infty$.*

Proof. This proof is similar to proof in [57] for analytic functions in the unit ball.

For every fixed $z^0 \in \overline{D}$ we expand the analytic function $F(z^0 + t\mathbf{b})$ in a power series by powers of t in the disc $\left\{t \in \mathbb{C} : |t| \le \frac{1}{L(z^0)}\right\}$

$$F(z^0 + t\mathbf{b}) = \sum_{m=0}^{\infty} \frac{\partial^m_{\mathbf{b}} F(z^0)}{m!} t^m. \tag{2.16}$$

The quantity $\frac{|\partial^m_{\mathbf{b}} F(z^0)|}{m!}$ is the modulus of a coefficient of the power series (2.16) at the point $t \in \mathbb{C}$ such that $|t| = \frac{1}{L(z^0)}$. Since $F(z)$ is analytic function, for every $z_0 \in \overline{D}$

$$\frac{|\partial^m_{\mathbf{b}} F(z^0)|}{m! L^m(z^0)} \to 0 \quad (m \to \infty),$$

i.e., there exists $m_0 = m(z^0, \mathbf{b})$ such that inequality (2.2) holds at the point $z = z^0$ for all $m \in \mathbb{Z}_+$.

We prove that $\sup\{m_0\colon z^0 \in \overline{D}\} < +\infty$. On the contrary, we assume that the set of all values m_0 is unbounded in z^0, i.e., $\sup\{m_0\colon z^0 \in \overline{D}\} = +\infty$. Hence, for every $m \in \mathbb{Z}_+$ there exists $z^{(m)} \in \overline{D}$ and $p_m > m$

$$\frac{|\partial^{p_m}_{\mathbf{b}} F(z^{(m)})|}{p_m! L^{p_m}(z^{(m)})} > \max\left\{\frac{|\partial^k_{\mathbf{b}} F(z^{(m)})|}{k! L^k(z^{(m)})}\colon \ 0 \le k \le m\right\}. \tag{2.17}$$

Since $\{z^{(m)}\} \subset \overline{D}$, there exists subsequence $z'^{(m)} \to z' \in \overline{G}$ as $m \to +\infty$. By Cauchy's integral formula

$$\frac{\partial_{\mathbf{b}}^p F(z)}{p!} = \frac{1}{2\pi i}\int_{|t|=r} \frac{F(z+t\mathbf{b})}{t^{p+1}}dt$$

for any $p \in \mathbb{N}$, $z \in D$. Rewrite (2.17) as following

$$\max\left\{\frac{|\partial_{\mathbf{b}}^k F(z^{(m)})|}{k! L^k(z^{(m)})} \colon\ 0 \le k \le m\right\} <$$

$$< \frac{1}{L^{p_m}(z^{(m)})}\int_{|t|=r/L(z^{(m)})} \frac{|F(z^{(m)}+t\mathbf{b})|}{|t|^{p_m+1}}|dt| \le \frac{1}{r^{p_m}}\max\{|F(z)| : z \in D_r\}, \quad (2.18)$$

where $D_r = \bigcup_{z^*\in\overline{D}}\{z \in \mathbb{C}^n : |z-z^*| \le \frac{|\mathbf{b}|r}{L(z^*)}\}$. We can choose $r \in (1,\beta)$, because $g_{z^{(m)}}(t) = F(z^{(m)}+t\mathbf{b})$ is an analytic function in $S_{z^{(m)}}$. Evaluating the limit for every directional derivative of fixed order in (2.18) as $m \to \infty$ we obtain

$$(\forall k \in \mathbb{Z}_+)\colon \quad \frac{|\partial_{\mathbf{b}}^k F(z')|}{k! L^k(z')} \le \varlimsup_{m\to\infty}\frac{1}{r^{p_m}}\max\{|F(z)| : z \in D_r\} \le 0.$$

Thus, all derivatives in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ of the function F at the point z' equals 0 and $F(z') = 0$. In view of (2.16) $F(z'+t\mathbf{b}) \equiv 0$. It is a contradiction.

□

2.5 Estimate of maximum modulus by minimum modulus

Using Theorem 2.4, we will prove the next criterion of L-index boundedness in direction. Similar results was firstly deduced by G. H. Fricke [40] for analytic functions of bounded index. Further it was generalized for various classes of holomorphic functions [16, 25, 31].

Theorem 2.9. *Let $L \in Q_{\mathbf{b}}^n$. A function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if and only if for any $r_{1,2} \in (0;\beta]$ $(r_1 < r_2)$, there exists $P_1 = P_1(r_1, r_2) \ge 1$ such that for every $z^0 \in \mathbb{D}^n$*

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{r_2}{L(z^0)}\right\} \le P_1 \max\left\{|F(z^0+t\mathbf{b})| :\ |t| = \frac{r_1}{L(z_0)}\right\}. \quad (2.19)$$

Proof. Our proof is based on the proof of appropriate theorem for analytic functions of bounded L-index in direction [28, 32].

Necessity. Let $N_{\mathbf{b}}(F,L) < +\infty$. On the contrary, suppose that there exist number r_1 and r_2, $0 < r_1 < r_2 \le \beta$, such that for each $P_* \ge 1$ there exists $z^* = z^*(P_*) \in \mathbb{D}^n$ satisfying inequality

$$\max\left\{|F(z^*+t\mathbf{b})| : |t| = \frac{r_2}{L(z^*)}\right\} > P_* \max\left\{|F(z^*+t\mathbf{b})| :\ |t| = \frac{r_1}{L(z^*)}\right\}.$$

By Theorem 2.4 there exist $n_0 = n_0(r_2) \in \mathbb{Z}_+$ and $P_0 = P_0(r_2) \geq 1$ such that for all $z^* \in \mathbb{D}^n$ and some $k_0 = k_0(t^*, z^*) \in \mathbb{Z}_+$, $0 \leq k_0 \leq n_0$, the following inequality holds

$$\max\left\{\left|\partial_{\mathbf{b}}^{k_0} F(z^* + t\mathbf{b})\right| : |t| = r_2/L(z^*)\right\} \leq P_0|\partial_{\mathbf{b}}^{k_0} F(z^*)|. \tag{2.20}$$

One should observe that for $k_0 = 0$ the proof of necessity is obvious, because (2.20) implies $\max\left\{|F(z^*+t\mathbf{b})| : \ |t| = r_2/L(z^*)\right\} \leq P_0|F(z^*)| \leq P_0 \max\left\{|F(z^*+t\mathbf{b})| : |t| = r_1/L(z^*)\right\}$.

Suppose that $k_0 > 0$ and

$$P_* = n_0!\left(\frac{r_2}{r_1}\right)^{n_0}\left(P_0 + \frac{r_1}{r_2 - r_1}\right) + 1. \tag{2.21}$$

We choose $t_0 \in \mathbb{C}$ such that $|t_0| = r_1/L(z^*)$ and

$$|F(z^* + t_0\mathbf{b})| = \max\left\{|F(z^* + t\mathbf{b})| : |t| = r_1/L(z^*)\right\} > 0,$$

and $t_{0j} \in \mathbb{C}$, $|t_{0j}| = r_2/L(z^*)$, be such that

$$|\partial_{\mathbf{b}}^{j} F(z^* + t_{0j}\mathbf{b})| = \max\{|\partial_{\mathbf{b}}^{j} F(z^* + t\mathbf{b})| : |t| = r_2/L(z^*)\},$$

$j \in \mathbb{Z}_+$. In the case $|F(z^* + t_0\mathbf{b})| = 0$ by uniqueness theorem for all $t \in \mathbb{C}$ one has $F(z^* + t\mathbf{b}) = 0$. But it contradicts inequality (2.5). By Cauchy's inequality we deduce

$$\frac{|\partial_{\mathbf{b}}^{j} F(z^*)|}{j!} \leq \left(\frac{L(z^*)}{r_1}\right)^{j} |F(z^* + t_0\mathbf{b})|, j \in \mathbb{Z}_+ \tag{2.22}$$

$$\begin{aligned}\left|\partial_{\mathbf{b}}^{j} F(z^* + t_{0j}\mathbf{b}) - \partial_{\mathbf{b}}^{j} F(z^*)\right| &= \left|\int_0^{t_{0j}} \partial_{\mathbf{b}}^{j+1} F(z^* + t\mathbf{b})\, dt\right| \leq \\ &\leq \left|\partial_{\mathbf{b}}^{j+1} F(z^* + t_{0(j+1)}\mathbf{b})\right| \frac{r_2}{L(z^*)}.\end{aligned} \tag{2.23}$$

From inequalities (2.22) and (2.23) it follows that

$$\begin{aligned}|\partial_{\mathbf{b}}^{j+1} F(z^*+t_{0(j+1)}\mathbf{b})| &\geq \frac{L(z^*)}{r_2}\left\{|\partial_{\mathbf{b}}^{j} F(z^*+t_{0j}\mathbf{b})| - |\partial_{\mathbf{b}}^{j} F(z^*)|\right\} \geq \\ &\geq \frac{L(z^*+t^*\mathbf{b})}{r_2}\left|\partial_{\mathbf{b}}^{j} F(z^* + t_{0j}\mathbf{b})\right| - \frac{j! L^{j+1}(z^*)}{r_2 (r_1)^j}|F(z^* + t_0\mathbf{b})|,\end{aligned}$$

$j \in \mathbb{Z}_+$. Hence, for $k_0 \geq 1$ we obtain

$$|\partial_{\mathbf{b}}^{k_0} F(z^* + t_{0k_0}\mathbf{b})| \geq \frac{L(z^*)}{r_2}|\partial_{\mathbf{b}}^{k_0-1} F(z^* + t_{0(k_0-1)}\mathbf{b})| -$$

$$\frac{(k_0-1)!L^{k_0}(z^*)}{r_2(r_1)^{k_0-1}}|F(z^*+t_0\mathbf{b})|\geq\ldots\geq\frac{L^{k_0}(z^*)}{(r_2)^{k_0}}|F(z^*+t_{00}\mathbf{b})|-$$
$$-\left(\frac{0!}{(r_2)^{k_0}}+\frac{1!}{(r_2)^{k_0-1}r_1}+\ldots+\frac{(k_0-1)!}{r_2(r_1)^{k_0-1}}\right)L^{k_0}(z^*)|F(z^*+t_0\mathbf{b})|=$$
$$=\frac{L^{k_0}(z^*)}{(r_2)^{k_0}}|F(z^*+t_0\mathbf{b})|\left(\frac{|F(z^*+t_{00}\mathbf{b})|}{|F(z^*+t_0\mathbf{b})|}-\sum_{j=0}^{k_0-1}j!\left(\frac{r_2}{r_1}\right)^j\right). \tag{2.24}$$

From (2.5) one has $|F(z^*+t_{00}\mathbf{b})|/|F(z^*+t_0\mathbf{b})|>P_*$. Besides, the following inequality is true

$$\sum_{j=0}^{k_0-1}j!\left(\frac{r_2}{r_1}\right)^j\leq k_0!\left(\frac{(r_2/r_1)^{k_0}-1}{r_2/r_1-1}\right)\leq n_0!\frac{r_1}{r_2-r_1}\left(\frac{r_2}{r_1}\right)^{n_0}.$$

Applying (2.21), we get

$$\frac{|F(z^*+t_{00}\mathbf{b})|}{|F(z^*+t_0\mathbf{b})|}-\sum_{j=0}^{k_0-1}j!\frac{r_2^j}{r_1^j}>P_*-\frac{n_0!r_1}{r_2-r_1}\left(\frac{r_2}{r_1}\right)^{n_0}=n_0!\left(\frac{r_2}{r_1}\right)^{n_0}P_0+1.$$

In view of (2.20) and (2.22), from (2.24) it follows that

$$\left|\partial_{\mathbf{b}}^{k_0}F(z^*+t_{0k_0}\mathbf{b})\right|>\frac{L^{k_0}(z^*)}{(r_2)^{k_0}}\left(P_*-n_0!\frac{r_1}{r_2-r_1}\left(\frac{r_2}{r_1}\right)^{n_0}\right)\left(\frac{r_1}{L(z^*)}\right)^{k_0}\times$$
$$\times\frac{|\partial_{\mathbf{b}}^{k_0}F(z^*)|}{k_0!}\geq\left(\frac{r_1}{r_2}\right)^{n_0}\left(P_*-n_0!\frac{r_1}{r_2-r_1}\left(\frac{r_2}{r_1}\right)^{n_0}\right)\frac{|\partial_{\mathbf{b}}^{k_0}F(z^*+t_{0k_0}\mathbf{b})|}{n_0!P_0}.$$

Hence, $P_*<n_0!\left(\frac{r_2}{r_1}\right)^{n_0}\left(P_0+\frac{r_1}{r_2-r_1}\right)$, and it contradicts (2.21).

Sufficiency. Choose any $r_1\in(0,1)$ and $r_2\in(1,\beta]$. For given $z^0\in\mathbb{D}^n$ we develop the function $F(z^0+t\mathbf{b})$ in power series by powers t

$$F(z^0+t\mathbf{b})=\sum_{m=0}^{\infty}b_m(z^0)t^m,\ b_m(z^0)=\frac{\partial_{\mathbf{b}}^m F(z^0)}{m!}$$

in the disc $\{t:|t|\leq\beta/L(z^0)\}$. For $r>0$ we define

$$M_{\mathbf{b}}(r,z^0,F)=\max\{|F(z^0+t\mathbf{b})|:|t|=r\},$$
$$\mu_{\mathbf{b}}(r,z^0,F)=\max\{|b_m(z^0)|r^m:m\geq0\},$$
$$\nu_{\mathbf{b}}(r,z^0,F)=\max\{|b_m(z^0)|r^m:|b_m(z^0)|r^m=\mu_{\mathbf{b}}(r,z^0,F)\}.$$

By Cauchy's inequality $\mu_{\mathbf{b}}(r,z^0,F)\leq M_{\mathbf{b}}(r,z^0,F)$. But for $r=\dfrac{1}{L(z^0)}$ one has

$$M_{\mathbf{b}}(r_1r,z^0,F)\leq\sum_{m=0}^{\infty}|b_m(z^0)|r^mr_1^m\leq\mu_{\mathbf{b}}(r,z^0,F)\sum_{m=0}^{\infty}r_1^m=\frac{\mu_{\mathbf{b}}(r,z^0,F)}{1-r_1}$$

and, applying monotonicity of $\nu_{\mathbf{b}}(r,z^0,F)$ in r, we get

$$\ln\mu_{\mathbf{b}}(r_2r,z^0,F)-\ln\mu_{\mathbf{b}}(r,z^0,F)=\int_r^{r_2r}\frac{\nu_{\mathbf{b}}(t,z^0,F)}{t}dt\geq\nu_{\mathbf{b}}(r,z^0,F)\ln r_2.$$

Hence, it follows that

$$\nu_{\mathbf{b}}(r,z^0,F)\leq\frac{1}{\ln r_2}(\ln\mu_{\mathbf{b}}(r_2r,z^0,F)-\ln\mu_{\mathbf{b}}(r,z^0,F))\leq$$
$$\leq\frac{1}{\ln r_2}\{\ln M_{\mathbf{b}}(r_2r,z^0,F)-\ln((1-r_1)M_{\mathbf{b}}(r_1r,z^0,F))\}=$$
$$=-\frac{\ln(1-r_1)}{\ln r_2}+\frac{1}{\ln r_2}\{\ln M_{\mathbf{b}}(r_2r,z^0,F)-\ln M_{\mathbf{b}}(r_1r,z^0,F)\}\tag{2.25}$$

Let $N_{\mathbf{b}}(F,L,z^0)$ be the L-index in direction of the function F at the point z^0, i.e, $N_{\mathbf{b}}(F,L,z^0)$ is the least number m_0, for which inequality (2.2) holds at the point $z=z^0$. Obviously that $N_{\mathbf{b}}(F,L,z^0)\leq\nu_{\mathbf{b}}(1/L(z^0),z^0,F)=\nu_{\mathbf{b}}(r,z^0,F)$. But inequality (2.19) can be rewritten in the following form $M_{\mathbf{b}}\left(\frac{r_2}{L(z^0)},z^0,F\right)\leq P_1(r_1,r_2)M_{\mathbf{b}}\left(\frac{r_1}{L(z^0)},z^0,F\right)$. Thus, from (2.25) one has $N_{\mathbf{b}}(F,L,z^0)\leq-\frac{\ln(1-r_1)}{\ln r_2}+\frac{\ln P_1(r_1,r_2)}{\ln r_2}$ for every $z^0\in\mathbb{D}^n$, that is $N_{\mathbf{b}}(F,L)\leq-\frac{\ln(1-r_1)}{\ln r_2}+\frac{\ln P_1(r_1,r_2)}{\ln r_2}$. Theorem 2.9 is proved. □

In view of proof of sufficiency in Theorem 2.9 the following lemma is valid.

Lemma 2.2. *Let $L\in Q^n_{\mathbf{b}}$, $F\in\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$. If there exist numbers r_1 and r_2, $0<r_1<1<r_2\leq\beta$, and $P_1\geq1$ such that for every $z^0\in\mathbb{D}^n$ inequality (2.19) holds then the function F is of bounded L-index in the direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$.*

We can relax sufficient conditions of Lemma 2.2, replacing the condition $0<r_1<1<r_2\leq\beta$ by $0<r_1<r_2\leq\beta$.

Proposition 2.7. *Let $L\in Q_{\mathbf{b}}(\mathbb{D}^n)$, $F\in\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$. If there exist r_1 and r_2, $0<r_1<r_2\leq\beta$, and $P_1\geq1$ such that for all $z^0\in\mathbb{D}^n$ inequality* (2.19) *holds, then the function has F bounded L-index in the direction* $\mathbf{b}$.

Proof. Our proof uses the idea of A.D. Kuzyk and M. M. Sheremeta [21].

Inequality (2.19) for $0<r_1<r_2<+\infty$ implies

$$\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{2r_2}{r_1+r_2}\frac{r_1+r_2}{2L(z^0)}\right\}\leq$$
$$\leq P_1\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{2r_1}{r_1+r_2}\frac{r_1+r_2}{2L(z_0)}\right\}.$$

Defining $L^*(z) = \frac{2L(z)}{r_1+r_2}$, we obtain

$$\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{2r_2}{(r_1+r_2)L^*(z^0)}\right\}\le$$
$$\le P_1\max\left\{|F(z^0+t\mathbf{b})|:|t|=\frac{2r_1}{(r_1+r_2)L^*(z^0)}\right\},$$

where $0<\frac{2r_1}{r_1+r_2}<1<\frac{2r_2}{r_1+r_2}<+\infty$. This means that F has bounded L^*-index in the direction $\mathbf{b}$. And by Proposition 2.5 the function F has bounded L-index in the direction $\mathbf{b}$. □

The following theorem gives estimate of maximum modulus by minimum modulus. It was firstly obtained by G. H. Fricke [40] for analytic functions of bounded index.

Theorem 2.10. *Let $L\in Q^n_{\mathbf{b}}$. If the function $F\in\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ is of bounded L-index in the direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ then for each $R>0$ there exist $P_2(R)\ge 1$ and $\eta(R)\in(0,R)$ such that for every $z^0\in\mathbb{D}^n$ and some $r=r(z^0)\in[\eta(R),R]$ the inequality holds*

$$\max\{|F(z^0+t\mathbf{b})|:|t|=r/L(z^0)\}\le P_2\min\{|F(z^0+t\mathbf{b})|:\ |t|=r/L(z^0)\}. \tag{2.26}$$

Proof. Our proof is based on the proof of appropriate theorem for analytic functions of bounded L-index in direction [28].

Let $N_{\mathbf{b}}(F,L)=N<+\infty$ and $R\ge 0$. Put

$$R_0=1, r_0=\frac{R}{8(R+1)},\ R_j=\frac{R_{j-1}}{4N}r^N_{j-1},\ r_j=\frac{1}{8}R_j(j=1,2,\dots,N).$$

Let $z^0\in\mathbb{D}^n$ and $N_0=N_{\mathbf{b}}(z^0,L,F)$ be the L-index in the direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ of the function F at the point z^0, i.e., $N_{\mathbf{b}}(z^0,L,F)$ is the least number m_0, for which inequality (2.2) holds for $z=z^0$. The maximum in the right-hand side (2.2) is attained at m_0. But $0\le N_0\le N$. For given $z^0\in\mathbb{D}^n$ the function $F(z^0+t\mathbf{b})$ can be developed in power series by powers t

$$F(z^0+t\mathbf{b})=\sum_{m=0}^{\infty}b_m(z^0)t^m,\ b_m(z^0)=\frac{\partial^m_{\mathbf{b}}F(z^0)}{m!}.$$

Put $a_m(z^0)=\frac{|b_m(z^0)|}{L^m(z^0)}=\frac{|\partial^m_{\mathbf{b}}F(z^0)|}{m!L^m(z^0)}$. For every $m\in\mathbb{Z}_+$ the inequlity $a_{N_0}(z^0)\ge a_m(z^0)=R_0a_m(z^0)$ is true. Then there exists the least number $n_0\in\{0,1,\dots,N_0\}$ such that for all $m\in\mathbb{Z}_+$ $a_{n_0}(z^0)\ge a_m(z^0)R_{N_0-n_0}$. Thus, $a_{n_0}(z^0)\ge a_{N_0}(z^0)R_{N_0-n_0}$ and $a_j(z^0)<a_{N_0}(z^0)R_{N_0-j}$ for $j<n_0$, because if $a_{j_0}(z^0)\ge a_{N_0}(z^0)R_{N_0-j_0}$ for some $j_0<n_0$ then $a_{j_0}(z^0)\ge a_m(z^0)R_{N_0-j_0}$ for all $m\in\mathbb{Z}_+$, but it contradicts the choice

of n_0. From inequalities $a_j(z^0) < a_{N_0}(z^0)R_{N_0-j}$ $(j < n_0)$ and $a_m(z^0) \le a_{N_0}(z^0)$ $(m > n_0)$ for $t \in S_{z^0}$ and $|t| = \frac{1}{L(z^0)} r_{N_0-n_0}$ we deduce

$$|F(z^0+t\mathbf{b})| =$$
$$= |b_{n_0}(z^0)t^{n_0} + \sum_{m\ne n_0} b_m(z^0)t^m| \ge |b_{n_0}(z^0)||t|^{n_0} - \sum_{m\ne n_0} |b_m(z^0)||t|^m =$$
$$= a_{n_0}(z^0)r^{n_0}_{N_0-n_0} - \sum_{m\ne 0} a_m(z^0)r^m_{N_0-n_0} = a_{n_0}(z^0)r^{n_0}_{N_0-n_0} - \sum_{j<n_0} a_j(z^0)r^j_{N_0-n_0} -$$
$$-\sum_{m>n_0} a_m(z^0)r^m_{N_0-n_0} \ge a_{N_0}(z^0)R_{N_0-n_0}r^{n_0}_{N_0-n_0} - \sum_{j<n_0} a_{N_0}(z^0)R_{N_0-j}r^j_{N_0-n_0} -$$
$$-\sum_{m>n_0} a_{N_0}(z^0)r^m_{N_0-n_0} \ge a_{N_0}(z^0)R_{N_0-n_0}r^{n_0}_{N_0-n_0} - n_0 a_{N_0}(z^0)R_{N_0-n_0+1} -$$
$$-a_{N_0}(z^0)r^{n_0+1}_{N_0-n_0}\frac{1}{1-r_{N_0-n_0}} = a_{N_0}(z^0)\left(R_{N_0-n_0}r^{n_0}_{N_0-n_0} - \frac{n_0}{4N}R_{N_0-n_0}r^N_{N_0-n_0} -\right.$$
$$\left.-r^{n_0}_{N_0-n_0}\frac{r_{N_0-n_0}}{1-r_{N_0-n_0}}\right) \ge a_{N_0}(z^0)\left(R_{N_0-n_0}r^{n_0}_{N_0-n_0} - \frac{1}{4}R_{N_0-n_0}r^{n_0}_{N_0-n_0} -\right.$$
$$\left.-\frac{1}{4}R_{N_0-n_0}r^{n_0}_{N_0-n_0}\right) = \frac{1}{2}a_{N_0}(z^0)R_{N_0-n_0}r^{n_0}_{N_0-n_0}. \tag{2.27}$$

Besides, for $t \in \mathbb{C}$ we have

$$|F(z^0+t\mathbf{b})| \le \sum_{m=0}^{+\infty} |b_m(z^0)||t|^m = \sum_{m=0}^{\infty} a_m(z^0)r^m_{N_0-n_0} \le$$
$$\le a_{N_0}(z^0)\sum_{m=0}^{+\infty} r^m_{N_0-n_0} = \frac{a_{N_0}(z^0)}{1-r_{N_0-n_0}} \le \frac{a_{N_0}(z^0)}{1-1/8} = \frac{8}{7}a_{N_0}(z^0). \tag{2.28}$$

From (2.27) and (2.28) it follows

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = r_{N_0-n_0}/L(z^0)\right\} \le \frac{8}{7}a_{N_0}(z^0) \le$$
$$\le \frac{16}{7}\frac{1}{R_{N_0-n_0}}r^{-n_0}_{N_0-n_0}\min\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{r_{N_0-n_0}}{L(z^0)}\right\} \le$$
$$\le \frac{16}{7}\frac{1}{R_N}r_N^{-N}\min\left\{|F(z^0+t\mathbf{b})| : |t| = r_{N_0-n_0}/L(z^0)\right\},$$

i.e., (2.26) holds with $P_2(R) = \dfrac{16}{7R_N r_N^N}$, $\eta(R) = r_N = \dfrac{1}{8R_N}$ and $r = r_{N_0-n_0}$. Theorem 2.10 is proved. □

Below we will prove the sufficint conditions which are symmetric to necessary conditions from Theorem 2.10

Theorem 2.11. *Let* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *and* $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$, $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$. *If there exist* $R \in (0, \beta]$, $P_2 \geq 1$ *and* $\eta \in (0, R)$ *such that for all* $z^0 \in \mathbb{D}^n$ *and some* $r = r(z^0) \in [\eta, R]$ *inequality* (2.26) *is valid, then the function* F *has bounded* L*-index in the direction* $\mathbf{b}$.

Proof. Directly this proposition was unknown for analytic functions of bounded index, i.e. for functions of one variable. Firstly, it was obtained for analytic functions of bounded L-index in direction in [47,48]. In view of Proposition 2.7 it is sufficient to show that there exists P_1 such that for all $z^0 \in \mathbb{D}^n$

$$\begin{aligned} &\max\left\{|F(z^0 + t\mathbf{b})| : |t| = (R+1)/L(z^0)\right\} \leq \\ &\leq P_1 \max\left\{|F(z^0 + t\mathbf{b})| : |t| = R/L(z^0)\right\}. \end{aligned} \tag{2.29}$$

Suppose that there exist $R \in (0, \beta]$, $P_2 \geq 1$ and $\eta \in \big(0, R\big)$ such that for all $z^0 \in \mathbb{D}^n$ and some $r = r(z^0) \in \big[\eta, R\big]$ one has

$$\max\left\{|F(z^0+t\mathbf{b})| : |t|=r/L(z^0)\right\} \leq P_2 \min\left\{|F(z^0 + t\mathbf{b})| : |t|=r/L(z^0)\right\}.$$

Denote $L^* = \max\left\{L(z^0 + t\mathbf{b}) : |t| \leq (2R+2)/L(z^0)\right\},\ \rho_0 = R/L(z^0)$, $\rho_k = \rho_0 + k\eta/L^*,\ k \in \mathbb{Z}_+$. We have

$$\frac{\eta}{L^*} < \frac{R}{L^*} \leq \frac{R}{L(z^0)} < \frac{2R+2}{L(z^0)} - \frac{R+1}{L(z^0)}.$$

Then there exists $n^* \in \mathbb{N}$, independent of z^0 such that

$$\rho_{p-1} < \frac{R+1}{L(z^0)} \leq \rho_p \leq \frac{2R+2}{L(z^0)},$$

for some $p = p(z^0) \leq n^*$, because $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$. Indeed,

$$\begin{aligned} &\left(\frac{2R+2}{L(z^0)} - \rho_0\right) \Big/ \left(\frac{\eta}{L^*}\right) = \frac{(R+2)L^*}{\eta L(z^0)} = \\ &= \frac{R+2}{\eta} \max\left\{\frac{L(z^0+t\mathbf{b})}{L(z^0)} : |t| \leq \frac{2R+2}{L(z^0)}\right\} \leq \frac{R+2}{\eta}\lambda_{\mathbf{b}}(2R+2). \end{aligned}$$

Therefore, $n^* = \left[\frac{R+2}{\eta}\lambda_{\mathbf{b}}(2R+2)\right]$, where $[a]$ is integer part of number $a \in \mathbb{R}$. Let $|F(z^0 + t_k^{**}\mathbf{b})| = \max\{|F(z^0 + t\mathbf{b})| : t \in c_k\}$, $c_k = \{t \in \mathbb{C} : |t| = \rho_k\}$, and t_k^* be an intersection point of the segment $[0, t_k^{**}]$ with the circle c_{k-1}. Then for every $r > \eta$ and for each $k \leq n^*$ the inequality holds $|t_k^{**} - t_k^*| = \frac{\eta}{L^*} \leq \frac{r}{L(z^0+t_k^*\mathbf{b})}$. Thus, for some $r = r(z^0 + t_k^*\mathbf{b}) \in [\eta, R]$ we deduce

$$\begin{aligned} &|F(z^0 + t_k^{**}\mathbf{b})| \leq \max\left\{|F(z^0 + t\mathbf{b})| : |t - t_k^*| = r/L(z^0 + t_k^*\mathbf{b})\right\} \leq \\ &\leq P_2 \min\left\{|F(z^0 + t\mathbf{b})| : |t - t_k^*| = r/L(z^0 + t_k^*\mathbf{b})\right\} \leq \end{aligned}$$

$$\leq P_2 \min\left\{|F(z^0+t\mathbf{b})| : |t-t_k^*| = r/L(z^0+t_k^*\mathbf{b}), |t-t_0| \leq \rho_{k-1}\right\} \leq \\ \leq P_2 \max\{|F(z^0+t\mathbf{b})| : t \in c_{k-1}\}.$$

Hence,

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = (R+1)/L(z^0)\right\} \leq \\ \leq \max\{|F(z^0+t\mathbf{b})| : t \in c_p\} \leq P_2 \max\{|F(z^0+t\mathbf{b})| : t \in c_{p-1}\} \leq \\ \leq \ldots \leq (P_2)^p \max\{|F(z^0+t\mathbf{b})| : t \in c_0\} \leq \\ \leq (P_2)^{n^*} \max\left\{|F(z^0+t\mathbf{b})| : |t| = R/L(z^0)\right\}.$$

We obtained (2.29) with $P_1 = (P_2)^{n^*}$. Theorem 2.11 is proved. □

2.6 Estimate of directional logarithmic derivative

In this section we deduce analog of logarithmic criterion for function from the class $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$. The one-dimensional analog of the criterion is efficient to investigate boundedness of l-index of infinite products [44, 46, 49]. As necessary conditions the criterion was obtained by G. H. Fricke [40, 42] for analytic functions of bounded index.

Below we prove the criterion of L-index boundedness in direction, which describes behavior of directional logarithmic derivative and distribution of zeros. We need additional denotations.

Denote

$$G_r(F) := G_r^{\mathbf{b}}(F) := \bigcup_{z\colon F(z)=0} \{z+t\mathbf{b}\colon |t| < r/L(z)\}, \tag{2.30}$$

where a_k^0 are zeros of the function $F(z^0+t\mathbf{b})$ for given $z^0 \in \mathbb{D}^n$.

By $n\big(r, z^0, 1/F\big) = \sum_{|a_k^0| \leq r} 1$ we denote counting function of zeros a_k^0.

Theorem 2.12. *Let $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$, $L \in Q_{\mathbf{b}}^n$. If the function F has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, then*

1) for each $r \in (0, \beta]$ there exists $P = P(r) > 0$ such that for every $z \in \mathbb{D}^n \setminus G_r^{\mathbf{b}}(F)$

$$\left|\frac{\partial_{\mathbf{b}} F(z)}{F(z)}\right| \leq PL(z); \tag{2.31}$$

2) for any $r \in (0, \beta]$ there exists $\widetilde{n}(r) \in \mathbb{Z}_+$ such that for all $z^0 \in \mathbb{D}^n$ such that $F(z^0+t\mathbf{b}) \not\equiv 0$ one has

$$n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{F}\right) \leq \widetilde{n}(r). \tag{2.32}$$

Proof. Out proof is based on the proof of appropriate proposition for analytic functions of bounded L-index in direction [28, 32].

Firstly, we will show that the condition "$F(z)$ is of bounded L-index in the direction" implies that for every $z^0 \in \mathbb{D}^n \backslash G_r^{\mathbf{b}}(F)$ $(r > 0)$ and for each $\widetilde{a}^k = z^0 + a_k^0\mathbf{b}$ one has

$$|z^0 - \widetilde{a}_k| > \frac{r|\mathbf{b}|}{2L(\widetilde{z}^0)\lambda_{\mathbf{b}}(r)}. \tag{2.33}$$

On the contrary, suppose that there exist $z^0 \in \mathbb{D}^n \backslash G_r^{\mathbf{b}}(F)$ and $\widetilde{a}^k = z^0 + a_k^0\mathbf{b}$ such that $|z^0 - \widetilde{a}_k| \leq \dfrac{r|\mathbf{b}|}{2L(\widetilde{z}^0)\lambda_{\mathbf{b}}(r)} \leq \dfrac{r|\mathbf{b}|}{2L(z^0)} < \dfrac{r|\mathbf{b}|}{L(z^0)}$. Hence, $|a_k^0| < \frac{r}{L(z^0)}$. But for $\lambda_2^{\mathbf{b}}$ the following estimates hold $L(\widetilde{a}^k) \leq \lambda_{\mathbf{b}}(r) L(z^0)$. Then $|z^0 - \widetilde{a}^k| = |\mathbf{b}| \cdot |a_k^0| \leq \frac{r|\mathbf{b}|}{2L(\widetilde{a}^k)}$, i.e., $|a_k^0| \leq \dfrac{r}{2L(\widetilde{a}^k)}$. This contradicts $\widetilde{z}^0 \in \mathbb{D}^n \backslash G_r^{\mathbf{b}}(F)$.

Put in Theorem 2.10 $R = \dfrac{r}{2\lambda_{\mathbf{b}}(r)}$. Then there exist $P_2 \geq 1$ and $\eta \in (0, R)$ such that for every $\widetilde{z^0} = z^0 \in \mathbb{D}^n$ and for some $r^* \in [\eta, R]$ inequality (2.26) holds with r^* instead r. Therefore, by Cauchy's inequality

$$\begin{aligned} |\partial_{\mathbf{b}} F(z^0)| &\leq \frac{L(z^0)}{r^*} \max\Big\{|F(z^0 + t\mathbf{b}) : |t| = r^*/L(z^0)\Big\} \leq \\ &\leq P_2 \frac{L(z^0)}{\eta} \min\{|F(z^0 + t\mathbf{b})| : |t| = r^*/L(z^0)\}. \end{aligned} \tag{2.34}$$

In view of (2.33) for every point $z^0 + t_0\mathbf{b} \in \mathbb{D}^n \backslash G_r^{\mathbf{b}}(F)$ the following set

$$\left\{z^0 + t\mathbf{b} : |t| \leq \frac{r}{2\lambda_2^{\mathbf{b}}(r)\, L(z^0)}\right\}$$

does not contain zeros of the function $F(z^0+t\mathbf{b})$. Therefore, applying the maximum modulus principle to $1/F$ as a function of variable t, we have

$$|F(z^0)| \geq \min\left\{|F(z^0 + t\mathbf{b})| : |t| = r^*/L(z^0)\right\} \tag{2.35}$$

From inequalities (2.34) and (2.35) it follows (2.31) with $P = \dfrac{P_2}{\eta}$.

Now we will prove that for a function F of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ there exists $P_3 > 0$ such that for every $z^0 \in \mathbb{D}^n$ $(F(z^0 + t\mathbf{b}) \not\equiv 0)$, $r \in (0, 1]$

$$\begin{aligned} n(r/L(z^0), z^0, 1/F) \min&\Big\{|F(z^0+t\mathbf{b})| : |t| = r/L(z^0)\Big\} \leq \\ &\leq P_3 \max\Big\{|F(z^0 + t\mathbf{b})| : |t| = 1/L(z^0)\Big\}. \end{aligned} \tag{2.36}$$

Applying Cauchy's inequality and Theorem 2.9 for all t on the circle $|t| = \frac{1}{L(z^0)}$, we have

$$\left|\partial_{\mathbf{b}} F(z^0+t\mathbf{b})\right| \leq \frac{L(z^0)}{r} \max\left\{|F(z^0+\theta\mathbf{b})| : |\theta - t| = \frac{r}{L(z^0)}\right\} \leq$$
$$\leq \frac{L(z^0)}{r} \max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{r+1}{L(z^0)}\right\} \leq$$
$$\leq \frac{P_1(1, r+1)}{r} L(z^0) \max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{1}{L(z^0)}\right\}. \tag{2.37}$$

If $F(z^0 + t\mathbf{b}) \neq 0$ on the circle $\{t \in \mathbb{C} : |t| = r/L(z^0)\}$, then

$$n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{F}\right) = \left|\frac{1}{2\pi i} \int\limits_{|t| = \frac{r}{L(z^0)}} \frac{\partial_{\mathbf{b}} F(z^0+t\mathbf{b})}{F(z^0+t\mathbf{b})} dt\right| \leq$$
$$\leq \frac{\max\left\{\left|\partial_{\mathbf{b}} F(z^0+t\mathbf{b})\right| : |t| = r/L(z^0)\right\}}{\min\left\{|F(z^0+t\mathbf{b})| : |t| = r/L(z^0)\right\}} \frac{r}{L(z^0)}. \tag{2.38}$$

From (2.37) and (2.38) we deduce

$$n\left(r/L(z^0), z^0, 1/F\right) \min\left\{|F(z^0+t\mathbf{b})| : |t| = r/L(z^0)\right\} \leq$$
$$\leq \frac{r}{L(z^0)} \max\left\{|\partial_{\mathbf{b}} F(z^0+t\mathbf{b})| : |t| = r/L(z^0)\right\} \leq$$
$$\leq \frac{1}{L(z^0)} \max\left\{|\partial_{\mathbf{b}} F(z^0+t\mathbf{b})| : |t| = 1/L(z^0)\right\} \leq$$
$$\leq P_1(1, r+1)/(r) \max\left\{|F(z^0+t\mathbf{b})| : |t| = 1/L(z^0)\right\}.$$

Thus, we obtained (2.36) with $P_3 = \frac{P_1(1, r+1)}{r}$. If the function $F(z^0 + t\mathbf{b})$ has zeros on the circle $\{t \in \mathbb{C} : |t| = r/L(z^0)\}$, then inequality (2.36) is obvious.

Set $R = 1$ in Theorem 2.10. Then there exist $P_2 = P_2(1) \geq 1$ and $\eta \in (0, 1)$ such that for every $z^0 \in \mathbb{D}^n$ and some $r^* = r^*(z^0, t_0) \in [\eta, 1]$

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{r^*}{L(z^0)}\right\} \leq P_2 \min\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{r^*}{L(z^0)}\right\}.$$

Next, by Theorem 2.9 there exists $P_1 \geq 1$ such that for all $z^0 \in \mathbb{D}^n$

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = 1/L(z^0)\right\} \leq$$
$$\leq P_1(1, \eta) \max\left\{|F(z^0+t\mathbf{b})| : |t| = \eta/L(z^0)\right\} \leq$$
$$\leq P_1(1, \eta) \max\left\{|F(z^0+t\mathbf{b})| : |t| = r^*/L(z^0)\right\} \leq$$

$$\leq P_1(1,\eta)P_2 \min\left\{|F(z^0+t\mathbf{b})| : |t| = r^*/L(z^0)\right\}.$$

Taking into account (2.36), we obtain

$$n\left(r^*/L(z^0), z^0, 1/F\right) \min\left\{|F(z^0+t\mathbf{b})| : |t| = r^*/L(z^0)\right\} \leq \\ \leq P_3 P_1(1,\eta)P_2 \min\left\{|F(z^0+t\mathbf{b})| : |t| = r^*/L(z^0)\right\},$$

i.e., $n\left(\frac{r^*}{L(z^0)}, z^0, \frac{1}{F}\right) \leq P_1(1,\eta)P_2P_3$. Hence,

$$n\left(\frac{r^*}{L(z^0)}, z^0, \frac{1}{F}\right) \leq P_4 = P_1(1,\eta)P_2P_3 = \frac{P_1(1,\eta)P_2(1)P_1(1,r+1)}{r}.$$

If $r \in (0,\eta]$, then property (2.32) is proved.

Let $r > \eta$ and $L^* = \max\left\{L(z^0+t\mathbf{b}) : |t| = \frac{r}{L(z^0)}\right\}$. Using properties $Q_{\mathbf{b}}(\mathbb{D}^n)$, we have $L^* \leq \lambda_{\mathbf{b}}(r)L(z^0)$. Put $\rho = \frac{\eta}{L(z^0)\lambda_{\mathbf{b}}(r)}$, $R = \frac{r}{L(z^0)}$. We can cover every set $\overline{K} = \{z^0+t\mathbf{b} : |t| \leq R\}$ by a finite number $m = m(r)$ of closed sets $\overline{K}_j = \{z^0+t\mathbf{b} : |t-t_j| \leq \rho\}$, where $t_j \in \overline{K}$. Since $\frac{\eta}{\lambda_{\mathbf{b}}(r)L(z^0)} \leq \frac{\eta}{L^*} \leq \frac{\eta}{L(z^0+t_j\mathbf{b})}$ every set $\overline{K}_j$ contains at least $[P_4]$ zeros of the function $F(z^0+t\mathbf{b})$. Therefore, $n\left(\frac{r}{L(z^0)}, z^0, 1/F\right) \leq \widetilde{n}(r) = [P_4]\, m(r)$ and property (2.32) is proved. □

By $n_{z^0}(r,F) = n_{\mathbf{b}}\big(r, z^0, 1/F\big) := \sum_{|a_k^0|\leq r} 1$ we denote counting function of zeros a_k^0 for the slice function $F(z^0+t\mathbf{b})$ in the disc $\{t \in \mathbb{C}\colon |t| \leq r\}$. If for given $z^0 \in \mathbb{D}^n$ and for all $t \in \mathbb{C}$ $F(z^0+t\mathbf{b}) \equiv 0$, then we put $n_{z^0}(r) = -1$. Denote $n(r) = \sup_{z\in\mathbb{D}^n} n_z(r/L(z))$. Below we relax conditions of previous theorem (see similar results for entire functions in [41]).

Theorem 2.13. *Let $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, $L \in Q^n_{\mathbf{b}}$, $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$. If the following conditions are satisfied*

1) *there exists $r_1 \in (0,\beta]$ such that $n(r_1) \in [-1;\infty)$;*

2) *there exist $r_2 \in (0,\beta]$, $P > 0$ such that $2r_2 \cdot n(r_1) < r_1/\lambda_{\mathbf{b}}(r_1)$ and for all $z \in \mathbb{D}^n \backslash G_{r_2}(F)$ inequality* (2.31) *is true;*

then the function F has bounded L-index in the direction $\mathbf{b}$.

Proof. Analog of the proposition was firstly deduced for analytic functions of bounded L-index in direction [47, 48]. Suppose that conditions 1) and 2) are true.

At first, we consider the case $n(r_1) \in \{-1; 0\}$. Then in the best case the function F can only identically equals zero on the complex line $z^*+t\mathbf{b}$ for some $z^* \in \mathbb{D}^n$, i.e., $F(z^*+t\mathbf{b}) \equiv 0$. For all points lying on such complex lines inequality (2.26) is obvious.

Let $z^0 \in \mathbb{D}^n \setminus G_{r_2}$. For any points t_1 and t_2 such that $|t_j| = \frac{r_2}{L(z_0)}$, $j \in \{1, 2\}$, one has

$$\ln\left|\frac{F(z^0+t_2\mathbf{b})}{F(z^0+t_1\mathbf{b})}\right| \le \int_{t_1}^{t_2} \left|\frac{\partial_{\mathbf{b}} F(z^0+t\mathbf{b})}{F(z^0+t\mathbf{b})}\right| |dt| \le$$

$$\le P\int_{t_1}^{t_2} L(z^0+t\mathbf{b})|dt| \le P\lambda_{\mathbf{b}}(r_2)\, L(z^0)\frac{\pi r_2}{L(z^0)} \le \pi r_2 P \lambda_{\mathbf{b}}(r_2)$$

(we also use that $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$). Hence,

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{r_2}{L(z^0)}\right\} \le P_2 \min\left\{|F(z^0+t\mathbf{b})| : \ |t| = \frac{r_1}{L(z^0)}\right\},$$

where $P_2 = \exp\{\pi r_2 P \lambda_2(r_2)\}$. Therefore, by Theorem 2.11 the function F has bounded L-index in the direction $\mathbf{b}$.

Let $r_1 \in (0, \beta]$ be a such that $n(r_1) \in [1; \infty)$ and $2n(r_1)r_2 < r_1/\lambda_{\mathbf{b}}(r_1)$. Put $c = \frac{r_1}{2r_2\lambda_{\mathbf{b}}(r_1)} - n(r_1) > 0$. Clearly, $r_2 = r_1/(2(n(r_1)+c)\lambda_{\mathbf{b}}(r_1))$.

Under condition 1) each set $\overline{K} = \left\{z^0+t\mathbf{b} : |t| \le \frac{r_1}{L(z^0)}\right\}$ has no more $n(r_1)$ zeros of the function F, where $F(z^0+t\mathbf{b}) \not\equiv 0$.

Under condition 2) there exists $P > 0$ such that $|\frac{\partial_{\mathbf{b}} F(z)}{F(z)}| \le PL(z)$ for every $z \in \mathbb{D}^n \setminus G_{r_2}$, i.e., for all $z \in \overline{K}$, lying outside the sets

$$\left\{z^0+t\mathbf{b} : \ |t-a_k^0| < \frac{r_2}{L(z^0+a_k^0\mathbf{b})}\right\},$$

where $a_k^0 \in \overline{K}$ are zeros of the slice function $F(z^0+t\mathbf{b}) \not\equiv 0$. By definition $\lambda_{\mathbf{b}}$ we obtain

$$L(z^0)/\lambda_{\mathbf{b}}(r_1) \le L(z^0+a_k^0\mathbf{b}).$$

Then $|\frac{\partial_{\mathbf{b}} F(z)}{F(z)}| \le PL(z)$ for every point $z \in \mathbb{D}^n$, lying outside union of the sets

$$c_k^0 = \left\{z^0+t\mathbf{b} : |t-a_k^0| \le \frac{r_2\lambda_{\mathbf{b}}(r_1)}{L(z^0)} = \frac{r_1}{2(n(r_1)+c)L(z^0)}\right\}.$$

The total sum of diameters of the sets c_k^0 does not exceed the value $\frac{r_1 n(r_1)}{(n(r_1)+c)L(z^0)} < \frac{r_1}{L(z^0)}$. Hence, there exists a set $\tilde{c}^0 = \left\{z^0+t\mathbf{b} : |t| = \frac{r}{L(z^0)}\right\}$, where $\frac{r_1 \min\{1,c\}}{2(n(r_1)+c)} = \eta < r < r_1$, such that, for all $z \in \tilde{c}^0$

$$\left|\frac{\partial_{\mathbf{b}} F(z)}{F(z)}\right| \le PL(z) \le P\lambda_{\mathbf{b}}(r)L(z^0) \le P\lambda_{\mathbf{b}}(r_1)\, L(z^0).$$

For any points $z_1 = z^0 + t_1\mathbf{b}$ and $z_2 = z^0 + t_2\mathbf{b}$ with $\tilde{c}^0$ one has

$$\ln\left|\frac{F(z^0+t_2\mathbf{b})}{F(z^0+t_1\mathbf{b})}\right| \le \int_{t_1}^{t_2} \left|\frac{\partial_{\mathbf{b}} F(z^0+t\mathbf{b})}{F(z^0+t\mathbf{b})}\right| |dt| \le$$

$$\leq P\lambda_2(r_1)\,L(z^0)\frac{\pi r}{L(z^0)} \leq \pi r_1\,P(r_2)\lambda_{\mathbf{b}}(r_1)\,.$$

Therefore,

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{r}{L(z^0)}\right\} \leq P_2 \min\left\{|F(z^0+t\mathbf{b})| : \ |t| = \frac{r}{L(z^0)}\right\}, \quad (2.39)$$

where $P_2 = \exp\{\pi r_1\,P(r_2)\lambda_{\mathbf{b}}(r_1)\}$. If $F(z^0+t\mathbf{b}) \equiv 0$, then inequality (2.39) is obvious. By Theorem 2.11 the function $F(z)$ has bounded L-index in the direction $\mathbf{b}$. Theorem 2.13 is proved. □

2.7 Analog of Hayman's Theorem

Below we formulate and prove criterion which is analog of Hayman's Theorem [51].

Theorem 2.14. *Let* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *and* $L \in Q^n_{\mathbf{b}}$. *A function* $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ *is of bounded* L*-index in the direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *if and only if there exist* $p \in \mathbb{Z}_+$ *and* $C > 0$ *such that for every* $z \in \mathbb{D}^n$ *one has*

$$\frac{|\partial^{p+1}_{\mathbf{b}}F(z)|}{L^{p+1}(z)} \leq C\max\left\{\frac{|\partial^k_{\mathbf{b}}F(z)|}{L^k(z)} : \ 0 \leq k \leq p\right\}. \quad (2.40)$$

Proof. The proof use ideas from the proof for analytic functions of bounded L-index in direction [28, 32]. Also there are known analogs of Hayman's theorem for other classes of analytic functions [13, 16].

Necessity. If $N_{\mathbf{b}}(F,L) < +\infty$, then by definition of boundedness of L-index in direction we obtain (2.40) with $p = N_{\mathbf{b}}(F,L)$ and $C = (N_{\mathbf{b}}(F,L)+1)!$

Sufficiency. Let inequality (2.40) be fulfilled, $z^0 \in \mathbb{D}^n$ and

$$K = \left\{t \in \mathbb{C} : |t| \leq 1/L(z^0)\right\}.$$

Since $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$, for every $t \in K$ from (2.40) it follows

$$\frac{|\partial^{p+1}_{\mathbf{b}}F(z^0+t\mathbf{b})|}{L^{p+1}(z^0)} \leq \left(\frac{L(z^0+t\mathbf{b})}{L(z^0)}\right)^{p+1}\frac{|\partial^{p+1}_{\mathbf{b}}F(z^0+t\mathbf{b})|}{L^{p+1}(z^0+t\mathbf{b})} \leq$$
$$\leq (\lambda_{\mathbf{b}}(1))^{p+1}\frac{|\partial^{p+1}_{\mathbf{b}}F(z^0+t\mathbf{b})|}{L^{p+1}(z^0+t\mathbf{b})} \leq C(\lambda_{\mathbf{b}}(1))^{p+1}\max_{0\leq k\leq p}\left\{\frac{|\partial^k_{\mathbf{b}}F(z^0+t\mathbf{b})|}{L^k(z^0+t\mathbf{b})}\right\} \leq$$
$$\leq C(\lambda_{\mathbf{b}}(1))^{p+1}\max\left\{\left(\frac{L(z^0)}{L(z^0+t\mathbf{b})}\right)^k\frac{|\partial^k_{\mathbf{b}}F(z^0+t\mathbf{b})|}{L^k(z^0)} : 0 \leq k \leq p\right\} \leq$$
$$\leq C(\lambda_{\mathbf{b}}(1))^{p+1}\max\left\{\frac{|\partial^k_{\mathbf{b}}F(z^0+t\mathbf{b})|}{L^k(z^0)}(\lambda_{\mathbf{b}}(1))^k : 0 \leq k \leq p\right\} \leq Bg_{z^0}(t), \quad (2.41)$$

where $B = C(\lambda_{\mathbf{b}}(1))^{2p+1}$ and $g_{z^0}(t) = \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z^0+t\mathbf{b})|}{L^k(z^0)} : 0 \le k \le p\right\}$.

Let us denote $\gamma_1 = \left\{t \in \mathbb{C} : |t| = \frac{1}{4L(z^0)}\right\}$, $\gamma_2 = \left\{t \in \mathbb{C} : |t| = \frac{2}{L(z^0)}\right\}$. Choose arbitrarily two points $t_1 \in \gamma_1$, $t_2 \in \gamma_2$ and connect them by a piecewise analytic curve $\gamma = \{t = t(s), 0 \le s \le T\}$ such that $g_{z^0}(t) \ne 0$ for $t \in \gamma$. We construct the curve γ such that its length $|\gamma|$ does not exceed $\dfrac{9}{2L(z^0)}$.

The function $g_{z^0}(t(s))$ is analytic on $[0, T]$. Without loss of generality we may assume that the function $t = t(s)$ is analytic on $[0, T]$. Otherwise, one can consider each interval of analyticity of this function separately and repeat the corresponding considerations, which are given below on $[0, T]$. First, we show that the function $g_{z^0}(t(s))$ is continuously differentiable on $[0, T]$ except possibly a finite set of points. For arbitrary $k_1, k_2, 0 \le k_1 \le k_2 \le p$, either $\frac{|\partial_{\mathbf{b}}^{k_1} F(z^0+t(s)\mathbf{b})|}{L^{k_1}(z^0)} \equiv \frac{|\partial_{\mathbf{b}}^{k_2} F(z^0+t(s)\mathbf{b})|}{L^{k_2}(z^0)}$ ot the equality $\frac{|\partial_{\mathbf{b}}^{k_1} F(z^0+t(s)\mathbf{b})|}{L^{k_1}(z^0)} = \frac{|\partial_{\mathbf{b}}^{k_2} F(z^0+t(s)\mathbf{b})|}{L^{k_2}(z^0)}$ is true for a finite set of points $s_k \in [0, T]$. Then we can split the segment $[0, T]$ onto a finite number of segments such that on each of them $g_{z^0}(t(s)) \equiv \frac{|\partial_{\mathbf{b}}^{k} F(z^0+t(z)\mathbf{b})|}{L^{k}(z^0)}$ for some k, $0 \le k \le p$. It means that the function $g_{z^0}(t(s))$ is continuously differentiable with exception, perhaps, of a finite sets of points. Taking into account (2.41), we obtain

$$\frac{dg_{z^0}(t(s))}{ds} \le \max\left\{\frac{d}{ds}\left(\frac{|\partial_{\mathbf{b}}^k F(z^0+t(s)\mathbf{b})|}{L^k(z^0)}\right) : 0 \le k \le p\right\} \le$$
$$\le \max\left\{|\partial_{\mathbf{b}}^{k+1} F(z^0+t(s)\mathbf{b})||t'(s)|/L^k(z^0) : 0 \le k \le p\right\} =$$
$$= L(z^0)|t'(s)| \max\left\{|\partial_{\mathbf{b}}^{k+1} F(z^0+t(s)\mathbf{b})|/L^{k+1}(z^0) :\ 0 \le k \le p\right\} \le$$
$$\le B g_{z^0}(t(s))|t'(s)|L(z^0).$$

Hence, we have

$$\left|\ln\frac{g_{z^0}(t_2)}{g_{z^0}(t_1)}\right| = \left|\int_0^T \frac{dg_{z^0}(t(s))}{g_{z^0}(t(s))}\right| \le BL(z^0)\int_0^T |t'(s)|ds = BL(z^0)|\gamma| \le 4.5B.$$

If we choose a point $t_2 \in \gamma_2$, such that

$$|F(z^0+t_2\mathbf{b})| = \max\left\{|F(z^0+t\mathbf{b})| : |t| = 2/L(z^0)\right\},$$

then we obtain

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{2}{L(z^0)}\right\} \le g_{z^0}(t_2) \le g_{z^0}(t_1)\exp\{9B/2\}. \tag{2.42}$$

Applying Cauchy's inequality and using that $t_1 \in \gamma_1$ we obtain for all $j \in \{1, \dots, p\}$

$$|\partial_{\mathbf{b}}^j F(z^0+t_1\mathbf{b})| \le j!(4L(z^0))^j \max\left\{|F(z^0+t\mathbf{b}) : |t-t_1| = \frac{1}{4L(z^0)}\right\} \le$$

$$\leq j!(4L(z^0))^j \max\left\{|F(z^0+t\mathbf{b}):|t-t_0|=\frac{1}{2L(z^0)}\right\},$$

$$\text{i.e. } g_{z^0}(t_1)\leq p!(4)^p\max\left\{|F(z^0+t\mathbf{b}):|t-t_0|=\frac{1}{2L(z^0)}\right\}.$$

Therefore, from (2.42) it follows that

$$|F(z^0+t_2\mathbf{b})|=\max\left\{|F(z^0+t\mathbf{b})|:|t|=\beta/L(z^0)\right\}\leq$$
$$\leq g_{z^0}(t_2)\leq g_{z^0}(t_1)\exp\left\{9B/2\right\}\leq p!(4)^p\exp\left\{9B/2\right\}\times$$
$$\times\max\left\{|F(z^0+t\mathbf{b})|:|t|=1/(2L(z^0))\right\}.$$

By Proposition 2.7 we conclude that the function F has bounded L-index in the direction $\mathbf{b}\in\mathbb{C}^n$. Theorem 2.14 is proved. □

Using Theorem 2.14 we prove the following

Theorem 2.15. *Let $L\in Q_{\mathbf{b}}(\mathbb{D}^n)$. A function $F\in\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ has bounded L-index in the direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ if and only if there exist numbers $C\in(0,+\infty)$ and $N\in\mathbb{N}$ such that for all $z\in\mathbb{D}^n$*

$$\sum_{k=0}^{N}\frac{|\partial_{\mathbf{b}}^kF(z)|}{k!L^k(z)}\geq C\sum_{k=N+1}^{\infty}\frac{|\partial_{\mathbf{b}}^kF(z)|}{k!L^k(z)}. \tag{2.43}$$

Proof. Proof of this theorem is similar to proof of its analogs for analytic functions of bounded L-index in direction [34] and for analytic functions of bounded l-index [31].

Let $0<\theta<1$. If the function F is of bounded L-index in the direction $\mathbf{b}$, then by Lemma 2.5 F is also of bounded L^*-index in the direction $\mathbf{b}$, where $L^*(z)=\theta L(z)$. Denote $N^*=N_{\mathbf{b}}(F,L_*)$ and $N=N_{\mathbf{b}}(F,L)$. Thus,

$$\max\left\{\frac{|\partial_{\mathbf{b}}^kF(z)|}{k!L^k(z)}:0\leq k\leq N^*\right\}=\max\left\{\frac{|\partial_{\mathbf{b}}^kF(z)|}{k!L_*^k(z)}\theta^k:0\leq k\leq N^*\right\}\geq$$
$$\geq\theta^{N^*}\max\left\{\frac{|\partial_{\mathbf{b}}^kF(z)|}{k!L_*^k(z)}:0\leq k\leq N^*\right\}\geq$$
$$\geq\theta^{N^*}\frac{|\partial_{\mathbf{b}}^jF(z)|}{j!L_*^j(z)}=\theta^{N^*-j}\frac{|\partial_{\mathbf{b}}^jF(z)|}{j!L^j(z)}$$

for all $j\geq 0$ and

$$\sum_{j=N^*+1}^{\infty}\frac{|\partial_{\mathbf{b}}^jF(z)|}{j!L^j(z)}\leq\max\left\{\frac{|\partial_{\mathbf{b}}^kF(z)|}{k!L^k(z)}:0\leq k\leq N^*\right\}\sum_{j=N^*+1}^{\infty}\theta^{j-N^*}=$$

$$= \frac{\theta}{1-\theta} \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)} : 0 \le k \le N^* \right\} \le \frac{\theta}{1-\theta} \sum_{k=0}^{N^*} \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)},$$

i.e. we obtain (2.43) with $N = N^*$ and $C = \frac{1-\theta}{\theta}$.

Now we prove the sufficiency. From (2.43) we obtain

$$\frac{|\partial_{\mathbf{b}}^{N+1} F(z)|}{(N+1)! L^{N+1}(z)} \le \sum_{k=N+1}^{\infty} \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)} \le \frac{1}{C} \sum_{k=0}^{N} \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)} \le$$
$$\le \frac{N+1}{C} \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)} : 0 \le k \le N \right\}.$$

Applying Theorem 2.14, we obtain the desired conclusion. □

Using Theorems 2.8 and 2.14 we obtain this corollary.

Corollary 2.2. *Let $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$, $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{D}^n)$, $(\forall p \in \mathbb{N})$ $\partial_{\mathbf{b}}^p F \in \mathcal{H}_{\mathbf{b}}(\mathbb{D}^n)$, G be a bounded domain in $\mathbb{D}^n$ such that $\forall z \in \overline{G}$ $F(z+t\mathbf{b}) \not\equiv 0$. The function F has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if and only if there exist $p \in \mathbb{Z}_+$ and $C > 0$ such that for all $z \in \mathbb{D}^n \setminus G$ inequality* (2.40) *holds.*

2.8 Functions having bounded value L-distribution in direction

An analytic function $f(z)$ $(z \in \mathbb{C})$ is said ot be of bounded value distribution [43, 51, 55], if there exist $p \ge 0$, $R > 0$ such that the equation $f(z) = w$ has at most p roots in any disc of radius R.

One of the remarkable properties generating big interest to functions of bounded index is the following fact proved by W. Hayman [51]: an analytic function has bounded value distribution if and only if it derivative has bounded index. Leter, there was introduced a concept of analytic function of bounded value l-distribution [20], and this property was generalized for analytic functions of bounded l-index [56]. For analytic bivariate functions of bounded index in joint variables similar results are partially obtained in [37].

Definition 2.2. *Function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ is said to be of bounded value L-distribution in a direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ if for all $p \in \mathbb{N}$ $\forall w \in \mathbb{C}$ $\forall z_0 \in \mathbb{D}^n$ such that $F(z^0 + t\mathbf{b}) \not\equiv w$, the inequality holds $n\big(\frac{1}{L(z^0)}, z^0, \frac{1}{F-w}\big) \le p$, i.e. the equation $F(z^0 + t\mathbf{b}) = w$ has at most p solutions in the disc $\{t : \ |t| \le \frac{1}{L(z^0)}\}$. In other words, the function $F(z^0 + t\mathbf{b})$ is p-valent in $\{t : \ |t| \le \frac{1}{L(z^0)}\}$.*

The corresponding Sheremeta result [56] we will generalize for the functions from the class $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$, which have bounded value L-distribution in direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$.

Proposition 2.8. *Let* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *and* $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$. *A function* $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ *is of bounded value* L*-distribution in the direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *if and only if the directional derivative* $\partial_{\mathbf{b}} F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ *has bounded* L*-index in the same direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$.

Proof. This is similar to the proof of the corresponding proposition for analytic functions in the unit ball [54].

Suppose that F is of bounded value L-distribution in direction $\mathbf{b}$, i.e. for all $z^0 \in \mathbb{D}^n$ such that $F(z^0 + t\mathbf{b}) \not\equiv const$ the function $F(z^0 + t\mathbf{b})$ is p-valent in every disc $\{t : \ |t| \le \frac{1}{L(z^0)}\}$.

To prove the proposition we need the following proposition from [25, p. 48, Theorem 2.8].

Theorem 2.16 ([25]). *Let* $D_0 = \{t : \ |t - t_0| < R\}$, $0 < R < \infty$. *If analytic in* D_0 *function* f *is* p*-valent in* D_0, *then for* $j > p$

$$\frac{|f^{(j)}(t_0)|}{j!} R^j \le (Aj)^{2p} \max\left\{ \frac{|f^{(k)}(t_0)|}{k!} R^k : \ 1 \le k \le p \right\}, \tag{2.44}$$

where $A = \sqrt[2p]{\frac{p+2}{2}} \sqrt{8e^{\pi^2}}$.

By Theorem 2.16 inequality (2.44) holds with $R = \frac{1}{L(z^0)}$ for the function $F(z^0 + t\mathbf{b})$, as a function of single variable $t \in \mathbb{C}$ for every fixed $z^0 \in \mathbb{D}^n$. Then it is easy to deduce that for every $m \in \mathbb{N}$ the following equality $g_{z^0}^{(p)}(t) = \partial_{\mathbf{b}}^p F(z^0 + t\mathbf{b})$ holds. Take $j = p + 1$ and $t_0 = 0$ in Theorem 2.16. From (2.44) it follows

$$\frac{\left|\partial_{\mathbf{b}}^{p+1} F(z^0)\right|}{(p+1)! L^{p+1}(z_0)} \le (A(p+1))^{2p} \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z^0)|}{k! L^k(z_0)} : \ 1 \le k \le p \right\} \Rightarrow$$

$$\frac{\left|\partial_{\mathbf{b}}^{p+1} F(z^0)\right|}{L^{p+1}(z_0)} \le (p+1)!(A(p+1))^{2p} \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z^0)|}{L^k(z_0)} : \ 1 \le k \le p \right\} \times$$

$$\times \max\left\{ \frac{1}{k!} : 1 \le k \le p \right\} \Rightarrow$$

$$\frac{|\partial_{\mathbf{b}}^p \partial_{\mathbf{b}} F(z^0)|}{L^p(z_0)} \le L(z^0) \cdot (p+1)! A^{2p} (p+1)^{2p} \max\left\{ \frac{|\partial_{\mathbf{b}}^{k-1} \partial_{\mathbf{b}} F(z^0)|}{L^k(z_0)} : \right.$$

$$\left. 0 \le k - 1 \le p - 1 \right\} \Rightarrow \frac{|\partial_{\mathbf{b}}^p \partial_{\mathbf{b}} F(z^0)|}{L^p(z_0)} \le$$

$$\le (p+1)! A^{2p} (p+1)^{2p} \max\left\{ \frac{|\partial_{\mathbf{b}}^{k-1} \partial_{\mathbf{b}} F(z^0)|}{L^{k-1}(z_0)} : 0 \le k - 1 \le p - 1 \right\}.$$

Now we need analog of Hayman's Theorem proved above. Thus, for $\partial_{\mathbf{b}}F$ inequality (2.40) holds with $p-1$ instead p and with $C=(p+1)!A^{2p}(p+1)^{2p}$. In Theorem 2.16 the constant $A \geq \max\limits_{j>p} \frac{p+2}{2}(8e^{\pi^2})^p(1-\frac{1}{j})^j$ does not depend from z^0, because the parameter p is independent of z^0. Hence, the quantity $C=(p+1)!A^{2p}(p+1)^{2p}$ does not depend of z^0. Therefore by Theorem 2.14 the function $\partial_{\mathbf{b}}F$ has bounded L-index in the direction $\mathbf{b}$.

Conversely, let $\partial_{\mathbf{b}}F$ be a function of bounded L-index in the direction $\mathbf{b}$. By Theorem 2.14 there exist $p \in \mathbb{Z}_+$ and $C \geq 1$ such that for every $z \in \mathbb{D}^n$ the following inequality holds

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L^{p+1}(z)} \leq C \max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{L^{k}(z)}:\ 1 \leq k \leq p\right\}. \tag{2.45}$$

Let us consider a disc $K_0=\left\{t \in \mathbb{C}: |t| \leq \frac{1}{L(z^0)}\right\}$, $z^0 \in \mathbb{D}^n$.

One should observe that if $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$, $z^0 \in \mathbb{D}^n$ then for all $r>0$ the inequality $|t| \leq \frac{r}{L(z^0)}$ and definition of class $Q_{\mathbf{b}}(\mathbb{D}^n)$ yield

$$L(z^0)/\lambda_{\mathbf{b}}(r) \leq L(z^0+t\mathbf{b}) \leq \lambda_{\mathbf{b}}(r)L(z^0). \tag{2.46}$$

Now from (2.45) and (2.46) for $z=z^0+t\mathbf{b}$, $t \in K$ one has

$$\begin{aligned}
\frac{|\partial_{\mathbf{b}}^{p+1}F(z^0+t\mathbf{b})|}{(p+1)!(C\lambda_{\mathbf{b}}(1)L(z^0))^{p+1}} &\leq \max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z^0+t\mathbf{b})|}{k!}\frac{1}{(C\lambda_{\mathbf{b}}(1)L(z^0))^{k}}\times\right.\\
&\times\left.\left(\frac{L(z^0+t\mathbf{b})}{C\lambda_{\mathbf{b}}(1)L(z^0)}\right)^{p+1-k}:\ 1\leq k\leq p\right\} \leq \frac{C}{p+1}\times\\
&\times \max_{1\leq k\leq p}\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z^0+t\mathbf{b})|}{k!}\frac{1}{(C\lambda_{\mathbf{b}}(1)L(z^0))^{k}}\frac{1}{C^{p+1-k}}\right\} \leq\\
&\leq \max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z^0+t\mathbf{b})|}{k!}\frac{1}{(C\lambda_{\mathbf{b}}(1)L(z^0))^{k}}:\ 1\leq k\leq p\right\}.
\end{aligned} \tag{2.47}$$

To prove the proposition we need such a statement from [25, p.44, Theorem 2.7].

Theorem 2.17 ([25, p.44, Theorem 2.7]). *Let $D_0=\{t \in \mathbb{C}:\ |t-t_0|<R\}$, $0<R<+\infty$, and $f(t)$ is an analytic function in D_0. If for all $t \in D_0$*

$$\left(\frac{R}{2}\right)^{p+1}\frac{|f^{(p+1)}(t)|}{(p+1)!} \leq \max\left\{\left(\frac{R}{2}\right)^{k}\frac{|f^{(k)}(t)|}{k!}:\ 1\leq k\leq p\right\}, \tag{2.48}$$

then $f(t)$ is p-valent in $\{t \in \mathbb{C}:\ |t-t_0| \leq \frac{R}{25\sqrt{p+1}}\}$, i.e., $f(t)$ attains every value at more p times.

From inequality (2.47) it follows inequality (2.48) with $R = \frac{2}{C\lambda_{\mathbf{b}}(1)L(z^0)}$ and $t_0 = 0$. By Theorem 2.17 the function $F(z^0 + t\mathbf{b})$ is p-valent in the disc $\{t \in \mathbb{C} : |t| \le \frac{\rho}{L(z^0)}\}$, $\rho = \frac{2}{25C\lambda_{\mathbf{b}}(1)\sqrt{p+1}}$.

Let t_j be an arbitrary point in K_0 and $K_j^* = \{t \in \mathbb{C} : |t - t_i| \le \frac{\rho}{L(z^0+t_j\mathbf{b})}\}$. Since by definition of class $Q_{\mathbf{b}}(\mathbb{D}^n)$ $L(z^0+t_j\mathbf{b}) \le \lambda_{\mathbf{b}}(1)L(z^0)$, one has $K_j = \left\{t \in \mathbb{C} : |t - t_j| \le \frac{\rho}{\lambda_{\mathbf{b}}(1)L(z^0)}\right\} \subset K_j^*$. We will repeat the similar considerations for the set $\left\{t \in \mathbb{C} : |t - t_j| \le \frac{1}{L(z^0+t_j\mathbf{b})}\right\}$. As a consequence, we deduce that $F(z^0 + t\mathbf{b})$ is p-valent in K_j^*. But $K_j \subset K_j^*$, then $F(z^0 + t\mathbf{b})$ is p-valent in K_j.

Finally, we note that every closed disc of radius R_* can be covered by a finite number m_* of closed discs of radius $\rho_* < R_*$ with the centers in the disc. Moreover, $m_* < B_*(R_*/\rho_*)^2$, where $B_* > 0$ is an absolute constant. Hence, K_0 can be covered finite number m of discs K_j, where $m \le 625B^*(p+1)C^2(\lambda_{\mathbf{b}}(1))^2/4$. Since the function $F(z^0 + t\mathbf{b})$ in K_j is p-valent, it is mp-valent in K_0.

In view of arbitrariness of z^0, the statement is proved. □

2.9 Existence theorem for functions of bounded L-index in direction

For the one-dimensional case, some time ago mathematicians were interested in the following two problems: the problem of the existence of an analytic function of bounded l-index for a given l, and the problem of the existence of a function l for a given analytic function f such that f is of bounded l-index [45, 49, 50, 52]. It is clear that similar problems can be posed for the multidimensional case [33, 53].

We note that the solution of the first problem for the one-dimensional case is given by a canonical product. The solution of the first problem in the multidimensional case also exists in the class of canonical products with "planar" zeros.

In particular, the following proposition is true.

We consider the function $F(z^0+t\mathbf{b})$ where $z^0 \in \mathbb{D}^n$ is fixed. If $F(z^0+t\mathbf{b}) \not\equiv 0$, then we denote by $p_{\mathbf{b}}(z^0 + a_k^0\mathbf{b})$ the multiplicity of the zero a_k^0 of the function $F(z^0 + t\mathbf{b})$. If $F(z^0 + t\mathbf{b}) \equiv 0$ for some $z^0 \in \mathbb{D}^n$, then we put $p_{\mathbf{b}}(z^0 + t\mathbf{b}) = -1$.

Theorem 2.18. *In order that for a function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ there exist a positive continuous function $L(z)$ such that $F(z)$ is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ it is necessary and sufficient that $\exists p \in \mathbb{Z}_+$ $\forall z^0 \in \mathbb{D}^n$ $\forall k$ $p_{\mathbf{b}}(z^0 + a_k^0\mathbf{b}) \le p$.*

Proof. Necessity. To simplify the notation we denote everywhere in the proof $p_k^0 \equiv p_{\mathbf{b}}(z^0 + a_0^k\mathbf{b})$. One can prove the necessity using the definition of bounded L-index in direction. Indeed, assume on the contrary that $\forall p \in \mathbb{Z}_+$ $\exists z^0$ $\exists k$ $p_k^0 > p$. It means that $\partial_{\mathbf{b}}^{p_k^0} F(z^0 + a_k^0\mathbf{b}) \ne 0$ and $\partial_{\mathbf{b}}^{j} F(z^0 + a_k^0\mathbf{b}) = 0$ for all $j \in \{1, \ldots, p_k^0 - 1\}$. Therefore, the L-index in the direction b at the point $z^0 + a_k^0\mathbf{b}$ is not less than

$p_k^0 > p$

$$N_{\mathbf{b}}(F, L, z^0 + a_k^0\mathbf{b}) > p.$$

If $p \to +\infty$, then we obtain that $N_{\mathbf{b}}(F, L, z^0 + a_k^0\mathbf{b}) \to +\infty$. But this contradicts the boundedness of L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ of the function F.

Sufficiency. If $F(z^0+t\mathbf{b}) \equiv 0$ for some $z^0 \in \mathbb{D}^n$, then inequality (2.2) is obvious.

Let p be the least integer such that $\forall z^0 \in \mathbb{D}^n$ $F(z^0+t\mathbf{b}) \not\equiv 0$, and $\forall k$ $p_k(z^0) \leq p$. For any point $z \in \mathbb{D}^n$ we choose $z^0 \in \mathbb{D}^n$ and $t_0 \in \mathbb{C}$ so that $z = z^0 + t_0\mathbf{b}$ and the point z^0 lies on the hyperplane $\langle z, m\rangle = 1$, where $\langle \mathbf{b}, m\rangle = 1$ (actually it is sufficient that $\langle \mathbf{b}, m\rangle \neq 0$, i.e. the hyperplane is not parallel to $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$). Therefore, $t_0 = \langle z, m\rangle - 1$, $z^0 = z - (\langle z, m\rangle - 1)\mathbf{b}$. We put $K_R = \{t \in \mathbb{C}\colon \max\{0, R-1\} \leq |t| \leq R+1\}$ for all $R \geq 0$ and

$$m_1(z^0, R) = \min_{a_k^0 \in K_R} \max_{0 \leq s \leq p} \left\{ \frac{|\partial_{\mathbf{b}}^s F(z^0 + a_k^0\mathbf{b})|}{s!} \right\}.$$

Since F is a slice analytic function, there exists $\varepsilon = \varepsilon(z^0, R) > 0$ such that

$$\frac{\left|\partial_{\mathbf{b}}^{s_0} F(z^0 + t\mathbf{b})\right|}{s_0!} \geq \frac{m_1(z^0, R)}{2}$$

for some $s_0 = s(a_k^0) \in \{0, \ldots, p\}$ and for all $t \in K_R \cap \{t \in \mathbb{C}\colon |t - a_k^0| < \varepsilon(R, z^0)\}$ and for all k. We denote $G_\varepsilon^0 = \bigcup_{a_k^0 \in K_R} \{t \in \mathbb{C}\colon |t - a_k^0| < \varepsilon\}$, $m_2(z^0, R) = \min\{|F(z^0 + t\mathbf{b})|\colon |t| \leq R+1, t \notin G_\varepsilon^0\}$,

$$Q(R, z^0) = \min\left\{ \frac{m_1(R, z^0)}{2}, m_2(R, z^0), 1 \right\}.$$

We take $R = |t_0|$. Then at least one of the numbers

$$|F(z^0 + t_0\mathbf{b})|,\ \left|\partial_{\mathbf{b}} F(z^0 + t_0\mathbf{b})\right|, \ldots,\ \frac{|\partial_{\mathbf{b}}^p F(z^0 + t_0\mathbf{b})|}{p!}$$

is not less than $Q(R, z^0)$ (respectively, for $t_0 \in G_\varepsilon^0$ it is $\frac{|\partial_{\mathbf{b}}^{s_0} F(z^0+t_0\mathbf{b})|}{s_0!}$ and for $t_0 \notin G_\varepsilon^0$ it is $|F(z^0 + t_0\mathbf{b})|$).

Hence,

$$\max\left\{ \frac{|\partial_{\mathbf{b}}^j F(z^0 + t_0\mathbf{b})|}{j!} \colon 0 \leq j \leq p \right\} \geq Q(R, z^0). \tag{2.49}$$

On the other hand, for $|t_0| = R$ and $j \geq p+1$ Cauchy's inequality is valid

$$\frac{|\partial_{\mathbf{b}}^j F(z^0 + t_0\mathbf{b})|}{j!} \leq \frac{1}{2\pi} \int\limits_{|\tau - t_0| = 1} \frac{|F(z^0 + \tau\mathbf{b})|}{|\tau - t_0|^{j+1}} |d\tau| \leq$$

$$\leq \max\{|F(z^0+\tau\mathbf{b})|\colon |\tau|\leq R+1\}. \tag{2.50}$$

We choose a positive continuous function $L(z)$ such that

$$L(z^0+t_0\mathbf{b}) \geq \max\left\{\frac{\max\{|F(z^0+t\mathbf{b})|\colon |\tau|\leq R+1\}}{Q(R,z^0)}, 1\right\}.$$

From (2.49) and (2.50) with $|t_0|=R$ and $j\geq p+1$ we obtain

$$\frac{\frac{|\partial^j_{\mathbf{b}}F(z^0+t_0\mathbf{b})|}{j!L^j(z^0+t_0\mathbf{b})}}{\max\left\{\frac{|\partial^k_{\mathbf{b}}F(z^0+t_0\mathbf{b})|}{k!L^k(z^0+t_0\mathbf{b})}\colon 0\leq k\leq p\right\}} \leq \frac{L^{-j}(z^0+t\mathbf{b})}{Q(R,z^0)L^{-p}(z^0+t\mathbf{b})}\times$$
$$\times \max\{|F(z^0+t\mathbf{b})|\colon |\tau|\leq R+1\} \leq L^{p+1-j}(z^0+t\mathbf{b}) \leq 1.$$

Since $z=z^0+t_0\mathbf{b}$, we have

$$\frac{\left|\partial^j_{\mathbf{b}}F(z)\right|}{j!L^j(z)} \leq \max\left\{\frac{\left|\partial^k_{\mathbf{b}}F(z)\right|}{k!L^k(z)}\colon 0\leq k\leq p\right\}.$$

In view of arbitrariness of z the function F has bounded L-index in the direction $\mathbf{b}$. □

2.10 Product of functions of bounded L-index in direction

Now we consider an application of Theorem 2.12. The following proposition can be obtained using similar considerations as in the case of analytic functions [34].

Proposition 2.9. *Let* $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ *and* $L\in Q_{\mathbf{b}}(\mathbb{D}^n)$, $F\in\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ *be a function of bounded* L*-index in the direction* $\mathbf{b}$, $\Phi\in\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ *and* $\Psi(z)=F(z)\Phi(z)$. *The function* $\Psi(z)$ *is of bounded* L*-index in the direction* $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ *if and only if the function* $\Phi(z)$ *is of bounded* L*-index in the direction* $\mathbf{b}$.

Proof. The similar result was obtained for analytic functions of bounded L-index in direction in [34]. Our proof is similar to proof for analytic functions in [34] but now we use Theorem 2.12, deduced for functions holomorphic on the slices. Since an analytic function $F(z)$ has bounded L-index in the direction $\mathbf{b}$, by Theorem 2.12 for every $r>0$ there exists $\widetilde{n}(r)\in\mathbb{Z}_+$ such that for all $z^0\in\mathbb{D}^n$, satisfying $F(z^0+t\mathbf{b})\not\equiv 0$, the estimate $n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{F}\right)\leq\widetilde{n}(r)$ holds. Hence,

$$n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{\Phi}\right) \leq n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{\Psi}\right) \leq n\left(\frac{r}{L(z^0)}, z^0, \frac{1}{\Phi}\right)+\widetilde{n}(r).$$

Thus, condition 2 of Theorem 2.12 either holds or does not hold for functions $\Psi(z)$ and $\Phi(z)$ simultaneously. If $\Phi(z)$ has bounded L-index in the direction $\mathbf{b}$, then for every $r>0$ there exist numbers $P_F(r)>0$ and $P_\Phi(r)>0$ such that $\left|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}\right| \le P_f(r)L(z)$, $\left|\frac{\partial_{\mathbf{b}}\Phi(z)}{\Phi(z)}\right| \le P_\Phi(r)L(z)$ for each $z\in(\mathbb{D}^n\setminus G_r^{\mathbf{b}}(F))\cap(\mathbb{D}^n\setminus G_r^{\mathbf{b}}(\Phi))$. Since

$$\mathbb{D}^n\setminus G_r^{\mathbf{b}}(\Psi)\subset(\mathbb{D}^n\setminus G_r^{\mathbf{b}}(F))\cap(\mathbb{D}^n\setminus G_r^{\mathbf{b}}(\Phi)),$$

$$\left|\frac{\partial_{\mathbf{b}}\Psi(z)}{\Psi(z)}\right| \le \left|\frac{\partial_{\mathbf{b}}F(z)}{F(z)}\right| + \left|\frac{\partial_{\mathbf{b}}\Phi(z)}{\Phi(z)}\right|,$$

for all $z\in\mathbb{D}^n\setminus G_r^{\mathbf{b}}(\Psi)$ we have $\left|\frac{\partial_{\mathbf{b}}\Psi(z)}{\Psi(z)}\right| \le (P_F(r)+P_\Phi(r))L(z)$, i.e. by Theorem 2.13 the function $\Psi(z)$ is of bounded L-index in the direction $\mathbf{b}$.

On the contrary, let $\Psi(z)$ be of bounded L-index in the direction $\mathbf{b}$, $r>0$. At first we show that for every $z^0\in\mathbb{D}^n\backslash G_r^{\mathbf{b}}(F)$ $(r>0)$ and for every $\widetilde{d}^k=z^0+d_k^0\mathbf{b}$, where d_k^0 are zeros of function $\Phi(z^0+t\mathbf{b})$, we have

$$|z^0-\widetilde{d}^k| > \frac{r|\mathbf{b}|}{2L(z^0)\lambda_{\mathbf{b}}(r)}. \tag{2.51}$$

On the other hand, let there exist $z^0\in\mathbb{D}^n\backslash G_r^{\mathbf{b}}(\Phi)$ and $\widetilde{d}^k=z^0+d_k^0\mathbf{b}$ such that $|z^0-\widetilde{d}^k|\le\frac{r|\mathbf{b}|}{2L(z^0)\lambda_{\mathbf{b}}(r)}$. Then by the definition of $\lambda_{\mathbf{b}}$ we have the next estimate $L(\widetilde{d}^k)\le\lambda_{\mathbf{b}}(r)L(z^0)$, and hence $|z^0-\widetilde{d}^k|=|\mathbf{b}|\cdot|d_k^0|\le\frac{r|\mathbf{b}|}{2L(\widetilde{d}^k)}$, i.e. $|d_k^0|\le\frac{r}{2L(\widetilde{d}^k)}$, but it contradicts $z^0\in\mathbb{D}^n\backslash G_r^{\mathbf{b}}(\Phi)$.

We consider $\overline{K}_0=\left\{z^0+t\mathbf{b}:|t|\le\frac{r}{2L(z^0)\lambda_{\mathbf{b}}(r)}\right\}$. It does not contain zeros of $\Phi(z^0+t\mathbf{b})$, but it may contain zeros $\widetilde{c}^k=z^0+c_k^0\mathbf{b}$ of the function $\Psi(z^0+t\mathbf{b})$. Since $\Psi(z)$ is of bounded L-index in the direction $\mathbf{b}$, the set $\overline{K}_0$ by Theorem 2.12 contains at most $\widetilde{n}_1=\widetilde{n}_1\left(\frac{r}{2\lambda_{\mathbf{b}}(r)}\right)$ zeros c_k^0 of the function $\Psi(z^0+t\mathbf{b})$. For all $c_k^0\in\overline{K}_0$, using the definition of $Q_{\mathbf{b}}(\mathbb{D}^n)$, we obtain the following inequality $L(z^0+c_k^0\mathbf{b})\ge\frac{1}{\lambda_{\mathbf{b}}\left(\frac{r}{\lambda_{\mathbf{b}}(r)}\right)}L(z^0)$. Thus, every set $m_k^0=\left\{z^0+t\mathbf{b}:|t-c_k^0|\le\frac{r_1}{L(z^0+c_k^0\mathbf{b})}\right\}$ with $r_1=\frac{r}{4(\widetilde{n}_1+1)\lambda_{\mathbf{b}}\left(\frac{r}{\lambda_{\mathbf{b}}(r)}\right)\lambda_{\mathbf{b}}(r)}$ is contained in the set $s_k^0=\left\{z^0+t\mathbf{b}:|t-c_k^0|\le\frac{r_1\lambda_{\mathbf{b}}\left(\frac{r}{\lambda_{\mathbf{b}}(r)}\right)}{L(z^0)}\right\}$. The total sum of diameters of these sets does not exceed

$$\frac{2\widetilde{n}_1r_1\lambda_{\mathbf{b}}\left(\frac{r}{\lambda_{\mathbf{b}}(r)}\right)}{L(z^0)}=\frac{r}{2\lambda_{\mathbf{b}}(r)L(z^0)}\cdot\frac{\widetilde{n}_1}{(\widetilde{n}_1+1)}<\frac{r}{2\lambda_{\mathbf{b}}(r)L(z^0)}.$$

Therefore, there exists $r^*\in\left(0,\frac{r}{2\lambda_{\mathbf{b}}(r)}\right)$ such that if $|t|=\frac{r^*}{L(z^0)}$, then $z^0+t\mathbf{b}\notin G_{r_1}^{\mathbf{b}}(\Psi)$, and therefore $z^0+t\mathbf{b}\notin G_{r_1}^{\mathbf{b}}(F)$. By Theorem 2.12 for all these points $z^0+t\mathbf{b}$ we obtain

$$\left|\frac{\partial_{\mathbf{b}}\Phi(z^0+t\mathbf{b})}{\Phi(z^0+t\mathbf{b})}\right| \le \left|\frac{\partial_{\mathbf{b}}\Psi(z^0+t\mathbf{b})}{\Psi(z^0+t\mathbf{b})}\right| + \left|\frac{\partial_{\mathbf{b}}F(z^0+t\mathbf{b})}{F(z^0+t\mathbf{b})}\right| \le (P_\Psi^*+P_F^*)L(z^0+t\mathbf{b}), \tag{2.52}$$

where P^*_Ψ and P^*_F depend only on r_1, i.e. only of r. Since the function $\frac{\partial_{\mathbf{b}}\Phi(z)}{\Phi(z)}$ is analytic in $\overline{K}_0$, applying the maximum modulus principle to the function $\frac{\partial_{\mathbf{b}}\Phi(z^0+t\mathbf{b})}{\Phi(z^0+t\mathbf{b})}$ as a function of variable t, we obtain that the modulus of this function at the point $t = 0$ does not exceed the maximum modulus of this function on the circle $\left\{t \in \mathbb{C} : |t| = \frac{r^*}{L(z^0)}\right\}$. It means that obtained inequality (2.52) also holds for z^0 instead $z^0 + t\mathbf{b}$.

Thus, for arbitrary $r > 0$ and $z^0 \in \mathbb{D}^n \backslash G^{\mathbf{b}}_r(F)$ we have proved the first condition of Theorem 2.13. Above we have already shown that the second condition of Theorem 2.13 is true. Hence, by the mentioned theorem the function $\Phi(z)$ has bounded L-index in the direction $\mathbf{b}$. □

2.11 Sum of functions of bounded L-index in direction

It is known that the product of analytic functions of bounded L-index in direction is function with the same class ([34]). But the class of analytic functions of bounded index is not closed under the addition. The corresponding example was constructed by W. Pugh (see [25, 35]). A generalization of Pugh's example for enire functions of bounded L-index in direction is proposed in [34]. The generalization is also applicable to the class $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$.

Let us consider arbitrary hyperplane $A = \{z \in \mathbb{D}^n : \langle z, c\rangle = 1\}$, where $\langle c, \mathbf{b}\rangle \neq 0$, and fixed it. Obviously that $\bigcup_{z^0\in A}\{z^0 + t\mathbf{b} : t \in \mathbb{C}\} = \mathbb{D}^n$.

Let $z^0 \in A$ be a given point. If $F(z^0 + t\mathbf{b}) \not\equiv 0$ as a function of variable $t \in \mathbb{C}$, then there exists a point $t_0 \in \mathbb{C}$ such that $F(z^0 + t_0\mathbf{b}) \neq 0$. Thus, for every line $\{z^0 + t\mathbf{b} : F(z^0 + t\mathbf{b}) \not\equiv 0\}$ we fix one point t_0 with this property. In this section, we will denote by B the union of these points $z^0 + t_0\mathbf{b}$, i.e.,

$$B = \bigcup_{\substack{z^0\in A\\ F(z^0+t\mathbf{b})\not\equiv 0}} \{z^0 + t_0\mathbf{b}\}.$$

Clearly, that for every $z \in \mathbb{D}^n$ there exist $z^0 \in A$ and $t \in \mathbb{C}$ obeying $z = z^0+t\mathbf{b}$. Indeed, $z^0 = z + \frac{1-\langle z,c\rangle}{\langle \mathbf{b},c\rangle}\mathbf{b},\ t = \dfrac{\langle z, c\rangle - 1}{\langle \mathbf{b}, c\rangle}$.

The following propositions can be proved by analogy to [36] and [5].

Proposition 2.10. *Let L be a positive continuous function, and the functions F and G belong to the class $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ and satisfy the conditions*

1) *$G(z)$ has bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ with $N_{\mathbf{b}}(G, L) = N < +\infty$;*

2) *there exists* $\alpha \in (0,1)$ *such that for all* $z \in \mathbb{D}^n$ *and* $p \geq N+1$ $(p \in \mathbb{N})$ *one has*

$$\frac{|\partial_{\mathbf{b}}^p G(z)|}{p!L^p(z)} \leq \alpha \max\left\{\frac{|\partial_{\mathbf{b}}^k G(z)|}{k!L^k(z)} : 0 \leq k \leq N\right\}; \tag{2.53}$$

3) *for each* $z = z^0 + t\mathbf{b} \in \mathbb{D}^n$, *where* $z^0 \in A$, $z^0 + t_0\mathbf{b} \in B$ *and* $r = |t-t_0|L(z^0 + t\mathbf{b})$, *the following inequality holds*

$$\max\left\{|F(z^0+t'\mathbf{b})| : \ |t'-t_0| = \frac{2r}{L(z^0+t\mathbf{b})}\right\} \leq$$
$$\leq \max\left\{\frac{|\partial_{\mathbf{b}}^k G(z^0+t\mathbf{b})|}{k!L^k(z^0+t\mathbf{b})} : \ 0 \leq k \leq N\right\}; \tag{2.54}$$

4) *either* $(\exists c > 0)(\forall z^0 + t_0\mathbf{b} \in B)(\forall t \in \mathbb{C}, \ |t-t_0|L(z^0+t\mathbf{b}) \leq 1)$:

$$\max\left\{|F(z^0+t'\mathbf{b})| : \ |t'-t_0| = \frac{2}{L(z^0+t\mathbf{b})}\right\} \Big/ |F(z^0+t_0\mathbf{b})| \leq c < +\infty,$$

or for $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$ $(\exists c > 0)(\forall z^0 + t_0\mathbf{b} \in B)$:

$$\max\left\{|F(z^0+t'\mathbf{b})| : \ |t'-t_0| = \frac{2\lambda_2^{\mathbf{b}}(1)}{L(z^0+t_0\mathbf{b})}\right\} \Big/ |F(z^0+t_0\mathbf{b})| \leq c < +\infty. \tag{2.55}$$

Then for every $\varepsilon \in \mathbb{C}$, $|\varepsilon| \leq \frac{1-\alpha}{2c}$, *the function*

$$H(z) = G(z) + \varepsilon F(z) \tag{2.56}$$

has bounded L*-index in the direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *and* $N_{\mathbf{b}}(H,L) \leq N$.

Proof. The proof uses ideas from [36]. We write Cauchy's formula for the function $F(z^0 + t\mathbf{b})$ as a function of single complex variable t

$$\frac{\partial_{\mathbf{b}} F(z^0+t\mathbf{b})}{p!} = \frac{1}{2\pi i}\int_{|t'-t|=\frac{r}{L(z^0+t\mathbf{b})}} \frac{F(z^0+t'\mathbf{b})}{(t'-t)^{p+1}}dt'. \tag{2.57}$$

For the chosen $r = |t-t_0|L(z^0+t\mathbf{b})$ the following inequality is valid

$$\frac{r}{L(z^0+t\mathbf{b})} = |t'-t| \geq |t'-t_0| - |t-t_0| = |t'-t_0| - \frac{r}{L(z^0+t\mathbf{b})}.$$

Hence,

$$|t'-t_0| \leq \frac{2r}{L(z^0+t\mathbf{b})}. \tag{2.58}$$

Equality (2.57) gives the following estimates

$$\frac{|\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})} \leq \frac{1}{2\pi L^p(z^0+t\mathbf{b})} \cdot \frac{L^{p+1}(z^0+t\mathbf{b})}{r^{p+1}} \times$$
$$\times \frac{2\pi r}{L(z^0+t\mathbf{b})} \cdot \max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t| = \frac{r}{L(z^0+t\mathbf{b})}\right\} \leq$$
$$\leq \frac{1}{r^p} \max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0| = \frac{2r}{L(z^0+t\mathbf{b})}\right\}. \tag{2.59}$$

If $r = |t-t_0|L(z^0+t\mathbf{b}) > 1$, then (2.59) implies

$$\frac{|\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})} \leq \max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0| = \frac{2r}{L(z^0+t\mathbf{b})}\right\}. \tag{2.60}$$

Let $r = |t-t_0|L(z^0+t\mathbf{b}) \in (0;1]$. Setting $r=1$ in (2.57) and (2.58), it is possible to deduce

$$\frac{|\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})} \leq \max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0| = \frac{2}{L(z^0+t\mathbf{b})}\right\} =$$
$$= \frac{\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0| = \frac{2}{L(z^0+t\mathbf{b})}\right\}}{\max\left\{|F(z^0+t'\mathbf{b})|: |t'-t_0| = \frac{2r}{L(z^0+t\mathbf{b})}\right\}} \times$$
$$\times \max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0| = \frac{2r}{L(z^0+t\mathbf{b})}\right\} \leq$$
$$\leq \frac{\max\left\{|F(z^0+t'\mathbf{b})|: |t'-t_0| = \frac{2}{L(z^0+t\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|} \times$$
$$\times \max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0| = \frac{2r}{L(z^0+t\mathbf{b})}\right\} \leq$$
$$\leq c \max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0| = \frac{2r}{L(z^0+t\mathbf{b})}\right\}, \tag{2.61}$$

where

$$c = \sup_{z^0+t_0\mathbf{b}\in B} \sup_{\substack{t\in\mathbb{C},\\ |t-t_0|L(z^0+t\mathbf{b})\leq 1}} \frac{\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0| = \frac{2}{L(z^0+t\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|} \geq 1.$$

If $L \in Q$, then $\sup\left\{\frac{L(z^0+t_0\mathbf{b})}{L(z^0+t\mathbf{b})}:\ |t-t_0| \leq \frac{1}{L(z^0+t\mathbf{b})}\right\} \leq \lambda_{\mathbf{b}}(1)$. This means that $L(z^0+t\mathbf{b}) \geq \frac{L(z^0+t_0\mathbf{b})}{\lambda_{\mathbf{b}}(1)}$. Using the obtained inequality we choose

$$c := \sup_{z^0+t_0\mathbf{b}\in B} \frac{\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0| = \frac{2\lambda_{\mathbf{b}}(1)}{L(z^0+t_0\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|} \geq 1$$

in (2.61). Taking into account (2.60) and (2.61), one has

$$\frac{|\partial_{\mathbf{b}}^{p}F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})}\le c\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\} \tag{2.62}$$

for all $n\in\mathbb{N}\cup\{\mathbf{0}\}$, $r\ge 0$, $z^0\in A$, $t\in\mathbb{C}$.

We differentiate (2.56) p times in the direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$, $p\ge N+1$, and apply (2.53), (2.62) and (2.54) to the obtained equality

$$\begin{gathered}\frac{|\partial_{\mathbf{b}}^{p}H(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})}\le\frac{|\partial_{\mathbf{b}}^{p}G(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})}+\frac{|\varepsilon||\partial_{\mathbf{b}}^{p}F(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})}\le\\ \le\alpha\max\left\{\frac{|\partial_{\mathbf{b}}^{k}G(z^0+t\mathbf{b})|}{k!L^k(z^0+t\mathbf{b})}:0\le k\le N\right\}+\\ +c|\varepsilon|\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}\le\\ \le(\alpha+c|\varepsilon|)\max\left\{\frac{|\partial_{\mathbf{b}}^{k}G(z^0+t\mathbf{b})|}{k!L^k(z^0+t\mathbf{b})}:0\le k\le N\right\}.\end{gathered} \tag{2.63}$$

If $s\le N$, then (2.62) is valid for $p=s$, but (2.53) is false. Therefore differentiation of equality (2.56) provides the following estimate

$$\begin{gathered}\frac{|\partial_{\mathbf{b}}^{s}H(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})}\ge\frac{|\partial_{\mathbf{b}}^{S}G(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})}-\frac{|\varepsilon||\partial_{\mathbf{b}}^{s}F(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})}\ge\\ \ge\frac{|\partial_{\mathbf{b}}^{s}G(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})}-c|\varepsilon|\max\left\{|F(z^0+t'\mathbf{b})|:|t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\},\end{gathered} \tag{2.64}$$

where $0\le s\le N$. From (2.54) and (2.64) we get

$$\max_{0\le s\le N}\left\{\frac{|\partial_{\mathbf{b}}^{s}H(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})}\right\}\ge(1-c|\varepsilon|)\max_{0\le s\le N}\left\{\frac{|\partial_{\mathbf{b}}^{s}G(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})}\right\}. \tag{2.65}$$

If $c|\varepsilon|<1$, then from (2.63) and (2.65) it implies

$$\frac{|\partial_{\mathbf{b}}^{p}H(z^0+t\mathbf{b})|}{p!L^p(z^0+t\mathbf{b})}\le\frac{\alpha+c|\varepsilon|}{1-c|\varepsilon|}\max_{0\le s\le N}\left\{\frac{|\partial_{\mathbf{b}}^{s}H(z^0+t\mathbf{b})|}{s!L^s(z^0+t\mathbf{b})}\right\} \tag{2.66}$$

for $p\ge N+1$. Suppose that $\frac{\alpha+c|\varepsilon|}{1-c|\varepsilon|}\le 1$. Hence, $|\varepsilon|\le\frac{1-\alpha}{2c}$.

Let $N_{\mathbf{b}}(z^0+t\mathbf{b},L,F)$ be the L-index in direction of the function F at the point $z^0+t\mathbf{b}$, i.e., $N_{\mathbf{b}}(z^0+t\mathbf{b},L,F)$ is the least integer m_0, for which inequality (2.2) holds with $z=z^0+t\mathbf{b}$.

For $|\varepsilon|\le\frac{1-\alpha}{2c}$ validity of inequality (2.66) means that for any $z^0\in A$ and for all $t\in\mathbb{C}$ such that $F(z^0+t\mathbf{b})\ne 0$ the L-index in the direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ at the point $z^0+t\mathbf{b}$ does not exceed N, that is $N_{\mathbf{b}}(z^0+t\mathbf{b},F,L)\le N$. In those points t, for which $F(z^0+t\mathbf{b})=0$, but $F(z^0+t\mathbf{b})\not\equiv 0$, we will use that $H(z^0+t\mathbf{b})=G(z^0+t\mathbf{b})$ and the function G has bounded L-index in the direction $\mathbf{b}$.

When for $z^0\in A\quad F(z^0+t\mathbf{b})\equiv 0$, one has $H(z^0+t\mathbf{b})\equiv G(z^0+t\mathbf{b})$ and $N_{\mathbf{b}}(z^0+t\mathbf{b},F,L)=N_{\mathbf{b}}(z^0+t\mathbf{b},G,L)\le N$. Hence, $H(z)$ is of bounded L-index in the direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ with $N_{\mathbf{b}}(H,L)\le N$. This completes the proof of Theorem 2.10. □

Every function $F\in\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ with $N_{\mathbf{b}}(F,L)=0$ satisfies inequality (2.55). (see proof of necessity of corresponding Theorem 4 in [10]).

If $L\in Q_{\mathbf{b}}(\mathbb{D}^n)$, then condition 2) in Proposition 2.10 is always satisfied.

Proposition 2.11. *Let $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ and $L\in Q_{\mathbf{b}}(\mathbb{D}^n)$, $\alpha\in(0,1)$, F and G be functions belonging to the class $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ and obeying the conditions*

1) $G(z)$ has bounded L-index in the direction $\mathbf{b}\in\mathbb{D}^n\setminus\{\mathbf{0}\}$.

2) for every $z=z^0+t\mathbf{b}\in\mathbb{D}^n$, where $z^0\in A$, $z^0+t_0\mathbf{b}\in B$, and $r=|t-t_0|L(z^0+t\mathbf{b})$ the following inequality hold

$$\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0|=\frac{2r}{L(z^0+t\mathbf{b})}\right\}\le$$
$$\le\max\left\{\frac{|\partial_{\mathbf{b}}^kG(z^0+t\mathbf{b})|}{k!L^k(z^0+t\mathbf{b})}:0\le k\le N_{\mathbf{b}}(G_\alpha,L_\alpha)\right\}.$$

3) $$c:=\sup_{z^0+t_0\mathbf{b}\in B}\frac{\max\left\{|F(z^0+t'\mathbf{b})|:\ |t'-t_0|=\frac{2\lambda_2^{\mathbf{b}}(1)}{L(z^0+t_0\mathbf{b})}\right\}}{|F(z^0+t_0\mathbf{b})|}<\infty.$$

If $|\varepsilon|\le\frac{1-\alpha}{2c}$, then the function $H(z)=G(z)+\varepsilon F(z)$ has bounded L-index in the direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ with $N_{\mathbf{b}}(H,L)\le N_{\mathbf{b}}(G_\alpha,L_\alpha)$, where $G_\alpha(z)=G(z/\alpha)$, $L_\alpha(z)=L(z/\alpha)$.

Proof. This proof is based on the proof of appropriate theorem for analytic functions of bounded L-index in direction [36]. The condition 2) in Theorem 2.10 is always satisfied for $N=N_b(G_\alpha,L_\alpha)$ instead $N=N_{\mathbf{b}}(G,L)$. Indeed by Theorem 2.9 inequality (2.19) holds for the function G. Substituting $\frac{z^0}{\alpha}$, $\frac{t}{\alpha}$ and $\frac{t_0}{\alpha}$ instead z^0, t and t_0 in (2.19) we obtain

$$\max\left\{|G((z^0+t\mathbf{b})/\alpha)|:|t-t_0|=\frac{r_2\alpha}{L((z^0+t_0\mathbf{b})/\alpha)}\right\}\le$$
$$\le P_1\max\left\{|G((z^0+t\mathbf{b})/\alpha)|:|t-t_0|=\frac{r_1\alpha}{L((z_0+t_0\mathbf{b})/\alpha)}\right\}.\qquad(2.67)$$

By Theorem 2.9 inequality (2.67) means that $G_\alpha = G(z/\alpha)$ has bounded L_α-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ and vice versa. Hence, for all $p \geq N_{\mathbf{b}}(G_\alpha, L_\alpha) + 1$ and $\alpha \in (0, 1)$

$$\frac{|\partial_{\mathbf{b}}^p G_\alpha(z)|}{p! L_\alpha^p(z)} = \frac{|\partial_{\mathbf{b}}^p G(z/\alpha)|}{p! \alpha^p L^p(z/\alpha)} \leq \max\left\{\frac{|\partial_{\mathbf{b}}^s G_\alpha(z)|}{s! L_\alpha^s(z)} :\ 0 \leq s \leq N_{\mathbf{b}}(G_\alpha, L_\alpha)\right\} =$$
$$= \max\left\{\frac{|\partial_{\mathbf{b}}^s G(z/\alpha)|}{s! \alpha^s L^s(z/\alpha)} :\ 0 \leq s \leq N_{\mathbf{b}}(G_\alpha, L_\alpha)\right\}.$$

Multiplying by α^p, we deduce

$$\frac{|\partial_{\mathbf{b}}^p G(z/\alpha)|}{p! L^p(z/\alpha)} \leq \max\left\{\frac{\alpha^{p-s}|\partial_{\mathbf{b}}^s G(z/\alpha)|}{s! L^s(z/\alpha)} :\ 0 \leq s \leq N_{\mathbf{b}}(G_\alpha, L_\alpha)\right\} \leq$$
$$\leq \alpha \max\left\{\frac{|\partial_{\mathbf{b}}^s G(z/\alpha)|}{s! L^s(z/\alpha)} :\ 0 \leq s \leq N_{\mathbf{b}}(G_\alpha, L_\alpha)\right\}.$$

In view of arbitrariness z, the last inequality yields (2.53). □

2.12 Boundedness of L-index in direction of analytic solutions of some linear partial differential equations

Denote

$$G_r(F) := G_r^{\mathbf{b}}(F) := \bigcup_{z:\ F(z)=0} \{z + t\mathbf{b} \colon |t| < r/L(z)\}.$$

By $n_{z^0}(r, F) = n_{\mathbf{b}}\big(r, z^0, 1/F\big) := \sum_{|a_k^0| \leq r} 1$ we denote counting function of zeros a_k^0 for the slice function $F(z^0 + t\mathbf{b})$ in the disc $\{t \in \mathbb{C} \colon |t| \leq r\}$. If for given $z^0 \in \mathbb{D}^n$ and for all $t \in \mathbb{C}$ $F(z^0 + t\mathbf{b}) \equiv 0$, then we put $n_{z^0}(r) = -1$. Denote $n(r) = \sup_{z \in \mathbb{D}^n} n_z(r/L(z))$.

From the proof of Theorem 2.12 in [11] it follows the following lemma:

Lemma 2.3. *If a function $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, then for each $z^0 \in \mathbb{D}^n \setminus G_r^{\mathbf{b}}(F)$ $(r > 0)$ and for every $\widetilde{a}^k = z^0 + a_k^0\mathbf{b}$ such that $F(\widetilde{a}^k) = 0$ one has*

$$|z^0 - \widetilde{a}_k| > \frac{r|\mathbf{b}|}{2L(z^0)\lambda_{\mathbf{b}}(r)}. \tag{2.68}$$

Let us consider the following directional differential equation

$$g_0(z)\partial_{\mathbf{b}}^p w + g_1(z)\partial_{\mathbf{b}}^{p-1} w + \ldots + g_p(z) w = h(z), \tag{2.69}$$

where g_j, h are functions from the class $\mathcal{H}_{\mathbf{b}}^n$, $j \in \{0, 1, \ldots, p\}$. For analytic functions of bounded L-index in direction the equation was investigated in [26, 27].

Also there were presented results for entire functions of bounded l-index [18] and for functions which are slice holomorphic in the unit ball [14]. A system of directional differential equations was also studied in [17], non-linear differential equation within the notion of bounded index was studied in [24]. Here we will consider the weaker assumption that the coefficients of equation (2.69) are slice analytic functions, i.e. they are functions from the class $\widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$.

We need the following proposition. Its proof is based on the proof of its analog for analytic functions [28].

Lemma 2.4. *Let* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *and* $F \in \widetilde{\mathcal{H}}_{\mathbf{b}}(\mathbb{D}^n)$ *be a function of bounded* L*-index in the direction* $\mathbf{b}$, $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$. *Then for every* $r \in (0, \beta]$ *and for every* $m \in \mathbb{N}$ *there exists* $P = P(r, m) > 1$ *such that for all* $z \in \mathbb{D}^n \setminus G_r^{\mathbf{b}}(F)$ *the inequality* $|\partial_{\mathbf{b}}^m F(z)| \le PL^m(z)|F(z)|$ *holds.*

Proof. We apply Theorem 2.10 with $R = \frac{r}{2\lambda_{\mathbf{b}}(r)}$. Then there exist $P_2 = P_2\big(R\big) \ge 1$ and $\eta = \eta\,(R) \in (0, R)$ such that for all $z^0 \in \mathbb{D}^n$ and some $r^* = r^*(z^0) \in [\eta(R), R]$ inequality (2.26) holds with r^* instead of r. Hence, by Cauchy's inequality we obtain

$$\frac{1}{m!}|\partial_{\mathbf{b}}^m F(z^0)| \le \left(\frac{L(z^0)}{r^*}\right)^m \max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{r^*}{L(z^0)}\right\}$$
$$\le P_2\left(\frac{L(z^0)}{\eta}\right)^m \min\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{r^*}{L(z^0)}\right\}.$$

But by (2.68) for every $z^0 \in \mathbb{D}^n \setminus G_r^{\mathbf{b}}(F)$ the set

$$\left\{z^0 + t\mathbf{b} : |t| \le \frac{r}{2\lambda_2^{\mathbf{b}}(r)L(z^0)}\right\}$$

does not contain zeros of function $F(z^0 + t\mathbf{b})$. Therefore, applying to $\frac{1}{F(z^0+t\mathbf{b})}$ the maximum modulus principle in variable $t \in \mathbb{C}$, we have

$$|F(z^0)| \ge \min\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{r^*}{L(z^0)}\right\}.$$

Thus,

$$\big|\partial_{\mathbf{b}}^m F(z^0)\big| \le m!\frac{P_2}{\eta^m}L^m(z^0)|F(z^0)|.$$

Hence, in view of arbitrariness of z^0, we obtain the desired inequality with $P = P_2 m!\eta^{-m}$. □

Denote $g^*(z) = h(z) \cdot \prod_{j=0}^p g_j(z)$, $n(r, g^*) = \sup\limits_{z\in\mathbb{D}^n} n_{\mathbf{b}}(r/L(z), z, 1/g^*)$,

$$r^* = \sup_{s\ge 1} \frac{(s-1)\lambda_1(s)}{8(n(s,g^*)+1)}, \quad H(F) := \bigcup_{\substack{z\in Z_F \\ \forall t\in\mathbb{C}\ F(z+t\mathbf{b})\equiv 0}} \{z + t\mathbf{b} \colon t \in \mathbb{C}\},$$

where Z_F is zero set of the function F. Using Lemma 2.4, we prove the following theorem. It was firstly obtained for analytic functions of bounded L-index in direction [27].

Denote $G_r = (G_r(h) \setminus H(h)) \cup G_r(g_0) \cup \bigcup_{j=1}^{p}(G_r(g_j) \setminus H(g_j))$.

Theorem 2.19. *Let* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *and* $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$, *and* $g_0(z), \dots, g_p(z), h(z) \in \mathcal{H}_{\mathbf{b}}(\mathbb{D}^n)$ *be functions of bounded* L*-index in the direction* $\mathbf{b}$. *Suppose that there exist* $r \in (0; r^*)$ *and* $T > 0$ *such that* G_r *is an unbounded set and for every* $z \in \mathbb{D}^n \setminus G_r(g_0)$ *and* $j = 1, \dots, p$

$$|g_j(z)| \leq T L^j(z) |g_0(z)|. \tag{2.70}$$

Then every function $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{D}^n)$ *such that* (2.69) *is valid, has bounded* L*-index in the direction* $\mathbf{b}$.

Proof. One should observe that the condition $F \in \mathcal{H}_{\mathbf{b}}(\mathbb{D}^n)$ implies $\partial_{\mathbf{b}}^m F \in \mathcal{H}_{\mathbf{b}}(\mathbb{D}^n)$ for all $m \in \mathbb{N}$. Besides, Theorem 2.12 and restrictions of Theorem 2.19 provides validity of inequalities $n(r, g^*) < +\infty$ and $r^* > 0$.

From conditions of Theorem it follows that $\mathbb{D}^n \setminus G_r \neq \emptyset$. Lemma 2.4 and inequality (2.70) yield that there exist $r \in (0; r^*]$ and $T^* > 0$ such that for all $z \in \mathbb{D}^n \setminus G_r$

$$|\partial_{\mathbf{b}} h(z)| \leq T^* |h(z)| L(z), \quad |g_j(z)| \leq T^* |g_0(z)| L^j(z), \ \ j \in \{1, 2, \dots, p, \},$$
$$|\partial_{\mathbf{b}} g_j(z)| \leq P(r) L(z) |g_j(z)| \leq T^*(r) |g_0(z)| L^{j+1}(z), \ j \in \{0, 1, 2, \dots, p, \}.$$

Evaluate the derivative in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ in Equation (2.69)

$$g_0(z)\partial_{\mathbf{b}}^{p+1} F(z) + \sum_{j=1}^{p} g_j(z) \partial_{\mathbf{b}}^{p+1-i} F(z) + \sum_{j=0}^{p} \partial_{\mathbf{b}} g_j(z) \partial_{\mathbf{b}}^{p-i} F(z) = \partial_{\mathbf{b}} h(z).$$

The obtained equality means that for all $z \in \mathbb{D}^n \setminus G_r$:

$$|g_0(z)| \left|\partial_{\mathbf{b}}^{p+1} F(z)\right| \leq |\partial_{\mathbf{b}} h(z)| + \sum_{j=1}^{p} |g_j(z)| \left|\partial_{\mathbf{b}}^{p+1-j} F(z)\right|$$
$$+ \sum_{j=0}^{p} |\partial_{\mathbf{b}} g_j(z)| \left|\partial_{\mathbf{b}}^{p-j} F(z)\right| \leq T^* |h(z)| L(z) + \sum_{j=1}^{p} |g_j(z)| \left|\partial_{\mathbf{b}}^{p+1-j} F(z)\right|$$
$$+ \sum_{j=0}^{p} |\partial_{\mathbf{b}} g_j(z)| \left|\partial_{\mathbf{b}}^{p-j} F(z)\right| \leq T^* L(z) \sum_{j=0}^{p} |g_j(z)| \left|\partial_{\mathbf{b}}^{p-j} F(z)\right|$$
$$+ \sum_{j=1}^{p} |g_j(z)| \left|\partial_{\mathbf{b}}^{p+1-j} F(z)\right| + \sum_{j=0}^{p} |\partial_{\mathbf{b}} g_j(z)| \left|\partial_{\mathbf{b}}^{p-j} F(z)\right|$$

$$\leq T^*|g_0(z)| \left(T^* L(z) \sum_{j=0}^{p} L^j(z) \left| \partial_{\mathbf{b}}^{p-j} F(z) \right| + \sum_{j=1}^{p} L^j(z) \left| \partial_{\mathbf{b}}^{p+1-i} F(z) \right| \right.$$
$$\left. + \sum_{j=0}^{p} L^{j+1}(z) \left| \partial_{\mathbf{b}}^{p-j} F(z) \right| \right) = T^*|g_0(z)| L^{p+1}(z)| ((T^*+1)$$
$$\times \sum_{j=0}^{p} \frac{1}{L^{p-j}(z)} \left| \partial_{\mathbf{b}}^{p-j} F(z) \right| + \sum_{j=1}^{p} \frac{1}{L^{p+1-j}(z)} \left| \partial_{\mathbf{b}}^{p+1-i} F(z) \right| \Bigg)$$
$$\leq T^*((T^*+1)(p+1)+p)|g_0(z)| L^{p+1}(z) \max_{0\leq j\leq p} \frac{|\partial_{\mathbf{b}}^{j} F(z)|}{L^j(z)}.$$

Thus, there exists $P_3 > 0$ such that for all $z \in \mathbb{D}^n \setminus G_r$

$$\frac{|\partial_{\mathbf{b}}^{p+1} F(z)|}{L^{p+1}(z)} \leq P_3 \max \left\{ \frac{|\partial_{\mathbf{b}}^{j} F(z)|}{L^j(z)} : 0 \leq j \leq p \right\}. \tag{2.71}$$

If $z' \in A := H(g_0) \setminus \bigcup_{j=1}^{p} (G_r(g_j) \setminus H(g_j))$, then there exists a sequence of points $z^m \in \mathbb{D}^n \setminus G_r$, satisfying (2.71) and such that $z^m \to z'$ as $m \to \infty$. Substituting $z = z^m$ in (2.71) and passing to the limit as $m \to \infty$, we obtain that inequality is valid for all $z \in A \cup (\mathbb{D}^n \setminus G_r)$. Here we used joint continuity of the function F for passing to the limit. If $\mathbb{D}^n = A \cup (\mathbb{D}^n \setminus G_r)$ (i.e., all zeros of the function g^* belongs to $H(g^*)$), then by Theorem 2.14 the function from the class $\mathcal{H}_{\mathbf{b}}(\mathbb{D}^n)$, obeying (2.69), has bounded $\mathbf{L}$-index in the direction $\mathbf{b}$. Otherwise, $n(s, g^*) \geq 1$.

Since $r \in (0, r^*)$ and $r^* = \sup_{s\geq 1} \frac{(s-1)\lambda_1(s)}{8(n(s,g^*)+1)}$, there exists $r' \geq 1$ such that $r \leq \frac{(r'-1)\lambda_1(r')}{8(n(r',g^*)+1)}$. Let z^0 be an arbitrary point from $\mathbb{D}^n$ and

$$K^0 = \left\{ z^0 + t\mathbf{b} : |t| \leq r'/L(z^0) \right\}.$$

Since the analytic functions $g_0, g_1, \ldots, g_p, h$ has bounded L-index in the direction $\mathbf{b}$, by Theorem 2.12 the set K^0 contains at more $n(r', g^*)$ zeros of these functions or $K^0 \subset Z_{g^*}$. Let c_m^0 be zeros of the slice function g^* (that is $g^*(z^0 + c_m^0\mathbf{b}) = 0$) such that $z^0 + c_m^0\mathbf{b} \in K^0 \cap ((Z_h \setminus H(h)) \cup \bigcup_{j=0}^{p} (Z_{g_j} \setminus H(g_j))$, where $m \in \mathbb{N}$, $m \leq n(r', g^*)$. From $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$ it follows that $L(z^0 + c_m^0\mathbf{b}) \geq \lambda_1(r')L(z^0)$. Obviously,

$$\widetilde{K}_m^0 := \left\{ z^0 + t\mathbf{b} : |t - c_m^0| \leq \frac{r}{L(z^0 + c_m^0\mathbf{b})} \right\}$$
$$\subset \left\{ z^0 + t\mathbf{b} : |t - c_m^0| \leq \frac{(r'-1)\lambda_1(r')}{8(n(r',g^*)+1)L(z^0 + c_m^0\mathbf{b})} \right\}$$
$$\subset K_m^0 := \left\{ z^0 + t\mathbf{b} : |t - c_m^0| \leq \frac{r'-1}{8(n(r',g^*)+1)L(z^0)} \right\}.$$

Therefore, for $z^0+t\mathbf{b}\in K^0\setminus\bigcup_{z^0+c_m^0\mathbf{b}\in K^0}K_m^0$ (2.71) is true. Hence, for there points $z^0+t\mathbf{b}$ inequalities $L(z^0)\geq\frac{L(z^0+t\mathbf{b})}{\lambda_2(r')}$ and (2.71) give us

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z^0+t\mathbf{b})|}{L^{p+1}(z^0)}\leq(\lambda_2(r'))^{p+1}\frac{|\partial_{\mathbf{b}}^{p+1}F(z^0+t\mathbf{b})|}{L^{p+1}(z^0+t\mathbf{b})}\leq P_3(\lambda_2(r'))^{p+1}$$
$$\times\max_{0\leq j\leq p}\left\{\frac{|\partial_{\mathbf{b}}^{j}F(z^0+t\mathbf{b})|}{L^{j}(z^0+t\mathbf{b})}\right\}\leq P_3(\lambda_2(r'))^{p+1}\max_{0\leq j\leq p}\left\{\frac{|\partial_{\mathbf{b}}^{j}F(z^0+t\mathbf{b})|}{L^{j}(z^0)}\left(\frac{1}{\lambda_1(r')}\right)^j\right\}$$
$$\leq P_3\left(\frac{\lambda_2(r')}{\lambda_1(r')}\right)^p\lambda_2(r')\max_{0\leq j\leq p}\left\{\frac{|\partial_{\mathbf{b}}^{j}F(z^0+t\mathbf{b})|}{L^{j}(z^0)}\right\}=P_4w_{z^0}(t),\qquad(2.72)$$

where $P_4=P_3\lambda_2(r')\left(\frac{\lambda_2(r')}{\lambda_1(r')}\right)^p$ and

$$w_{z^0}(t)=\max\left\{\frac{|\partial_{\mathbf{b}}^{j}F(z^0+t\mathbf{b})|}{L^{j}(z^0)}:0\leq j\leq p\right\}.$$

Let D be a sum of diameters K_m^0. Then $D\leq\frac{2(r'-1)n(r',g^*)}{8(n(r',g^*)+1)L(z^0)}<\frac{r'-1}{4L(z^0)}$. Therefore there exist numbers $r_1\in\left[\frac{r'}{4},\frac{r'}{2}\right]$ and $r_2\in\left[\frac{3r'+1}{4};r'\right]$ such that for $z^0+t\mathbf{b}\in C_j=\left\{z^0+t\mathbf{b}:|t|=\frac{r_j}{L(z^0)}\right\}$, $j\in\{1,2\}$, one has $z^0+t\mathbf{b}\in K^0\setminus\bigcup_{c_m^0\in K^0}K_m^0$. Choose arbitrary points $z^0+t_1\mathbf{b}\in C_1$ and $z^0+t_2\mathbf{b}\in C_2$ and connect them by a smooth curve $\gamma=\{z^0+t\mathbf{b}:t=t(s),0\leq s\leq 1\}$ such that $w_{z^0}(t)\neq 0$ and $\gamma\subset K^0\setminus\bigcup_{c_m^0\in K^0}K_m^0$.

In the following detailed description of construction of the curve we will use ideas from proof of Theorem 8 in [29] with adaptation for slice functions. For a construction of the curve γ we connect t_1 and t_2 by a line $t(s)=(t_2-t_1)s+t_1$, $s\in[0,1]$. Let t_k^* be points on the line $t(s)$ such that $w_{z^0}(t_k^*)=0$. The number of such points $m_0=m(z^0+t_1\mathbf{b},z^0+t_2\mathbf{b})$ is finite. Let (t_k^*) be a sequence of these points in ascending order of the value $|t_1-t_k^*|$, $k\in\{1,2,...,p\}$. We choose

$$r_0<\min_{1\leq k\leq m_0-1}\left\{|t_k^*-t_{k+1}^*|,|t_1^*-t_1|,|t_{m_0}^*-t_2|,\frac{r'}{4\pi L(z^0)}\right\}.$$

Now we construct circles with centers at the points t_k^* and corresponding radii $r_k^{'}<\frac{r_0}{2^k}$ such that $w_{z^0}(t(s))\neq 0$ for all t on the circles. It is possible, because $F\not\equiv 0$.

Every such circle is divided onto two semicircles by the line $t=t(s)$. The required piecewise-analytic curve consists with arcs of the constructed semicircles and segments of line $z_1^*(t)$, which connect the arcs in series between themselves or with the points t_1,t_2. If the curve intersects some set K_m^0 then we override the set by a semicircle with the center at the point c_m^0 and radius $\frac{r'}{8(n(r',g^*)+1)L(z^0)}$.

The description show that the curve can be chosen with the following estimate of its length

$$|\gamma|\leq|\mathbf{b}|\left(\frac{\pi r_1}{L(z^0)}+\frac{r_2-r_1}{L(z^0)}+\frac{\pi r_0}{L(z^0)}+\frac{\pi n(r',g^*)r'}{8(n(r',g^*)+1)L(z^0)}\right)$$
$$<\frac{|\mathbf{b}|}{L(z^0)}\left(\frac{\pi r'}{2}+r'+\frac{\pi r'}{8}\right)<\frac{3|\mathbf{b}|r'}{L(z^0)}.$$

Then on γ inequality (2.72) is valid, that is

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z^0+t(s)\mathbf{b})|}{L^{p+1}(z^0)}\leq P_4 w_{z^0}(t(s)),\ 0\leq s\leq 1.$$

In view of the described construction, the function $z^0+t(s)\mathbf{b}$ is piece-wise analytic on $[0,1]$. Hence, for arbitrary $k\in\mathbb{Z}_+$, $j\in\mathbb{Z}_+$, $k\leq p$, either

$$\frac{|\partial_{\mathbf{b}}^{k}F(z^0+t(s)\mathbf{b})|}{L^{k}(z^0)}\equiv\frac{|\partial_{\mathbf{b}}^{j}F(z^0+t(s)\mathbf{b})|}{L^{j}(z^0)},\tag{2.73}$$

or the equality

$$\frac{|\partial_{\mathbf{b}}^{k}F(z^0+t(s)\mathbf{b})|}{L^{k}(z^0)}=\frac{|\partial_{\mathbf{b}}^{j}F(z^0+t(s)\mathbf{b})|}{L^{j}(z^0)}\tag{2.74}$$

holds only for a finite set of points $s_k\in[0;1]$.

Then for the function $w_{z^0}(t(s))$ as maximum of such expressions $\frac{|\partial_{\mathbf{b}}^{j}F(z^0+t(s)\mathbf{b})|}{L^{j}(z^0)}$ by all $j\leq p$ two cases are possible:

1. In some interval of analyticity of the curve γ the function $w_{z^0}(t(s))$ identically equals simultaneously to some derivatives, that is (2.73) holds. It means that $w_{z^0}(t(s))\equiv\frac{|\partial_{\mathbf{b}}^{j}F(z^0+t(s)\mathbf{b})|}{L^{j}(z^0)}$ for some $j\leq p$. Clearly, the function $\partial_{\mathbf{b}}^{j}F(z^0+t(s)\mathbf{b})$ is analytic. Then $|\partial_{\mathbf{b}}^{j}F(z^0+t(s)\mathbf{b})|$ is continuously differentiable function on the interval of analyticity except points where this directional derivative equals zero $|\partial_{\mathbf{b}}^{j}F(z^0+t(s)\mathbf{b})|=0$. However, there are not the points, because in the opposite case $w_{z^0}(t(s))=0$. But it contradicts the construction of the curve γ.

2. In some interval of analyticity of the curve γ the function $w_{z^0}(t(s))$ equals simultaneously to some derivatives at a finite number of points s_k, that is (2.74) holds. Then the points s_k divide interval of analyticity onto a finite number of segments, in which of them $w_{z^0}(t(s))$ equals to one from the partial derivatives, i. e. $w_{z^0}(t(s))\equiv\frac{|\partial_{\mathbf{b}}^{j}F(z^0+t(s)\mathbf{b})|}{L^{j}(z^0)}$ for some $j\leq p$. As above, in each from these segments the functions $|\partial_{\mathbf{b}}^{j}F(z^0+t(s)\mathbf{b})|$, and $w_{z^0}(t(s))$ are continuously differentiable except the points s_k.

Therefore, the function $g_{z^0}(t(s))$ is continuous on $[0,1]$ and continuously differentiable except, possibly, finite set of points. Moreover, for complex-valued function of real variable the following inequality $\frac{d}{ds}|\varphi(s)| \leq \left|\frac{d}{ds}\varphi(s)\right|$ holds except the points s where $\varphi(s)=0$. Therefore in view of (2.72) we obtain

$$\frac{d}{ds}g_{z^0}(t(s)) \leq \max_{0\leq j\leq p}\{\frac{d}{ds}\frac{|\partial_{\mathbf{b}}^j F(z^0+t(s)\mathbf{b})|}{L^j(z^0))}\} \leq$$
$$\leq \max_{0\leq j\leq p}\{\frac{|\partial_{\mathbf{b}}^{j+1}F(z^0+t(s)\mathbf{b})|}{L^{j+1}(z^0)}|t'(s)|L(z^0)\}$$
$$\leq \max_{0\leq j\leq p+1}\left\{\frac{|\partial_{\mathbf{b}}^j F(z^0+t(s)\mathbf{b})|}{L^j(z^0)}\right\}|t'(s)|L(z^0) \leq P_5 g_{z^0}(t(s))|t'(s)|L(z^0),$$

where $P_5=\max\{1,P_4\}$. Integrating in variable s and using (2.12), we establish

$$\left|\ln\frac{g_{z^0}(t_2)}{g_{z^0}(t_1)}\right| = \left|\int_0^1 \frac{1}{g_{z^0}(t(s))}\frac{d}{ds}g_{z^0}(t(s))ds\right|$$
$$\leq P_5 L(z^0)\int_0^1 |t'(s)|ds \leq P_5 L(z^0)|\gamma| \leq 3|\mathbf{b}|r'P_5,$$

that is $g_{z^0}(t_2) \leq g_{z^0}(t_1)\exp\{3|\mathbf{b}|r'P_5\}$. Is is possible to choose t_2 such that

$$|F(z^0+t_2\mathbf{b})| = \max\{|F(z^0+t\mathbf{b})| : z^0+t\mathbf{b}\in C_2\}.$$

Hence,

$$\max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{3r'+1}{4L(z^0)}\right\} \leq |F(z^0+t_2\mathbf{b})|$$
$$\leq g_{z^0}(t_2) \leq g_{z^0}(t_1)\exp\{3|\mathbf{b}|r'P_5\}. \tag{2.75}$$

Since $z^0+t_1\mathbf{b} \in C_1 = \left\{z^0+t\mathbf{b} : |t| = \frac{r_1}{L(z^0)}\right\}$ and $r_1 \in [\frac{r'}{4},\frac{r'}{2}]$, for all $j \in \{1,2,\dots,p\}$, by Cauchy's inequality in variable t we obtain

$$\frac{r'^j}{(4L(z^0))^j}|\partial_{\mathbf{b}}^j F(z^0+t_1\mathbf{b})| \leq j!\max\left\{|F(z^0+t\mathbf{b})|: |t-t_1| = \frac{r'}{4L(z^0)}\right\}$$
$$\leq p!\max\left\{|F(z^0+t\mathbf{b})| : |t| = \frac{3r'}{4L(z^0)}\right\},$$

i.e.

$$g_{z^0}(t_1) \leq p!\max\{1,(4/r')^p\}\max\left\{|F(z^0+t\mathbf{b})|: |t| = \frac{3r'}{4L(z^0)}\right\} \tag{2.76}$$

From inequalities (2.75) and (2.76) it follows

$$\max\left\{|F(z^0+t\mathbf{b})|: |t|=\frac{3r'+1}{4L(z^0)}\right\}\le P_6\max\left\{|F(z^0+t\mathbf{b})|: |t|=\frac{3r'}{4L(z^0)}\right\},$$

where $P_6 = p!\max\{1,(4/r')^p\}\exp\{3|\mathbf{b}|r'P_5\}$. Therefore, by Proposition 2.7 the function F has bounded L-index in the direction $\mathbf{b}$. □

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Oleh SKASKIV, Andriy BANDURA, Tetyana SALO, and Andriy KURYLIAK

3 Wiman-Valiron type results for analytic functions in a bounded multiple circular domain

3.1 Introduction

The Wiman-Valiron theory is usually understood to encompass the whole set of results on the properties of analytic functions obtained using the properties of the maximal term or the central index of the series representing the function. In particular, it includes the analysis of entire functions of the form $f(z) = \sum\limits_{k=0}^{+\infty} a_k z^k$ using the maximal term $\mu_f(r) = \max\{|a_k|r^k \colon k \geq 0\}$ and the central index $\nu_f(r) = \max\{k \colon |a_k|r^k = \mu_f(r)\}$. The first results of the theory concerned generalizations of various asymptotic properties of algebraic polynomials to the class of entire functions of one complex variable. Subsequently, the main results of the theory were repeatedly transferred, in one form or another, to different classes of entire and analytic functions, both of one and several complex variables [5], and for random analytic functions [1, 17],

It is well-known that for every non-constant entire function f of one complex variable $z \in \mathbb{C}$ and arbitrary given $\varepsilon > 0$ there exist a set E of finite logarithmic measure (f.l.m.) $\int_{E\cap[1;+\infty)} d\ln r < +\infty$ and $r_0 \in (0, +\infty)$ such that the inequality (Wiman's inequality)

$$M_f(r) \leq \mu_f(r)(\ln \mu_f(r))^{1/2+\varepsilon} \tag{3.1}$$

holds for all $r \geq r_0$ ($r \notin E$), where $M_f(r) = \max\{|f(z)| \colon |z| = r\}$.

From (3.1) it immediately follows that (Borel's relation): $\ln M_f(r) \sim \ln \mu_f(r)$ $r \to +\infty$ $(r \notin E)$. The Wiman inequality [23, 40], the Borel relation, and their analogs in various classes of analytic functions, are the most fully studied among all the basic relations of the theory and its counterparts.

In article [21], there are proved the following statement.

Oleh SKASKIV, Ivan Franko National University of Lviv, 1 Universytetska street, 79000, Lviv, Ukraine, e-mail: oleh.skaskiv@lnu.edu.ua

Andriy BANDURA, Ivano-Frankivsk National Technical University of Oil and Gas, 15 Karpatska street, 76019, Ivano-Frankivsk, Ukraine, e-mail: andriy.bandura@nung.edu.ua

Tetyana SALO, Lviv Polytechnic National University, 12 Stepana stripery street, 79000, Lviv, Ukraine, e-mail: tetiana.m.salo@lpnu.ua

Andriy KURYLIAK, Ivan Franko National University of Lviv, 1 Universytetska street, 79000, Lviv, Ukraine, e-mail: andriy.kuryliak@lnu.edu.ua

Theorem 3.1 ([21]). *Let a non-decreasing function* $h\colon [0,R)\to(0,\infty)$ *be such that* $\int_{r_0}^{R} h(r)d\ln r=+\infty$ *for some* $r_0\in(0,R)$, $R\in(0,+\infty]$. *If* f *is an analytic function represented by power series of the form* $f(z)=\sum\limits_{k=0}^{+\infty}a_kz^k$ *with the radius of convergence* $R\in(0,+\infty]$, *then* $(\forall\ \delta>0)$ $(\exists E(\delta,f,h)=E\subset(0,R))$ $(\exists r_0\in(0,R))$ $(\forall\ r\in(r_0,R)\setminus E)$

$$M_f(r)\le h(r)\mu_f(r)\{\ln h(r)\ln(h(r)\mu_f(r))\}^{1/2+\delta}\ \textit{and}\ \int_{E\cap(r_0,R)}\frac{h(r)}{r}dr<+\infty.$$

The same can be said about Wiman's Theorem (and its analogs), which states [18, 19] that for every entire function f of one complex variable $z\in\mathbb{C}$ there exists a set E of f. l. m. $\int_{E\cap[1;+\infty)}d\ln r<+\infty$ such that the asymptotic relations

$$M_f(r)=(1+o(1))B_f(r)=-(1+o(1))C_f(r)\tag{3.2}$$

hold as $r\to+\infty$ ($r\notin E$), where

$$B_f(r)=\max\{\text{Re } f(z):\ |z|=r\},\quad C_f(r)=\min\{\text{Re } f(z):\ |z|=r\}.$$

The relations

$$f(ze^{\eta})\sim e^{\eta\nu_f(r)}f(z),\qquad \frac{f^{(k)}(z)}{f(z)}\sim\left(\frac{\nu_f(r)}{z}\right)^k,\tag{3.3}$$

are often called the main relations of this theory. They have been examined to a considerably lesser extent. The Wiman-Valiron Theorem [18, 19, 53] states that for every entire function f, for small $|\eta|$ in some sense and for any point z, $|z|=r$, such that $|f(z)|=M_f(r)$, we have relations (3.3) as $|z|=r\to+\infty$, $r\notin E$, and E is some set of f.l.m. But, even in the case of analytic functions in the unit disc, such statements are limited in a certain sense, because they cannot be true for the whole class of such functions.

3.2 Wiman-type theorems for analytic functions in the unit disc and in the strip: review of results

In the case of the class $\mathcal{A}_1$ of analytic functions f in the unit disc $\mathbb{D}=\{z\in\mathbb{C}\colon |z|<1\}$ we can use arguments similar to those in [49, p.266–270] and the scheme of proving Wiman's type theorem (see [19, 44]). Then it can be obtained that relations (3.2) hold for $r\to 1-0$ outside some set E of finite logarithmic measure on $(0,1)$, i.e.

$$\int_{E\cap(0,1)}\frac{dr}{1-r}<+\infty,$$

provided that

$$\varliminf_{r\to 1-0} \frac{\ln K_f(r)}{-\ln(1-r)} > 1,$$

where $K_f(r) = (\ln M_f(r))'_+$ is the right-hand derivative.

Actually, an analog of the last statement is true in the class $S(0;1)$ of functions F, analytic in the strip $\{z\colon 0 < \mathrm{Re}\ z < 1\}$, for which the condition

$$\varliminf_{x\to 1-0} \frac{\ln L_F(x)}{-\ln(1-x)} > 1 \tag{3.4}$$

holds, where $L_F(x) = (\ln M(x,F))'_+$ is the right-hand derivative of the convex function $\ln M(x,F)$ and $M(x,F) = \sup\{|F(x+iy)|\colon y \in \mathbb{R}\}$. For such functions relations

$$M(x,F) = (1+o(1))B(x,F) = -(1+o(1))C(x,F) \tag{3.5}$$

hold as $x \to 1-0$ outside some set E of finite logarithmic measure on the interval $(0,1)$, i.e. $\int_{E\cap(0,1)} dx/(1-x) < +\infty$, where

$$B(x,F) = \sup\{\mathrm{Re}\ f(z):\ \mathrm{Re}\ z = x\},\ C(x,F) = \inf\{\mathrm{Re}\ f(z):\ \mathrm{Re}\ z = x\}.$$

It was established [48] that condition (3.4) in the class $S(0,1)$ can be replaced by the condition

$$(1-x)L_F(x) \to +\infty \quad (x \to 1-0). \tag{3.6}$$

Actually, if (3.6) is satisfied for the function $F \in S(0,1)$, then relation (3.5) is true for $x \to 1-0$ outside some set E of zero linear density at the point $x = 1$, i.e.

$$\mathcal{D}^1 E := \varlimsup_{x\to 1-0} \frac{\mathrm{meas}\ (E\cap[x,1))}{1-x} = 0.$$

In this case, the condition (3.6) is not improvable in the sense that it cannot be replaced by any condition of the form

$$\varliminf_{x\to 1-0} (1-x)L_F(x) > \alpha_1 > 0.$$

To simplify the formulations, it is convenient to consider instead of the class $S(0;1)$ the class $S(-1;0)$ of analytic functions in the strip $\{z : -1 < \mathrm{Re}\ z < 0\}$, which are bounded in $\{z : -1 < \mathrm{Re}\ z < x\}$ for each $x \in (-1;0)$ and such that

$$|x|L_F(x) \to +\infty \quad (x \to -0).$$

The following theorem is proved in paper [48].

Theorem 3.2 ([48]). *Let Φ be a positive non-decreasing to $+\infty$ on $(-1,0)$ function such that*

$$|x|\Phi(x) \to +\infty \quad (x \to -0),$$

and the function $F \in S(-1;0)$ such that

$$L_F(x) \geq \Phi(x) \quad (x_0 \leq x < 0).$$

Then the relation (3.5) *holds for $x \to -0$ $(x \in (-1,0) \setminus E)$, and for every positive non-decreasing on $(-1,0)$ function h such that*

$$h(x) = o(\Phi(x)),$$
$$h\left(x + o\left(\frac{1}{h(x)}\right)\right) = O(h(x)) \quad (x \to -0),$$

the set E has zero asymptotic h-density at the point $x = 0$, i.e.

$$\mathcal{D}_h E := \overline{\lim_{x\to -0}}\, h(x) \operatorname{meas}(E \cap [x,0]) = 0.$$

Remark 3.1. *The above statement is meaningful under the condition $\overline{\lim\limits_{x\to-0}}\, h(x)|x| > 0$ (in the opposite case $\mathcal{D}_h E = 0$ for each set $E \subset [-1,0)$). Since $\mathcal{D}_h = 0 \iff \mathcal{D}_{h_1} = 0$ for $h_1(x) = ah(x)$ and any $a > 0$, here and further we assume that $h(x) \geq 1/|x|$ $(-1 < x < 0)$.*

In the interpretation, for analytic function f in $\mathbb{D}$ condition (3.6) will become the condition

$$(1-r)K_f(r) \to +\infty \quad (r \to 1-0), \tag{3.7}$$

and the exceptional set E in the relation (3.5) will have the description $\mathcal{D}^1 E = 0$.

Note [48] that for the function $f(z) = \frac{1}{1-z}$ we have

$$M_f(r) = \frac{1}{1-r}, \quad K_f(r) = \frac{1}{1-r}, \quad C_f(r) = \frac{1}{1+r}, \quad B_f(r) = \frac{1}{1-r},$$

therefore

$$(1-r)K_f(r) \equiv 1,$$
$$C_f(r) = O(1), \quad C_f(r) = o(B_f(r)) \quad (r \to 1-0),$$

i.e. condition (3.7) and relation (3.5) are not satisfied anywhere. It is clear that the same is true for each unique branch in $\mathbb{D}$ of the function $f(z) = \frac{1}{(1-z)^\alpha}$, $\alpha > 0$. From the above, it follows that condition (3.7) cannot be significantly improved.

3.3 Multidimensional notations and motivation

We use the following standard notations from [15, 29]. Let $\mathbb{C}^p$ be the p-dimensional ($p \geq 1$) complex vector space, and $\mathbb{R}^p$ be the p-dimensional ($p \geq 1$) real vector space, $\mathbb{Z}_+^p = (\mathbb{N} \cup \{0\})^p$, $z^n = z_1^{n_1} \cdots z_p^{n_p}$, $\|n\| = n_1 + \ldots + n_p$, $|z|^2 = |z_1|^2 + \ldots + |z_p|^2$, for $n = (n_1, \ldots, n_p) \in \mathbb{Z}_+^p$ and $z = (z_1, \ldots, z_p) \in \mathbb{C}^p$, $\mathbb{R}_+ = [0, +\infty)$.

We call a domain $\mathbf{G} \subset \mathbb{C}^p$ a complete Reinhardt domain if:

a) $z = (z_1, \ldots, z_p) \in \mathbf{G} \Longrightarrow (\forall R = (R_1, \ldots, R_p) \in [0,1]^p)\colon\ Rz = (R_1 z_1, \ldots, R_p z_p) \in \mathbf{G}$ (*a complete domain*);
b) $(z_1, \ldots, z_p) \in \mathbf{G} \Longrightarrow (\forall (\theta_1, \ldots, \theta_p) \in \mathbb{R}^p) : (z_1 e^{i\theta_1}, \ldots, z_p e^{i\theta_p}) \in \mathbf{G}$ (*a multiple circular domain*).

The Reinhardt domain $\mathbf{G}$ is called logarithmically-convex, if the image of the set $\mathbf{G}^* = \{z \in \mathbf{G}\colon z_1 \cdot \ldots \cdot z_p \neq 0\}$ under the mapping $\mathrm{Ln}\colon z \to \mathrm{Ln}(z) = (\ln|z_1|, \ldots, \ln|z_p|)$ is a convex set in the space $\mathbb{R}^p$.

We denote by $\mathcal{A}^p(\mathbf{G})$, $p \in \mathbb{N}$, the class of an analytic functions f in $\mathbf{G} \subset \mathbb{C}^p$, represented by power series of the form

$$f(z) = f(z_1, \ldots, z_p) = \sum_{\|n\|=0}^{+\infty} a_n z^n, \quad z = (z_1, \ldots, z_p), \tag{3.8}$$

with the domain of convergence $\mathbf{G}$, where $z^n = z_1^{n_1} \ldots z_p^{n_p}$, $n = (n_1, \ldots, n_p) \in \mathbb{Z}_+^p$, $\|n\| = \sum_{j=1}^{p} n_j$; $\mathcal{A}^p := \mathcal{A}^p(\mathbb{C}^p)$ is the class of entire functions of several variables (i.e., analytic functions in $\mathbb{C}^p$). It is well known, that from the one hand, every analytic function f in the complete Reinhardt domain $\mathbf{G}$ with centre $z = 0$ can be represented in $\mathbf{G}$ by the series of form (3.8). On the other hand, the domain of convergence of the each series of form (3.8) is the logarithmically-convex complete Reinhardt domain with center at the point $z = 0$. Everywhere we will assume that the domain $\mathbf{G}$ is bounded.

Denote $\mathbf{G}_r = r \cdot \mathbf{G}$, $0 \leq r < 1$. We consider the *exhaustion* of the domain $\mathbf{G}$ by a system $(\mathbf{G}_r)_{r \in [0,1)}$. Then i) $\bigcup_{r \in [0,1)} \mathbf{G}_r = \mathbf{G}$; ii) $(\forall r_1 < r_2 < 1)\colon \mathbf{G}_{r_1} \subset \mathbf{G}_{r_2}$;
iii) $(z_1, \ldots, z_p) \in \mathbf{G}_1 \Longleftrightarrow (\forall\, r \in (0,1)) : (rz_1, \ldots, rz_p) \in \mathbf{G}_r$;
iv) $(z_1, \ldots z_p) \in \mathbf{G}_r \Longrightarrow (\forall\, \theta = (\theta_1, \ldots, \theta_p) \in \mathbb{R}^p)\colon (z_1 e^{i\theta_1}, \ldots, z_p e^{i\theta_p}) \in \mathbf{G}_r$.

Remark, that the following domains are the complete Reinhardt domains:

i) $C_{r,a} = \{(z_1, \ldots, z_p) \in \mathbb{C}^p :\ |z_j| < a_j r,\ 1 \leq j \leq p\}$;

ii) $\mathbb{B}_{r,a} = \{(z_1, \ldots, z_p) \in \mathbb{C}^p :\ a_1|z_1|^2 + \ldots + a_p|z_p|^2 < r^2\}$;

iii) $\Pi_{r,a} = \{(z_1, \ldots, z_p) \in \mathbb{C}^p :\ a_1|z_1| + \ldots + a_p|z_p| < r\}$;

iv) $\mathbf{G}_r = \{(z_1, \dots, z_p) \in \mathbb{C}^p : |z_1|^{a_1} \cdot \ldots \cdot |z_2|^{a_p} < r^{a_1+\dots+a_p}\}$;

where $a = (a_1, \dots, a_p)$, $a_j > 0$ $(1 \le j \le p)$, $r \in (0, 1)$.

In one complex variable ($p = 1$), a logarithmically-convex Reinhardt domain is a disc.

To prove Theorem 3.2 in [48], the following counterpart of the Borel-Nevanlinna lemma was used.

Lemma 3.1 ([48]). *Let $u(x)$ be a positive non-decreasing on $[-1, 0)$ function, and let Φ and h be two functions on $[-1, 0)$ such that as in Theorem 3.2. If*

$$u(x) \ge \Phi(x) \quad (x_0 \le x < 0),$$

then there exist a function $\delta(u) \nearrow +\infty$ $(u \uparrow -0)$ and the set E has zero asymptotic h-density at the point $x = 0$, i.e. $\mathcal{D}_h(E) = 0$, such that the inequality

$$|u(x + \tau) - u(x)| < \frac{u(x)}{\delta(x)}$$

holds for all $x \in (-1, 0) \setminus E$ and for every $\tau \in \mathbb{R}$, $|\tau| \le \psi(x) \stackrel{\text{def}}{=} \delta(x)/u(x)$.

Various analogs of Wiman's Theorem for an entire Dirichlet series with a positive monotonically increasing to infinity sequence of exponents can be found in [35, 36, 44], and for absolutely convergent in a half-plane Dirichlet series in [31, 48], for graph [26].

There are also some known analogs of Wiman's inequality for entire multiple Dirichlet series with arbitrary complex exponents [20], for series in systems of functions [38].

Other interesting property in Wiman-Valiron's theory concerns the asymptotic behavior of entire functions in certain discs around points of maximum modulus [14]. There are many papers estimating the size of these discs from above and below as for entire functions [4], so for subharmonic functions [12] and meromorphic mappings [24, 50]. Extension of the Wiman-Valiron-type considerations to complex differences gives difference variants of Wiman-Valiron theory (see key results in [25]). This theory was also partially developed for fractional derivatives [7].

The distance between a maximum modulus point and the zero set of entire function within Wiman-Valiron's theory were asymptotically estimated in [9, 28]. P. Fenton [10, 13] developed another approach to deduce main relations of Wiman-Valiron's theory in the multidimensional case.

Note that W.K. Hayman [18, 19] used Wiman's Theorem to prove for the harmonic functions $u(x, y)$ in the whole complex plane one counterpart of Wiman's Theorem.

The chapter is devoted to establishment of Wiman's Theorem analogs for an analytic functions of several complex variables in bounded multiple circular domains. The statements describe the asymptotic behavior of such analytic functions $F(z)$ of several complex variables $z = (z_1, \dots, z_p)$ and its directional derivatives at the neighborhood of a point w, where the value $F(w)$ is close to the supremum of its modulus on the boundaries of multiple circular domain. There are known similar theorems for analytic functions in polylinear domains. For analytic functions in bounded multiple circular domains such statements are unknown. For entire functions of several complex variables similar results we found in papers ([2, 10, 11, 16, 27, 32, 33, 45, 47, 49]. The number of publications concerning generalizations to the multidimensional case of the Borel relation theorems [6] and Wiman inequality is much larger. Their list is not limited to the publications listed at the end of this text ([20–22, 37–39, 41–43, 51, 52]).

3.4 Asymptotic relations in polylinear domains.

Let us now introduce some additional notation and concepts.

For $a = (a_1, \dots, a_p) \in \mathbb{R}^p$, $b = (b_1, \dots, b_p) \in \mathbb{R}^p$ we write $a < b$, respectively $a \le b$, if $(\forall j, 1 \le j \le p)\colon a_j < b_j$, and $(\forall j, 1 \le j \le p)\colon a_j \le b_j$, respectively. For $z = (z_1, \dots, z_p) \in \mathbb{C}^p$, $w = (w_1, \dots, w_p) \in \mathbb{C}^p$, we denote $\langle z, w\rangle = z_1w_1 + \ldots + z_pw_p$, $\|z\| = z_1 + \ldots + z_p$, $\mathrm{Re}\, z = (\mathrm{Re}\, z_1, \dots, \mathrm{Re}\, z_p)$, and for $R = (r_1, \dots, r_p) \in \mathbb{R}^p$ we denote $\Pi_R = \{z \in \mathbb{C}^p \colon \mathrm{Re}\, z < R\}$.

A domain $\mathbf{D} \subset \mathbb{C}^p$ is called *a polylinear domain* in $\mathbb{C}^p$, $p \ge 1$ ([49, p.294]), if it is satisfied the following condition: if $z = (z_1, \dots, z_p) \in \mathbf{D}$, then for every $y = (y_1, \dots, y_p) \in \mathbb{R}^p$ we get $z + iy = (z_1 + iy_1, \dots, z_p + iy_p) \in \mathbf{D}$. Let $A = (A_1, \dots, A_p) \in \mathbb{R}^p_+$ be a fixed vector and $\mathbf{D}$ be a polylinear domain. Denote $\mathbf{D}(r, A) = \mathbf{D} + rA$, $r \in \mathbb{R}$ and by $D := \mathrm{Re}\,\mathbf{D}$ the image of the domain $\mathbf{D}$ at the mapping $\mathrm{Re}\ z\colon \mathbf{D} \to \mathbb{R}^p$. The domain $\mathbf{D}(r, A)$ also is a polylinear domain and it is clear $\mathbf{D}(0, A) = \mathbf{D}$.

We assume that a polylinear domain $\mathbf{D}$ is such that the system of domains

$$\{\mathbf{D}(r, A)\}_{r\in(-1,0)}$$

is the geometric exhaustion of $\mathbf{D}$, that is

a) $\bigcup\limits_{r\in(-1,0)} \mathbf{D}(r, A) = \mathbf{D}$; b) $\mathbf{D}(r_1, A) \subset \mathbf{D}(r_2, A)$ $(-1 \le r_1 < r_2 < 0)$.

We say that such a polylinear domain $\mathbf{D}$ belongs to the class Σ (in [49, p.299] see the definition of the class Σ_π), if there exist $R_*, R^* \in \mathbb{R}^p$ such that $\Pi_{R_*} \subset \mathbf{G} \subset \Pi_{R^*}$. It is easy to see that

$$\bigcup_{r\in(-1,0)} D(r, A) = D,\ D(0, A) = D \text{ and } D(r_1, A) \subset D(r_2, A)\ (-1 \le r_1 < r_2 < 0),$$

and also $\Pi_{\mathbf{0}} \in \Sigma$, where $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^p$.

Let us consider the class $\mathcal{A}^p(\mathbf{D})$ of an analytic functions in $\mathbf{D}^p$, which are bounded in an arbitrary domain Π_R, $R = (R_1, \dots, R_p) < \mathbf{0}$. For a function $F \in \mathcal{A}^p(\mathbf{D})$ and $x < \mathbf{0}$, $x \in \mathbb{R}^p$ it is obvious that

$$M(x, F) := \sup\{|F(x + iy)| : y \in \mathbb{R}^p\} < +\infty.$$

Moreover, if an analytic function F is bounded in a polylinear domain $\mathbf{D}$, then for every x such that $\{z \in \mathbb{C}^p\colon \ \mathrm{Re}\ z_1 = x_1, \dots, \mathrm{Re}\ z_p = x_p\} \subset \mathbf{D}$ one has $M(x, F) < +\infty$.

For a function $F \in \mathcal{A}^p(\mathbf{D})$ and $r \in [-1, 0)$ we denote

$$S_F(r, D) := \sup\{|F(z)| : \ z \in \mathbf{D}(r, A)\}.$$

Repeating the reasoning from the proof of [49, Theorem 1.4.26], we obtain: if $\mathbf{D} \in \Sigma$, and the function F such that $M(x, F) < +\infty$, and in each polylinear domain Π_R the function F is bounded, then $\ln S_F(r, D)$ is a convex function of $r \in [-1, 0)$.

We note that $S_F(r, D) < +\infty$ $(\forall r \in [-1, 0))$, in the case, when $F \in \mathcal{A}^p(\mathbf{D})$, and the exhaustion is such that $\Pi_{R_*} \subset \mathbf{D}(r, A) \subset \Pi_{R^*}$ for some $R_* < R^*$.

Since $\ln S_F(r, D)$ is a convex function, it has a right-hand derivative everywhere

$$L_F(r, D) := \big(\ln S_F(r, D)\big)'_+,$$

which is a non-decreasing function. It is obvious that for every $r_0 \in (-1, 0)$ there exists

$$L_+(r_0) := \lim_{r \to r_0 + 0} L_F(r, D) \geq \lim_{r \to r_0 - 0} L_F(r, D).$$

Therefore, without loss of the generality, we can assume that the function $L_F(r, D)$ is right semi-continuous. Then one has $L_F(r_0, D) = L_+(r_0)$ at every point $r_0 \in (-1, 0)$.

The main results of this chapter will concern analytic functions $f\colon \mathbf{G} \to \mathbb{C}$ in bounded complete multiple circular domains $\mathbf{G} \subset \mathbb{C}^p$. For convenience, we will move on to the class of analytic functions, which, in particular, contains all analytic functions of the form $F(z) = f(e^{z_1}, \dots, e^{z_p})$, $z = (z_1, \dots, z_p)$.

We prove the following theorem (see also [8]).

Theorem 3.3. *Let Φ be a positive non-decreasing to $+\infty$ on $(-1, 0)$ function such that*

$$|r|\Phi(r) \to +\infty \quad (r \to -0),$$

and the function $F \in \mathcal{A}^p(\mathbf{D})$ such that

$$L_F(r, D) \geq \Phi(r) \quad (r_0 \leq r < 0). \tag{3.9}$$

Then relations

$$S_F(r,D)=(1+o(1))B_F(r,D)=-(1+o(1))C_F(r,D) \tag{3.10}$$

hold as $r\to-0$ $(r\in(-1,0)\setminus E)$, *where*

$$B_F(r,D)=\sup\{\mathrm{Re}\ f(z)\colon\ z\in\mathbf{D}(r,A)\},\ C_F(r,D)=\inf\{\mathrm{Re}\ f(z)\colon\ z\in\mathbf{D}(r,A)\}$$

for every positive non-decreasing on $(-1,0)$ *function* h *such that*

$$h(x)=o(\Phi(x)),$$
$$h\left(x+o\left(\frac{1}{h(x)}\right)\right)=O(h(x))\quad(x\to-0),$$

the set E *has zero asymptotic* h*-density at the point* $x=0$, *i.e.*

$$\mathcal{D}_hE:=\overline{\lim_{x\to-0}}\,h(x)\,\mathrm{meas}\,(E\cap[x,0])=0.$$

Here meas E_1 *is the Lebesgue measure of the measurable set* E_1 *on the real line.*

At first, we prove the following statement.

Theorem 3.4. *Let the functions* Φ *and* h *be as in Theorem 3.3. If for the function* $F\in\mathcal{A}^p(\mathbf{D})$ *condition* (3.9) *is satisfied, and* $\varepsilon(r)\downarrow 0$ $(r\to-0)$ *is an arbitrary function, then there exists a function* $\delta(r)\nearrow+\infty$ $(r\to-0)$ *and a set* $E\subset(-1,0)$, $\mathcal{D}_hE=0$ *such that for all* $w\in\partial\mathbf{D}(r,A)$ *chosen from*

$$|F(w)|\ge\frac{S_F(r,D)}{1+\varepsilon(r)},$$

and for all $s\in\mathbb{C}$, $|s|\le r_0<\frac{\delta(r)}{L_F(r,D)c(r)}$, $c(r)=1+e(1+\varepsilon(r))$, *we get*

$$F(w+sA)=F(w)\exp\left\{(L_F(r,D)+\Delta_1)s+\sum_{n=2}^{+\infty}\Delta_ns^n\right\},\ \textit{as}\ r\to-0\ (r\notin E)$$

where

$$|\Delta_n|\le 2r_0^{-n}\ln\left(1+\frac{r_0c(r)L_F(r,D)}{\delta(r)}\right)\quad(n\ge0).$$

Proof of Theorem 3.4. We will use Lemma 3.1 with $u(r)=L_F(r,D)$. For all $r\in(-1,0)\setminus E$, $\mathcal{D}_hE=0$ and $\tau\in\mathbb{R}$, $|\tau|\le\psi(r)=\delta(r)/L_F(r,D)$, we have

$$|L_F(r+\tau,D)-L_F(r,D)|\le\frac{1}{\psi(r)}. \tag{3.11}$$

Let $\varepsilon(r)\downarrow 0$ $(r\to-0)$, and the point $z_0\in\partial\mathbf{D}_r$ be such that

$$|F(z_0)|\ge S_F(r,F)/(1+\varepsilon(r)).$$

From the monotonicity of $L_F(r,D)$ it follows ([49, c.147]) that for all $r \in (-1,0)$, $h \in \mathbb{R}$, $|r|-1 < h < |r|$,

$$\ln S_F(r+h,D) - \ln S_F(r,D) \le hL_F(r+h,D).$$

Hence and from (3.11) for $r \in (-1,0) \setminus E$, $h \in \mathbb{R}$, $|h| \le \psi(r)$, we get

$$\ln S_F(r+h,D) - \ln S_F(r,D) - hL_F(r,D) \le h(L_F(r+h,D) - L_F(r,D)) =$$

$$= |h||L_F(r+h,D) - L_F(r,D)| \le 1.$$

Therefore, for $r \notin E$, $\eta \in \mathbb{C}$, $|\operatorname{Re}\ \eta| \le \psi(r)$

$$\left| \frac{F(z_0 + A\eta)}{F(z_0)} e^{-\eta L_F(r,D)} \right| \le$$

$$\le (1+\varepsilon(r)) \exp\{\ln S_F(r + \operatorname{Re}\ \eta, F) - \ln S_F(r,D) - \operatorname{Re}\ \eta\, L_F(r,D)\} \le$$

$$\le e(1+\varepsilon(r)).$$

By the Schwartz Lemma ([34, p.103]), applied in the disc $\{\eta : |\eta| \le \psi(r)\}$ to the function

$$g(\eta) := \frac{F(z_0 + A\eta)}{F(z_0)} e^{-\eta L_F(r,D)} - 1,$$

for all $\eta \in \mathbb{C}$, $|\eta| < \psi(r)$ we have

$$|g(\eta)| \le (1 + e(1+\varepsilon(r))) \frac{|\eta|}{\psi(r)} = c(r) \frac{|\eta|}{\psi(r)}. \tag{3.12}$$

Since $|g(\eta)| < 1$ for $|\eta| < \frac{\psi(r)}{c(r)}$, $r \in (-1,0) \setminus E$, with (3.12) at $|\eta| < \frac{\psi(r)}{c(r)}$, $r \in (-1,0) \setminus E$ we get

$$\left| \frac{F(z_0 + A\eta)}{F(z_0)} e^{-\eta L_F(r,D)} \right| \ge 1 - |g(\eta)| > 0.$$

Therefore $F(z_0 + A\eta) \ne 0$ for $|\eta| < \frac{\psi(r)}{c(r)}$ and, the function

$$G(\eta) = \int_{[0,\eta]} \frac{F'(z_0 + A\tau)}{F(z_0 + A\tau)} d\tau - \eta L_F(r,D) \tag{3.13}$$

is analytic in the disc $\{\eta : |\eta| < \psi(r)/c(r)\}$, $G(0) = 0$.

Let

$$G(\eta) = \sum_{j=1}^{+\infty} \Delta_j \eta^j.$$

In view of condition $|\eta| \le q < \psi(r)/c(r)$ from inequality (3.12) we have

$$\mathrm{Re}\ G(\eta) = \ln|1+g(\eta)| \le \ln(1+|g(\eta)|) \le \ln\left(1+\frac{qc(r)}{\psi(r)}\right). \tag{3.14}$$

In the solution of Problem **236** (see, [30, Part III, Ch. 5, § 2, **236**, p.355]) we find the formulation of the Cauchy Modified Inequality.

Lemma 3.2. *Let* $f(z) = \sum\limits_{n=0}^{+\infty} f_n z^n$ *be an analytic function in the disc* $\mathbb{D}_R = \{z\colon |z| < R\}$, $R > 0$. *If* $\mathrm{Re}\, f(z) < M$ *for all* $z \in \mathbb{D}_R$, *then* $|f_n|R^n \le 2(M - \mathrm{Re}\, f_0)$ *for all* $z \in \mathbb{D}_R$.

Therefore, by the modified Cauchy inequality (Lemma 3.2), in the disc $\{\eta : |\eta| \le q\}$ we have

$$|\Delta_j| \le 2q^{-j} \max\{\mathrm{Re}\ G(\eta) : |\eta| = q\} \le 2q^{-j} \ln\left(1+\frac{qc(r)}{\psi(r)}\right).$$

It remains to note that for $|\eta| < \psi(r)/c(r)$ from (3.13) it follows that

$$F(z_0 + A\eta) = F(z_0)\exp\left\{\eta L_F(r,D) + \sum_{j=1}^{+\infty} \Delta_j \eta^j\right\}.$$

Theorem 3.4 is proved. □

Proof of Theorem 3.3. To prove our theorem we will use the scheme from [44, 45] (see also [18, 19, 35]). Let us choose $\eta = i(\pi - \arg F(w))/L_F(r,D)$, $w \in \partial\mathbf{D}(r,A)$ such that

$$|F(w)| = (1+o(1))S_F(r,D) \quad (r \to +\infty)$$

and make sure that the application of Theorem 3.4 will ultimately lead us to the formulated statement. Indeed, it is clear that $|\eta| \le \delta(r)/L_F(r,D)$ and

$$|\omega(\eta)| \le c(r)|\pi - \arg F(w)|/\delta(r) \to 0$$

as $r \to +\infty$. Thus, by Corollary 3.3,

$$F(w + A\eta) = (1+o(1))|F(w)|e^{i\pi} = -(1+o(1))|F(w)|.$$

Hence, Re $F(w + A\eta) = -(1+o(1))S_F(r,D)$. So,

$$C_F(r,D) \le \mathrm{Re}\ F(w + A\eta) = -(1+o(1))S_F(r,D).$$

But, $|C_F(r,D)| \le S_F(r,D)$, therefore

$$C_F(r,D) = -(1+o(1))S_F(r,D) \quad (r \to +\infty). \tag{3.15}$$

Let us choose now, $F^*(w) := -iF(w)$. Then, on the one hand, Re $F = -$Im F^*. Hence, $B_F(r, D) = -C_{F^*}(r, D)$. On the other hand,

$$S_F(r, D) = S_{F^*}(r, D),\ |F(w)| = |F^*(w)|,\ F^* \in \mathcal{A}^p(\mathbf{D}).$$

Therefore, applying relation (3.15) to the function F^*, finally we obtain that

$$B_F(r, D) = -C_{F^*}(r, D) = (1+o(1))S_{F^*}(r, D) = (1+o(1))S_F(r, D) \quad (r \to +\infty).$$

□

3.5 Asymptotic relation for higher-order directional derivative

In this subsection, we consider the statement for functions F in the class $\mathcal{A}^p(\mathbf{D})$ about the relations of the form

$$F_A^{(k)}(w) = (1 + o(1))L_F^k(r, D)F(w) \tag{3.16}$$

as $r \to +\infty$ outside some exceptional sets. Here our proof again makes substantial use of Theorem 3.4, which we established in the previous section.

For $A \in \mathbb{R}^p$ and a function $F \in \mathcal{A}^p(\mathbf{D})$ by $F_A^{'}(w)$ we denote the derivative of F in the direction A at the point $w \in \mathbf{D}$; $F_A^{(k)}(w) = (F_A^{(k-1)}(w))_A^{'}$ denotes the k-th derivative in the direction A at the point $w \in \mathbf{D}$, $F_A^{'}(w + A\tau)$ is the derivative of the function F in the direction A at the point $w + A\tau$.

Theorem 3.5. *Let the functions Φ and h be as in Theorem 3.3, for the function $F \in \mathcal{A}^p(\mathbf{D})$ condition* (3.9) *is satisfied, and $\varepsilon(r) \downarrow 0$ $(r \to -0)$ is an arbitrary function. Then there exists a set $E \subset (-1, 0)$, $\mathcal{D}_h E = 0$ such that for each $k \in \mathbb{N}$ asymptotic relation* (3.16) *holds as $r \to -0$ $(r \notin E)$ for every point $w \in \partial G(r, A)$ provided the inequality $|F(w)| \geq S_F(r, D)/(1 + \varepsilon(r))$.*

Proof of Theorem 3.5. The proof of Theorem 3.5 repeats the scheme of the proof of similar theorem from [2]. We adapt some places in our proof with additions and some necessary clarifications, and rewrite other places from the proof almost verbatim, so that in general we can obtain a completely correct proof. Let $w \in \partial G(r, A)$ be a given point such that condition $|F(w)| \geq S_F(r, D)/(1 + \varepsilon(r))$ is satisfied, and consider the function

$$\omega(\eta) = \frac{F(w + A\eta)}{F(w)} e^{-\eta L_F(r,D)} - 1$$

of the variable $\eta \in \mathbb{C}$, $|\eta| \leq \psi(r) = \delta(r)/L_F(r, D)$, for fixed $r \geq r_0$, as above.

For all η, $|\eta| < \psi(r)/c(r)$, $r \geq r_0$, by inequalities (3.12) and (3.14) one has

$$|\omega(\eta)| \leq \ln(1 + |g(\eta)|) \leq \ln\left(1 + \frac{|\eta|c(r)}{\psi(r)}\right) \leq c(r)|\eta|/\psi(r) = |\eta|c(r)\frac{L_F(r, D)}{\delta(r)}.$$

Hence, we have

$$|\omega(\eta)| \leq |\eta| c(r) L_F(r,D)/\delta(r) < 1. \tag{3.17}$$

Thus,

$$\Big|\frac{F(w+A\eta)}{F(w)} e^{-\eta L_F(r,D)}\Big| = |1+\omega(\eta)| \geq 1 - |\omega(\eta)| > 0$$

for all η, $|\eta| < \psi(r)/c(r)$, $r \geq r_0$. So, for the fixed A and a given point $w \in \partial G(r,A)$ such that $|F(w)| \geq S_F(r,D)/(1+\varepsilon(r))$, we get $F(w+A\eta) \neq 0$ for all η, $|\eta| < \psi(r)/c(r)$, $r \geq r_0$. Therefore, the function $F_A^{'}(w+A\tau)/F(w+A\tau)$ is an analytic function of η, $|\eta| < \psi(r)/c(r)$, $r \geq r_0$. Hence, the function

$$f(\eta) = \int_0^\eta \frac{F_A^{'}(w+A\tau)}{F(w+A\tau)} d\tau - \eta L_F(r,D), \quad f(0) = 0, \tag{3.18}$$

is an analytic function in the disc $\mathbb{D}_R = \{\eta \in \mathbb{C}\colon |\eta| < R\}$, $R = \psi(r)/c(r)$. Let the function f have a Taylor series expansion of the form $f(z) = \sum\limits_{k=0}^{+\infty} f_k z^k$ in the disk $\mathbb{D}_R$.

Clearly,

$$f_1 = f^{'}(0) = F_A^{'}(w)/F(w) - L_F(r,D),$$

and, by inequality (3.17), one has

$$\begin{aligned} \text{Re } f(\eta) &= \ln\Big|\frac{F(w+A\eta)}{F(w)} e^{-\eta L_F(r,D)}\Big| = \ln|1+\omega(\eta)| \leq \\ &\leq \ln(1+|\omega(\eta)|) \leq (1+c(r)|\eta|/\psi(r)) \leq \ln\left(1+\frac{qc(r)}{\psi(r)}\right) \end{aligned} \tag{3.19}$$

for all η, $|\eta| \leq q < \psi(r)/c(r)$.

Let us apply Lemma 3.2 to the function f in the disc $\mathbb{D}_q = \{\eta\colon |\eta| \leq q\}$, $q < \psi(r)/c(r)$. Using inequlity (3.19), we have

$$\begin{aligned} \left|\frac{F'_A(w)}{F(w)} - L_F(r,D)\right| &= |f'(0)| = |f_1| \leq 2\max\{\text{Re } f(\eta)\colon |\eta| = q\} \leq \\ &\leq 2\ln\left(1+\frac{qc(r)}{\psi(r)}\right) \leq \frac{2c(r)}{\psi(r)}. \end{aligned}$$

Hence, for all $w \in \partial G_F(r,D)$ such that $|F(w)| \geq S_F(r,D)(1+\varepsilon(r))^{-1}$ one has

$$\left|\frac{F'_A(w)}{F(w)} \frac{1}{L_F(r,D)} - 1\right| \leq \frac{2c(r)}{L_F(r,D)\psi(r)} = \frac{c(r)}{\delta(r)} = o(1) \tag{3.20}$$

as $r \to +\infty$.

Let us apply now again, as above, Lemma 3.2 in the disc $\mathbb{D}_q = \{\eta\colon |\eta| < q\}$, $q < \psi(r)/c(r)$ to the function f, which is defined in (3.18). For fixed $k \in \mathbb{N}$, we have

$$\frac{1}{k!}\left|f^{(k)}(0)\right| = |f_k| \le \frac{2}{q^k}\max\{\operatorname{Re} f(\eta)\colon |\eta| = q\} \le$$
$$\le 2q^{-k}\ln(1 + qc(r)/\psi(r)) \le 2q^{-k+1}\frac{c(r)}{\psi(r)} \tag{3.21}$$

for $|\eta| \le q < \psi(r)/c(r)$. Let us denote $F_0(\eta) := F(w+\eta A)/F(w)$. The function $F_0(\eta)$ is an analytic function in the variable η, $|\eta| \le q$, $F_0(0) = 1$. So, for $|\eta| \le q$ we get

$$F(w+\eta A) = F(w)\exp\{f(\eta) + \eta L_F(r,D)\} =$$
$$= F(w)\exp\left\{\frac{F'_A(w)}{F(w)}\eta + \sum_{k=2}^{+\infty}\frac{f^{(k)}(0)}{k!}\eta^k\right\} = F(w)F_0(\eta). \tag{3.22}$$

Suggest that the function F_0 have the Taylor series expansion of the form

$$F_0(\eta) = 1 + \sum_{k=1}^{+\infty} F_k\eta^k, \quad |\eta| \le q. \tag{3.23}$$

in the disk $\mathbb{D}_q$. It is clear, that for the function $f_1(\eta)\colon = f(\eta) + \eta L_F(r,D)$ we have $f_1^{(k)}(\eta) = f^{(k)}(\eta)$ $(k \ge 2)$, so

$$F_0(\eta) = 1 + \sum_{s=1}^{+\infty}\frac{1}{s!}\left(\frac{F'_A(z)}{F(z)}\eta + \sum_{k=2}^{+\infty}\frac{f^{(k)}(0)}{k!}\eta^k\right)^s = 1 + \sum_{s=1}^{+\infty}\frac{1}{s!}\left(\sum_{k=1}^{+\infty}\frac{f_1^{(k)}(0)}{k!}\eta^k\right)^s.$$

Therefore, the Taylor coefficients in (3.23) are formed by sums of the following form

$$F_k = \sum_{\|\alpha\|_0 = k} B_\alpha\,(f'_1(0))^{\alpha_1}\cdot\left(\frac{f''_1(0)}{2!}\right)^{\alpha_2}\cdot\ldots\cdot\left(\frac{f_1^{(k)}(0)}{k!}\right)^{\alpha_k}, \tag{3.24}$$

where $\#\{\alpha\colon \|\alpha\|_0 = k\} < +\infty$, $\|\alpha\|_0 := \sum_{j=1}^k j\alpha_j$ for a the multi-index $\alpha = (\alpha_1,\ldots,\alpha_k) \in \mathbb{Z}_+^k$, $B_\alpha \in \mathbb{R}$. That is finite number of the summands in (3.24).

From the equalities (3.22)–(3.23) and the Taylor expansion

$$F(w + A\eta) = \sum\nolimits_{k=0}^{+\infty}\frac{F_A^{(k)}(w)}{k!}\eta^k$$

we obtain that $F_A^{(k)}(w) = F(w)k!F_k$ $(k \ge 1)$. Thus,

$$\frac{F_A^{(k)}(w)}{F(w)}\frac{1}{L_F^k(r,D)} - 1 = \left(\frac{F'_A(w)}{F(w)}\frac{1}{L_F(r,D)}\right)^k - 1 + \sum_{\substack{\|\alpha\|_0 = k\\ \alpha_1 < k}} B_\alpha \prod_{j=1}^k\left(\frac{f_1^{(j)}(0)}{j!}\right)^{\alpha_j}. \tag{3.25}$$

Since $c(r) = O(1)$ and $\delta(r) \nearrow +\infty$ $(r \to +\infty)$, $c(r)/\delta(r) = o(1)$ $(r \to +\infty)$. Let us recall that $t^k - 1 = (t-1) \cdot \sum_{j=0}^{k-1} t^j$. Therefore, from the inequality (3.20) we have consistently,

$$\left| \left(\frac{F'(w)}{F(w)} \frac{1}{L_F(r,D)} \right)^k - 1 \right| = \left| \frac{F'(w)}{F(w)} \frac{1}{L_F(r,D)} - 1 \right| \sum_{j=0}^{k-1} \left| \frac{F'(w)}{F(w)} \frac{1}{L_F(r,D)} \right|^j \le$$

$$\le \frac{c(r)}{\delta(r)} \sum_{j=0}^{k-1} \left(1 + \frac{c(r)}{\delta(r)}\right)^j = \left(\left(1 + \frac{c(r)}{\delta(r)}\right) - 1 \right) \sum_{j=0}^{k-1} \left(1 + \frac{c(r)}{\delta(r)}\right)^j =$$

$$= \left(1 + \frac{c(r)}{\delta(r)}\right)^k - 1 = o(1) \tag{3.26}$$

as $r \to +\infty$. Let us take it now

$$q = \frac{1}{2} \frac{\psi(r)}{c(r)} = \frac{1}{2} \frac{\delta(r)}{c(r) L_F(r,D)}.$$

Using inequality (3.21), we get

$$\prod_{j=1}^{k} \left(\frac{f_1^{(j)}(0)}{j!} \right)^{\alpha_j} \le \prod_{j=1}^{k} \left(2q^{-j+1} \frac{c(r)}{\psi(r)} \right)^{\alpha_j} = q^{-\|\alpha\|_0} \left(\frac{2qc(r)}{\psi(r)} \right)^{\|\alpha\|} = q^{-\|\alpha\|_0}.$$

Hence,

$$\left| \sum_{\substack{\|\alpha\|_0 = k, \\ \alpha_1 < k}} B_\alpha \prod_{j=1}^{k} \left(\frac{f_1^{(j)}(0)}{j!} \right)^{\alpha_j} \right| \le \sum_{\substack{\|\alpha\|_0 = k \\ \alpha_1 < k}} |B_\alpha| q^{-k}. \tag{3.27}$$

But, $\delta(r)/L_F(r,D) \to 0$ as $r \to +\infty$ and $c(r) > 1 + e$. Therefore, applying relation (3.26) and inequality (3.27), from equality (3.25) we obtain

$$\left| \frac{F_A^{(k)}(w)}{F(w)} \frac{1}{L_F^k(r,D)} - 1 \right| = o(1)$$

as $r \to +\infty$. Thus, the statement of Theorem 3.5 is proved. □

3.6 Corollaries: analytic functions in the bounded domains.

From Theorem 3.3, choosing

$$h(x) = \frac{1}{|x|}, \quad \Phi(x) = L_F(x, D),$$

we obtain the following corollary.

Corollary 3.1. *For each function $F \in \mathcal{A}^p(\mathbf{D})$ such that $|x|L_F(x,D) \to +\infty$ $(x \to -0)$ relations* (3.10) *hold for $x \to -0$ $(x \in (-1,0) \setminus E)$, while*

$$\mathcal{D}_1(E) := \overline{\lim_{x \to -0}} \frac{1}{|x|} \operatorname{meas}\left(E \cap [x,0]\right) = 0.$$

Let f be an entire transcendental function of the form (3.8) with the domain of convergence $\mathbf{G}$, where $\mathbf{G}$ is a bounded multiple circular domain. Then the function

$$F(z) = F(z_1, \ldots, z_p) = f(e^z) = f(e^{z_1}, \ldots, e^{z_p})$$

is analytic in a polylinear domain $\mathbf{D}$. Clearly,

$$M_f(r,G) = \max\{|f(z)| \colon z \in \partial\mathbf{G}_r\} = \sup\{|F(w)| \colon w \in \partial\mathbf{D}_x\} = S_F(r,D)$$

at $x = (x_1, \ldots, x_p) = \ln r = (\ln r_1, \ldots, \ln r_p)$. Then, at $x = \ln r$

$$L_F(x,D) = (\ln M_f(r,G))'_+ := K_f(r,G).$$

Since, $r = e^x \to 1 - 0 \iff x \to -0$. Therefore, from Corollary 3.1 we obtain the following statement.

Corollary 3.2. *For each function $f \in \mathcal{A}^p(\mathbf{G})$ such that $(1-r)K_f(r,G) \to +\infty$ $(r \to 1-0)$ relations*

$$M_f(r,G) = (1+o(1))B_f(r,G) = -(1+o(1))C_f(r,G)$$

hold for $r \to 1-0$ $(r \in (0,1) \setminus E)$, while

$$\mathcal{D}^1(E) := \overline{\lim_{r \to 1-0}} \frac{1}{1-r} \operatorname{meas}\left(E \cap [r,1)\right) = 0.$$

Here

$$B_f(r,G) = \sup\{\operatorname{Re} f(z) \colon z \in \mathbf{G}_r\}, \quad C_F(r,G) = \inf\{\operatorname{Re} f(z) \colon z \in \mathbf{G}_r\}.$$

The following statement is a consequence of the inequality (3.12) established in the proof of Theorem 3.4.

Corollary 3.3. *Let Φ, h and $F \in S(-1;0)$ be as in Theorem 3.2, and the function $b(r) \colon (-1,0) \to \mathbb{R}_+$ be such that*

$$L_F(r,D)b(r) = O(1) \quad (r \to -0).$$

There exists a set $E \subset [-1,0)$ such that $\mathcal{D}_h E = 0$, and for all $z_0 \in \mathbf{D}$, $z_0 \in \partial\mathbf{D}_r$, such that

$$|F(z_0)| = (1+o(1))S_F(r,D)$$

and for all $\eta \in \mathbb{C}$ *such that* $|\eta| \le b(r)$ *for* $r \to -0$ $(r \notin E)$, *we have*

$$F(z_0 + A\eta) = F(z_0)(1+\omega(\eta))e^{\eta L_F(r,D)},$$

where the function $\omega(\eta) = \omega(\eta, z_0)$ *such that*

$$\omega(\eta) = o(1) \quad (\eta \to 0).$$

Proof of Corollary 3.3. Let $b_0(r) = O(1)$ $(r \to -0)$ and

$$b(r) = b_0(r)/L_F(r, D).$$

For disc $|\eta| \le b(r)$ within the proof of Theorem 3.4 we have

$$\left|\sum_{j=1}^{+\infty} \Delta_j \eta^j\right| \le 2\ln\left(1 + \frac{r_0 c(r) L_F(r,D)}{\delta(r)}\right)\sum_{j=1}^{+\infty}\left(\frac{b(r)}{r_0}\right)^j,$$

where $c(r) = 1 + e(1+\varepsilon(r))$, $\varepsilon(r) \downarrow 0$ $(r \to -0)$ is some arbitrary function. We choose $r_0 = (1+o(1))\frac{\delta(r)}{L_F(r,D)c(r)}$. Then as $r \to -0$

$$\left|\sum_{j=1}^{+\infty} \Delta_j \eta^j\right| \le 2(1+o(1))\frac{b(r)/r_0}{1 - b(r)/r_0} = 2(1+o(1))\frac{b(r)}{r_0 - b(r)} =$$

$$= 2(1+o(1))\frac{c(r)b_0(r)}{\delta(r)(1+o(1)) - c(r)b_0(r)} = o(1).$$

The application of Theorem 3.4 completes the proof of Corollary 3.3.

First, note that Inequality (3.12) holds for all $\eta \in \mathbb{C}$, $|\eta| < \psi(r) \equiv \frac{\delta(r)}{L_F(r,D)}$, as $r \to -0$ $(r \notin E, \mathcal{D}_h E = 0)$. Since $b_0(r) = o(\delta(r))$ $(r \to -0)$, one has

$$b(r) = \frac{b_0(r)}{L_F(r,D)} = o\left(\frac{\delta(r)}{L_F(r,D)}\right) = o(\psi(r)) \quad (r \to -0) \tag{3.28}$$

and therefore $b(r) < \psi(r)$ $(r_1 < r < 0)$. Hence, by Inequality (3.12) and Relation (3.28) we obtain that for $\omega(\eta, z_0) := g(\eta)$ at $r \to -0$ $(r \notin E, \mathcal{D}_h E = 0)$ and $|\eta| \le b(r)$

$$|\omega(\eta, z_0)| \le c(r)\frac{|\eta|}{\psi(r)} \le c(r)\frac{|b(r)|}{\psi(r)} = o(c(r)) = o(1).$$

Corollary 3.3 is proved. □

We put $A = (1, \dots, 1) \in \mathbb{R}^p_+$. It is easy to see,

$$F'_A(z) = \left(\frac{\partial}{\partial z_1} + \ldots + \frac{\partial}{\partial z_p}\right) f(e^z) = \left(e^{z_1}\frac{\partial}{\partial \tau_1} + \ldots + e^{z_p}\frac{\partial}{\partial \tau_p}\right) f(\tau)\Big|_{\tau = e^z}.$$

Similarly,

$$F_A^{(k)}(z) = \Big(e^{z_1}\frac{\partial}{\partial\tau_1} + \ldots + e^{z_p}\frac{\partial}{\partial\tau_p}\Big)^{(k)} f(\tau)\Big|_{\tau=e^z}.$$

Then, from Theorem 3.5 we obtain the following statement.

Corollary 3.4. *Let Φ be a positive non-decreasing to $+\infty$ on $(0,1)$ function such that*

$$(1-r)\Phi(r) \to +\infty \quad (r \to 1-0),$$

and the function $f \in \mathcal{A}^p(\mathbf{G})$ such that

$$K_f(r, G) \geq \Phi(r) \quad (r_0 \leq r < 1).$$

and $\varepsilon(r) \downarrow 0$ $(r \to 1-0)$ is an arbitrary function. Then there exists a set $E \subset (0,1)$, $\mathcal{D}_h^1 E = 0$ such that for each $n \in \mathbb{N}$ asymptotic relation

$$\Big(\tau_1\frac{\partial}{\partial\tau_1} + \ldots + \tau_p\frac{\partial}{\partial\tau_p}\Big)^{(n)} f(\tau) = (1+o(1))K_f^n(r, G) f(\tau)$$

holds as $r \to 1-0$ $(r \notin E)$ for every point $\tau \in \partial\mathbf{G}_r$ provided the inequality $|f(\tau)| \geq M_f(r)/(1+\varepsilon(r))$.

Discussion. In the case $p=1$, the finiteness of the logaritmic measure of exceptional set in the relations from Wiman's theorem is the best possible description. It follows from the examples in papers [3, 46]. The same situation is in the case of the main relation [3].

Problem 3.1. *Is the description $\mathcal{D}_h(E) = 0$ the best possible description of the exceptional set E in the case $p > 1$ in Theorem 3.5? And in the case $p = 1$?*

Problem 3.2. *Is the description $\mathcal{D}_h^1(E) = 0$ is the best possible description of the exceptional set E in the case $p > 1$ in Corollary 3.4? And in the case $p = 1$?*

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Iryna CHERNEGA and Andriy ZAGORODNYUK

4 Symmetric analytic functions on Banach spaces and their applications

Groups and semigroups of symmetries play a fundamental role in contemporary mathematics. Their invariants are important in operator theory, functional analysis, differential equations, algebraic geometry, Lee theory, and other areas of algebra and analysis. The question of the description of the invariants of a linear transformations group on $\mathbb{C}^n$ which naturally acts on the algebra of polynomials is a typical problem of the classical Invariant Theory. Such invariants form algebras of symmetric polynomials with respect to given groups and are well investigated in the classical cases (see e. g. [35,43]). It is very important for these studies to describe the spectra of the algebras of invariants. The cases when a group (or even a semigroup) of symmetry acts on infinite-dimensional Banach spaces where considered in [1, 5, 7, 11–14, 18, 20, 22, 24, 28, 38, 44–46] and in other papers.

For the infinite-dimensional case we need to work with a natural completion of the algebra of continuous polynomials, that is, the algebra of analytic functions of bounded type. In this case we can use some methods and ideas developed in [3, 4, 6, 9, 10, 47].

R. Aron, B. Cole and T. Gamelin introduced in [3] a convolution operation in the spectrum of the algebra $H_b(X)$ of the analytic functions of bounded type defined on a complex Banach space X. This convolution is defined relying on translations on X. Later, R. Aron et al. [6] discussed the commutativity of that convolution and proved that for $X = \ell_p$, it is not commutative.

By a *symmetric* function on ℓ_p, $1 \leq p < \infty$ we mean a function which is invariant under any reordering of the sequence in ℓ_p. The algebra of symmetric analytic functions with the topology of the uniform convergence on bounded sets will be denoted $\mathcal{H}_{bs}(\ell_p)$. We put $\mathcal{M}_{bs}(\ell_p)$ for its spectrum, that is, the set of all continuous scalar valued homomorphisms. The case of symmetric functions on ℓ_∞ was considered in [19]. As pointed out below, when dealing with symmetric analytic functions the translation operators are not well defined anymore. This is why in [11] it was introduced the so-called "intertwining" operators that lead them to define a "symmetric" convolution operation as is described in the next section.

Iryna CHERNEGA, Institute for Applied Problems of Mechanics and Mathematics, Ukrainian Academy of Sciences, 79060, Lviv, Ukraine, e-mail: icherneha@ukr.net

Andriy ZAGORODNYUK, Vasyl Stefanyk Carpathian National University, 57 Shevchenka Str., 76018 Ivano-Frankivsk, Ukraine, e-mail: andriy.zagorodnyuk@pnu.edu.ua

In this survey paper we gathered results about different representations of the spectra $\mathcal{M}_{bs}(\ell_p)$ and their some applications that were basically obtained in [11–16]. The most complete description of $\mathcal{M}_{bs}(\ell_p)$ was obtained for the case $p = 1$ in [14].

In Subsection 4.1, we show that, contrary to the non-symmetric case, the symmetric convolution is indeed commutative. Also a representation of $\mathcal{M}_{bs}(\ell_1)$ in terms of entire functions of exponential type is obtained. Such representation allows us to determine the invertible elements in $\mathcal{M}_{bs}(\ell_1)$ for such symmetric convolution. Finally, we present a description of the elements in the spectrum through certain points in ℓ_1^+. In Section 4.2, we introduce a multiplicative convolution operation on $\mathcal{M}_{bs}(\ell_1)$ that allows to assure that the representation of $\mathcal{M}_{bs}(\ell_1)$ in terms of entire functions of exponential type is not onto. This is achieved by appealing to a result from Number Theory on the number of divisors of a given positive integer. It is also shown that $\mathcal{M}_{bs}(\ell_1)$ becomes a commutative semi-ring with identity when endowed with both mentioned convolution operations. In Section 4.3, we provide a complete description of the spectrum $\mathcal{M}_{bs}(\ell_1)$ in terms of the convolution operation. Such description follows from Theorem 4.9. Its extension Theorem 4.10 should be compared with the analogous equality in [3, 3.1] In Section 4.4, there are some applications of symmetric polynomials in statistical quantum mechanics. Further applications can be found in [26].

In [22] it is proved that, similarly to the classical finite dimensional case, the polynomials

$$F_k(x) = \sum_{i=1}^{\infty} x_i^k, \quad k = \lceil p \rceil, \lceil p \rceil + 1 \ldots$$

form an *algebraic basis*—named *the power series* basis—in the algebra of all symmetric polynomials on ℓ_p (here $\lceil p \rceil$ is the smallest integer that is greater than or equal to p). This means that for every symmetric polynomial P of degree $\lceil p \rceil + n - 1$, $n \geq 1$ there is a polynomial q on $\mathbb{C}^n$ such that $P(x) = q(F_{\lceil p \rceil}(x), \ldots, F_{\lceil p \rceil + n-1}(x))$. Actually, q is unique as pointed out in [1].

For background on analytic functions on infinite-dimensional spaces, we refer the reader to [17] or to [33].

4.1 The convolution operation on the spectra of algebras of symmetric analytic functions

4.1.1 The symmetric convolution

We start with the following simple remark.

Remark 4.1. There is no $w \in \ell_p$, $w \neq 0$, such that $g(x) = f(x + w)$ is symmetric for every symmetric $f \in \mathcal{H}_{bs}(\ell_p)$.

Proof. There is $i_0 \in \mathbb{N}$, such that $|w_n| < 1/3$ if $n \geq i_0$. Assume that $f(\cdot + w)$ belongs to $\mathcal{H}_{bs}(\ell_p)$ for every symmetric $f \in \mathcal{H}_{bs}(\ell_p)$. Then for every fixed permutation σ and each element in the basis of ℓ_p, $f(e_{\sigma(i)} + w) = g(e_{\sigma(i)}) = g(e_i) = f(e_i + w), \forall f \in \mathcal{H}_{bs}(\ell_p)$. Thus $e_{\sigma(i)} + w$ is a permutation of $e_i + w$, that is, $1 + w_{\sigma(i)} = w_{j_i}$ for some index $j_i \in \mathbb{N}$.

Since σ is a bijection, the set $\{\sigma(i) > i_0\}$ is infinite, so there are infinite terms w_{j_i} with absolute value greater that $2/3$. Impossible. □

Next we recall some definitions.

Definition 4.1 ([11]). *Let $x, y \in \ell_p$, $x = (x_1, x_2, \dots,)$ and $y = (y_1, y_2, \dots,)$. We define the intertwining $x \bullet y \in \ell_p$ according to*

$$x \bullet y = (x_1, y_1, x_2, y_2, \dots,).$$

The mapping $f \mapsto T^s_y(f)$ where $T^s_y(f)(x) = f(x \bullet y)$ will be referred as to the *intertwining operator.* Observe that $T^s_x \circ T^s_y = T^s_{x \bullet y} = T^s_y \circ T^s_x$: Indeed, $[T^s_x \circ T^s_y](f)(z) = T^s_x[T^s_y(f)](z) = T^s_y(f)(z \bullet x) = f((z \bullet x) \bullet y)) = f(z \bullet (x \bullet y))$, since f is symmetric.

The above remark explains why we are led to use the intertwining operators to define the convolution in $\mathcal{M}_{bs}(\ell_p)$.

Definition 4.2 ([11]). *Given $f \in \mathcal{H}_{bs}(\ell_p)$ and $\theta \in \mathcal{H}_{bs}(\ell_p)'$, its symmetric convolution $\theta \star f$ is defined by $(\theta \star f)(x) = \theta[T^s_x(f)]$.*

As pointed out in [11], it turns out that $\theta \star f \in \mathcal{H}_{bs}(\ell_p)$.

Definition 4.3 ([11]). *For any ϕ and θ in $\mathcal{H}_{bs}(\ell_p)'$, its symmetric convolution is defined according to*

$$(\phi \star \theta)(f) = \phi(\theta \star f) = \phi(y \mapsto \theta(T^s_y f)).$$

Corollary 4.1 ([11]). *If $\phi, \theta \in \mathcal{M}_{bs}(\ell_p)$, then $\phi \star \theta \in \mathcal{M}_{bs}(\ell_p)$.*

Theorem 4.1. *a) For every $\varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$ the following holds:*

$$(\varphi \star \theta)(F_k) = \varphi(F_k) + \theta(F_k). \tag{4.1}$$

b) The semigroup $(\mathcal{M}_{bs}(\ell_p), \star)$ is commutative, the evaluation at 0, δ_0, is its identity and the cancellation law holds.

Proof. Observe that for each element F_k in the algebraic basis of polynomials, $\{F_k\}$, we have

$$(\theta \star F_k)(x) = \theta(T^s_x(F_k)) = \theta(F_k(x) + F_k) = F_k(x) + \theta(F_k).$$

Therefore,

$$(\varphi \star \theta)(F_k) = \varphi(F_k + \theta(F_k)) = \varphi(F_k) + \theta(F_k).$$

To check that the convolution is commutative, that is, $\phi \star \theta = \theta \star \phi$, it suffices to prove it for symmetric polynomials, hence for the basis $\{F_k\}$. Bearing in mind (4.1) and also by exchanging parameters $(\theta\star\varphi)(F_k) = \theta(F_k)+\varphi(F_k) = (\varphi\star\theta)(F_k)$ as we wanted.

It also follows from (4.1) that the cancellation rule is valid for this convolution: If $\varphi \star \theta = \psi \star \theta$, then $\varphi(F_k) + \theta(F_k) = \psi(F_k) + \theta(F_k)$, hence $\varphi(F_k) = \psi(F_k)$, and thus, $\varphi = \psi$. □

Example 4.1. *There exist nontrivial elements in the semigroup $(\mathcal{M}_{bs}(\ell_p), \star)$ that are invertible:*

In [1, Example 3.1] it was constructed a continuous homomorphism $\varphi = \Psi_1$ on the uniform algebra $A_{us}(B_{\ell_p})$ such that $\varphi(F_p) = 1$ and $\varphi(F_i) = 0$ for all $i > p$. In a similar way, given $\lambda \in \mathbb{C}$ we can construct a continuous homomorphism Ψ_λ on the uniform algebra $A_{us}(|\lambda| B_{\ell_p})$ such that $\Psi_\lambda(F_p) = \lambda$ and $\Psi_\lambda(F_i) = 0$ for all $i > p$: It suffices to consider for each $n \in \mathbb{N}$, the element

$$v_n = \left(\frac{\lambda}{n}\right)^{1/p} (e_1 + \cdots + e_n)$$

for which $F_p(v_n) = \lambda$, and $\lim_n F_j(v_n) = 0$. Now, the sequence $\{\delta_{v_n}\}$ has an accumulation point Ψ_λ in the spectrum of $A_{us}(|\lambda| B_{\ell_p})$. We use the notation ψ_λ for the restriction of Ψ_λ to the subalgebra $\mathcal{H}_{bs}(\ell_p)$ of $A_{us}(|\lambda| B_{\ell_p})$. It turns out that $\psi_\lambda \star \psi_{-\lambda} = \delta_0$ since for all elements F_j in the algebraic basis, $(\psi_\lambda \star \psi_{-\lambda})(F_j) = \psi_\lambda(F_j) + \psi_\lambda(F_j) = 0 = \delta_0(F_j)$.

Therefore, we obtain a complex line of invertible elements $\{\psi_\lambda \colon \lambda \in \mathbb{C}\}$.

As in the non-symmetric case [3] Theorem 5.5, the following holds:

Proposition 4.1. *Every $\varphi \in \mathcal{M}_{bs}(\ell_p)$ lies in a schlicht complex line through δ_0.*

Proof. For every $z \in \mathbb{C}$, consider the composition operator $L_z : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ defined according to $L_z(f)((x_n)) := f((zx_n))$, and then, the restriction L_z^* to $\mathcal{M}_{bs}(\ell_p)$ of its transpose map. Now put $\varphi^z := L_z^*(\varphi) = \varphi \circ L_z$. Observe that $\varphi^z(F_k) = \varphi \circ L_z(F_k) = \varphi((F_k(z\cdot))) = z^k \varphi(F_k)$. Also, $\varphi^0 = \delta_0$.

For each $f \in \mathcal{H}_{bs}(\ell_p)$ the self-map of $\mathbb{C}$ defined according to $z \rightsquigarrow \varphi^z(f)$ is entire by [3] Lemma 5.4.(i). Therefore, the mapping $z \in \mathbb{C} \rightsquigarrow \varphi^z \in \mathcal{M}_{bs}(\ell_p)$ is analytic.

Since $\varphi \neq \delta_0$, the set $\Sigma := \{k \in \mathbb{N} : \varphi(F_k) \neq 0\}$ is non-empty. Let m be the first element of Σ, so that $\varphi(F_m) \neq 0$. Then if $\varphi^z = \varphi^w$, one has $z^m \varphi(F_m) = w^m \varphi(F_m)$, hence $z^m = w^m$. Taking the principal branch of the m^{th} root, the map $\xi \rightsquigarrow \varphi^{\sqrt[m]{\xi}}$ is one-to-one. □

Recall that a linear operator $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ is said to be a *convolution operator* if there is $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta \star f$. Let us denote $H_{conv}(\ell_p) := \{T \in L(\mathcal{H}_{bs}(\ell_p)) : T \text{ is a convolution operator}\}$.

Proposition 4.2. *A continuous homomorphism $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ is a convolution operator if, and only if, it commutes with all intertwining operators $T_y^s,\ y \in \ell_p$.*

Proof. Assume there is $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta \star f$. Fix $y \in \ell_p$. Then $[T \circ T_y^s](f)(x) = [T(T_y^s(f))](x) = [\theta \star T_y^s(f)](x) = \theta[T_x^s(T_y^s(f)] = \theta[T_{x\bullet y}^s(f)]$. On the other hand, $[T_y^s \circ T](f)(x) = [T_y^s(Tf)](x) = Tf(x\bullet y) = (\theta \star f)(x\bullet y) = \theta[T_{x\bullet y}^s(f)]$.

Conversely, set $\theta = \delta_0 \circ T$. Clearly, $\theta \in \mathcal{M}_{bs}(\ell_p)$. Let us check that $Tf = \theta \star f$: Indeed, $(\theta \star f)(x) = \theta[T_x^s(f)] = [T(T_x^s(f))](0) = [T_x^s(T(f))](0) = Tf(0 \bullet x) = Tf(x)$. □

Consider the mapping Λ defined by $\Lambda(\theta)(f) = \theta \star f$, that is,

$$\begin{array}{rccc} \Lambda : & \mathcal{M}_{bs}(\ell_p) & \to & H_{conv}(\ell_p) \\ & \theta & \mapsto & f \rightsquigarrow \theta \star f \equiv \Lambda(\theta)(f) \end{array} .$$

Clearly, it is bijective. Moreover we obtain a representation of the convolution semigroup

Proposition 4.3. *The mapping Λ is an isomorphism from $(\mathcal{M}_{bs}(\ell_p), \star)$ into $(H_{conv}(\ell_p), \circ)$ where $\circ$ denotes the usual composition operation.*

Proof. First, notice that using the above proposition,

$$\Lambda(\varphi \star \theta)(f)(x) = [(\varphi \star \theta) \star f](x) = (\varphi \star \theta)(T_x^s f) = \varphi(\theta \star T_x^s f) = \\ = \varphi[\Lambda(\theta)(T_x^s f)] = \varphi[(\Lambda(\theta) \circ T_x^s)(f)] = \varphi[(T_x^s \circ \Lambda(\theta))(f)].$$

On the other hand,

$$[\Lambda(\varphi) \circ \Lambda(\theta)](f)(x) = \Lambda(\varphi)[\Lambda(\theta)(f)](x) = [\varphi \star \Lambda(\theta)(f)](x) = \varphi[T_x^s(\Lambda(\theta)(f))].$$

Thus the statement follows. □

As a consequence, the homomorphism θ is invertible in $(\mathcal{M}_{bs}(\ell_p), \star)$, if, and only if, the convolution operator $\Lambda(\theta)$ is an algebraic isomorphism.

We observe also that for $\psi \in \mathcal{M}_{bs}(\ell_p)$, one has

$$\psi \circ \Lambda(\theta) = \psi \star \theta,$$

because $[\psi \circ \Lambda(\theta)](f) = \psi[\Lambda(\theta)(f)] = \psi(\theta \star f) = (\psi \star \theta)(f)$.

Next we address the question of solving the equation $\varphi = \psi \star \theta$ for given $\varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$. We begin with a general lemma.

Lemma 4.1. *Let A, B be Fréchet algebras and $T : A \to B$ an onto homomorphism. Then T maps (closed) maximal ideals onto (closed) maximal ideals.*

Proof. Since T is onto, it maps ideals in A onto ideals in B. Let $\mathcal{J} \subset A$ be a maximal ideal, we prove that $T(\mathcal{J})$ is a maximal ideal in B : If $\mathcal{I}$ is another ideal with $T(\mathcal{J}) \subset \mathcal{I} \subset B$, it turns out that for the ideal $T^{-1}(\mathcal{I})$, $\mathcal{J} \subset T^{-1}(T(\mathcal{J})) \subset T^{-1}(\mathcal{I})$, hence either $\mathcal{J} = T^{-1}(\mathcal{I})$, or $A = T^{-1}(\mathcal{I})$. That is, either $T(\mathcal{J}) = \mathcal{I}$, or $B = \mathcal{I}$.

Let now $\varphi \in M(A)$ and $\mathcal{J} = Ker(\varphi)$, a closed maximal ideal. Then $T(\mathcal{J})$ is a maximal ideal in B, so there is a character ψ on B such that $Ker(\psi) = T(\mathcal{J})$. Then $Ker(\varphi) \subset Ker(\psi \circ T)$, because if $\varphi(a) = 0$, that is, $a \in \mathcal{J}$, we have $T(a) \in Ker(\psi)$. By the maximality, either $\varphi = \psi \circ T$, or $\psi \circ T = 0$, hence $\psi = 0$. In the former case, ψ is also continuous since being T an open mapping, if (b_n) is a null sequence in B, there is a null sequence $(a_n) \subset A$ such that $T(a_n) = b_n$; thus $\lim_n \psi(b_n) = \lim_n \psi \circ T(a_n) = \lim_n \varphi(a_n) = 0$. □

Remark 4.2. *Let A, B be Fréchet algebras and $T : A \to B$ an onto homomorphism. If $T(Ker(\varphi))$ is a proper ideal, then there is a unique $\psi \in M(B)$ such that $\varphi = \psi \circ T$.*

Corollary 4.2. *Let $\theta \in \mathcal{M}_{bs}(\ell_p)$. Assume that $\Lambda(\theta)$ is onto. If $\Lambda(\theta)(Ker\varphi)$ is a proper ideal, then the equation $\varphi = \psi \star \theta$ has a unique solution. In case $\Lambda(\theta)(Ker\varphi) = \mathcal{H}_{bs}(\ell_p)$, then the equation $\varphi = \psi \star \theta$ has no solution.*

Proof. The first statement is just an application of the remark, since $\psi \star \theta = \psi \circ \Lambda(\theta) = \varphi$. For the second statement, if some solution ψ exists, then again $\psi \circ \Lambda(\theta) = \psi \star \theta = \varphi$, so $\psi(\mathcal{H}_{bs}(\ell_p)) = (\psi \circ \Lambda(\theta))((Ker\varphi)) = \varphi(Ker\varphi) = 0$. Therefore, then also $\varphi = 0$. □

4.1.2 A weak polynomial topology on $\mathcal{M}_{bs}(\ell_p)$

Let us denote by w_p the topology in $\mathcal{M}_{bs}(\ell_p)$ generated by the following neighborhood basis:

$$U_{\varepsilon,k_1,\ldots,k_n}(\psi) = \{\psi \star \varphi \colon \varphi \in \mathcal{M}_{bs}(\ell_p) \quad |\varphi(F_{k_j})| < \varepsilon, \quad j = 1, \ldots, n\}.$$

It is easy to check that the convolution operation is continuous for the w_p topology, since thanks to (4.1),

$$U_{\varepsilon/2,k_1,\ldots,k_n}(\theta) \star U_{\varepsilon/2,k_1,\ldots,k_n}(\psi) \subset U_{\varepsilon,k_1,\ldots,k_n}(\theta \star \psi).$$

We say that a function $f \in \mathcal{H}_{bs}(\ell_p)$ is *finitely generated* if there are a finite number of the basis functions $\{F_k\}$ and an entire function q such that $f = q(F_1, \ldots, F_j)$.

Theorem 4.2. *A function $f \in \mathcal{H}_{bs}(\ell_p)$ is w_p-continuous if and only if it is finitely generated.*

Proof. Clearly, every finitely generated function is w_p-continuous. Let us denote by V_n the finite dimensional subspace in ℓ_p spanned by the basis vectors $\{e_1, \ldots, e_n\}$. First we observe that if there is a positive integer m such that the restriction $f_{|V_n}$ of f to V_n is generated by the restrictions of $F_1, \ldots, F_m$ to V_n for every $n \geq m$, then f is finitely generated. Indeed, for given $n \geq k \geq m$ we can write

$$f_{|V_k}(x) = q_1(F_1(x), \ldots, F_m(x)) \quad \text{and} \quad f_{|V_n}(x) = q_2(F_1(x), \ldots, F_m(x))$$

for some entire functions q_1 and q_2 on $\mathbb{C}^n$. Since

$$\{(F_1(x), \ldots, F_m(x)) \colon x \in V_k\} = \mathbb{C}^m$$

(see e. g. [1]) and $f|_{V_n}$ is an extension of $f|_{V_k}$ we have $q_1(t_1, \ldots, t_n) = q_2(t_1, \ldots, t_n)$. Hence, $f(x) = q_1(F_1(x), \ldots, F_m(x))$ on ℓ_p, because $f(x)$ coincides with

$$q_1(F_1(x), \ldots, F_m(x))$$

on the dense subset $\bigcup_n V_n$.

Let f be a w_p-continuous function in $\mathcal{H}_{bs}(\ell_p)$. Then f is bounded on a neighborhood $U_{\varepsilon,1,\ldots,m} = \{x \in \ell_p \colon |F_1(x)| < \varepsilon, \ldots, |F_m(x)| < \varepsilon\}$. For a given $n \geq m$ let

$$f|_{V_n}(x) = q(F_1(x), \ldots, F_m(x))$$

be the representation of $f|_{V_n}(x)$ for some entire function q on $\mathbb{C}^n$. Since

$$\{(F_1(x), \ldots, F_m(x)) \colon x \in V_n\} = \mathbb{C}^m, \;\; q(t_1, \ldots, t_n)$$

must be bounded on the set $\{|t_1| < \varepsilon, \ldots, |t_m| < \varepsilon\}$. The Liouville Theorem implies $q(t_1, \ldots, t_n) = q(t_1, \ldots, t_m, 0 \ldots, 0)$, that is, $f|_{V_n}$ is generated by $F_1, \ldots, F_m$. Since it is true for every n, f is finitely generated. □

For example, $f(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{n!}$ is not w_p-continuous.

Proposition 4.4. *w_p is a Hausdorff topology.*

Proof. If $\varphi \neq \psi$, then there is a number k such that

$$|\varphi(F_k) - \psi(F_k)| = \rho > 0.$$

Let $\varepsilon = \rho/3$. Then for every θ_1 and θ_2 in $U_{\varepsilon,k}(0)$,

$$|(\varphi \star \theta_1)(F_k) - (\varphi \star \theta_2)(F_k)| = |(\varphi(F_k) - \psi(F_k)) - (\theta_2(F_k)) - \theta_1(F_k)| \geq \rho/3.$$

□

Proposition 4.5. *On bounded sets of $\mathcal{M}_{bs}(\ell_p)$ the topology w_p is finer than the weak-star topology $w(\mathcal{M}_{bs}(\ell_p), \mathcal{H}_{bs}(\ell_p))$.*

Proof. Since $(\mathcal{M}_{bs}(\ell_p), w_p)$ is a first-countable space, it suffices to verify that for a bounded sequence $(\varphi_i)_i$ which is w_p convergent to some ψ, we have $\lim_i \varphi_i(f) = \psi(f)$ for each $f \in \mathcal{H}_{bs}(\ell_p)$: Indeed, by the Banach-Steinhaus theorem, it is enough to see that $\lim_i \varphi_i(P) = \psi(P)$ for each symmetric polynomial P. Being $\{F_k\}$ an algebraic basis for the symmetric polynomials, this will follow once we check that $\lim_i \varphi_i(F_k) = \psi(F_k)$ for each F_k. To see this, notice that given $\varepsilon > 0$, $\varphi_i \in U_{\varepsilon,k}$ for i large enough, that is, there is θ_i such that $\varphi_i = \psi \star \theta_i$ with $|\theta_i(F_k)| < \varepsilon$. Then, $|\varphi_i(F_k) - \psi(F_k)| = |\theta_i(F_K)| < \varepsilon$ for i large enough. □

Proposition 4.6. *If $(\mathcal{M}_{bs}(\ell_p), \star)$ is a group, then w_p coincides with the weakest topology on $\mathcal{M}_{bs}(\ell_p)$ such that for every polynomial $P \in \mathcal{H}_{bs}(\ell_p)$ the Gelfand extension $\widehat{P}$ is continuous on $\mathcal{M}_{bs}(\ell_p)$.*

Proof. The sets $F_k^{-1}(B(F_k(\psi), \varepsilon))$ generate the weakest topology such that all $\widehat{P}$ are continuous. Let $\theta \in \mathcal{M}_{bs}(\ell_p)$ be such that $|F_k(\theta) - F_k(\psi)| < \varepsilon$. Set $\varphi = \theta \star \psi^{-1}$. Then $|F_k(\varphi)| = |F_k(\theta) - F_k(\psi)| < \varepsilon$ and $\theta = \psi \star \varphi$. □

4.1.3 Representations of the convolution semigroup $(\mathcal{M}_{bs}(\ell_1), \star)$

Let us consider the case $\mathcal{H}_{bs}(\ell_1)$. This algebra admits besides the power series basis $\{F_n\}$, another natural basis that is useful for us: It is given by the sequence $\{G_n\}$ defined by $G_0 = 1$, and

$$G_n(x) = \sum_{k_1 < \cdots < k_n}^{\infty} x_{k_1} \cdots x_{k_n},$$

and we refer to it as the *basis of elementary* symmetric polynomials.

Lemma 4.2. *We have that $\|G_n\| = 1/n!$*

Proof. To calculate the norm, it is enough to deal with vectors in the unit ball of ℓ_1 whose components are non-negative. And we may reduce ourselves to calculate it on L_m the linear span of $\{e_1, \dots, e_m\}$ for $m \geq n$. We do the calculation in an inductive way over m.

Since $G_{n|_{L_m}}$ is homogeneous, its norm is achieved at points of norm 1. If $m = n$, then G_n is the product $x_1 \cdots x_n$. By using the Lagrange multipliers rule, we deduce that the maximum is attained at points with equal coordinates, that is at $\frac{1}{n}(e_1 + \cdots + e_n)$. Thus $|G_n(\frac{1}{n}, \overset{n}{\ldots}, \frac{1}{n}, 0, \dots)| = 1/n^n \leq \frac{1}{n!}$.

Now for $m > n$, and $x \in L_m$, we have $G_n(x) = \sum_{k_1 < \cdots < k_n \leq m}^{\infty} x_{k_1} \cdots x_{k_n}$. Again the Lagrange multipliers rule leads to either some of the coordinates vanish or they are all equal, hence they have the same value $\frac{1}{m}$. In the first case, we are led back to some the previous inductive steps, with L_k with $k < m$, so the aimed inequality holds. While in the second one, we have

$$G_n\left(\frac{1}{m}, \overset{m}{\ldots}, \frac{1}{m}, 0, \ldots\right) = \binom{m}{n}\frac{1}{m^n} \leq \frac{1}{n!}.$$

Moreover,

$$\|G_n\| \geq \lim_m \binom{m}{n}\frac{1}{m^n} = \frac{1}{n!}.$$

This completes the proof. □

Let $\mathbb{C}\{t\}$ be the space of all power series over $\mathbb{C}$. We denote by $\mathcal{F}$ and $\mathcal{G}$ the following maps from $\mathcal{M}_{bs}(\ell_1)$ into $\mathbb{C}\{t\}$

$$\mathcal{F}(\varphi) = \sum_{n=1}^{\infty} t^{n-1}\varphi(F_n) \quad \text{and} \quad \mathcal{G}(\varphi) = \sum_{n=0}^{\infty} t^n \varphi(G_n).$$

Let us recall that every element $\varphi \in \mathcal{M}_{bs}(\ell_1)$ has a radius-function

$$R(\varphi) = \limsup_{n\to\infty} \|\varphi_n\|^{\frac{1}{n}} < \infty,$$

where φ_n is the restriction of φ to the subspace of n-homogeneous polynomials [11].

Proposition 4.7. *The mapping $\varphi \in \mathcal{M}_{bs}(\ell_1) \xrightarrow{\mathcal{G}} \mathcal{G}(\varphi) \in \mathcal{H}(\mathbb{C})$ is one-to-one and ranges into the subspace of entire functions on $\mathbb{C}$ of exponential type. The type of $\mathcal{G}(\varphi)$ is less than or equal to $R(\varphi)$.*

Proof. Using Lemma 4.2,

$$\limsup_{n\to\infty} \sqrt[n]{n!|\varphi_n(G_n)|} \leq \limsup_{n\to\infty} \sqrt[n]{n!\|\varphi_n\|\|G_n\|} = \limsup_{n\to\infty} \sqrt[n]{\|\varphi_n\|} = R(\varphi) < \infty,$$

hence $\mathcal{G}(\varphi)$ is entire and of exponential type less than or equal to $R(\varphi)$. That $\mathcal{G}$ is one-to-one follows from the fact $\{G_n\}$ is a basis. □

Theorem 4.3. *The following identities hold:*

1. $\mathcal{F}(\varphi \star \theta) = \mathcal{F}(\varphi) + \mathcal{F}(\theta)$.
2. $\mathcal{G}(\varphi \star \theta) = \mathcal{G}(\varphi)\mathcal{G}(\theta)$.

Proof. The first statement is a trivial conclusion of the properties of the convolution. To prove the second we observe that

$$G_n(x \bullet y) = \sum_{k=0}^{n} G_k(x) G_{n-k}(y).$$

Thus

$$(\theta \star G_n)(x) = \theta(T_x^s(G_n)) = \theta\left(\sum_{k=0}^{n} G_k(x)G_{n-k}\right) = \sum_{k=0}^{n} G_k(x)\theta(G_{n-k}).$$

Therefore,

$$(\varphi \star \theta)(G_n) = \varphi\left(\sum_{k=0}^{n} G_k(x)\theta(G_{n-k})\right) = \sum_{k=0}^{n} \varphi(G_k)\theta(G_{n-k}).$$

Hence, being the series absolutely convergent,

$$\mathcal{G}(\varphi)\mathcal{G}(\theta) = \sum_{k=0}^{\infty} t^k \varphi(G_k) \sum_{m=0}^{\infty} t^m \theta(G_m) = \sum_{n=0}^{\infty} \sum_{k+m=n} t^n \varphi(G_k)\theta(G_m) =$$

$$= \sum_{n=0}^{\infty} t^n \sum_{k+m=n} \varphi(G_k)\theta(G_m) = \sum_{n=0}^{\infty} t^n (\varphi \star \theta)(G_n) = \mathcal{G}(\varphi \star \theta).$$

□

Example 4.2. *Let ψ_λ be as defined in Example 4.1. We know that $\mathcal{F}(\psi_\lambda) = \lambda$. To find $\mathcal{G}(\psi_\lambda)$ note that*

$$G_k(v_n) = \left(\frac{\lambda}{n}\right)^k \binom{n}{k}, \quad \textit{hence} \quad \varphi(G_k) = \lim_n G_k(v_n) = \frac{\lambda^k}{k!}$$

and so

$$\mathcal{G}(\psi_\lambda)(t) = \lim_{n\to\infty} \sum_{k=0}^{n} (\lambda t)^k \psi_\lambda(G_n) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!} = e^{\lambda t}.$$

According to well-known Newton's formula we can write for $x \in \ell_1$,

$$nG_n(x) = F_1(x)G_{n-1}(x) - F_2(x)G_{n-2}(x) + \cdots + (-1)^{n+1}F_n(x).$$

Moreover, if ξ is a complex homomorphism (not necessarily continuous) on the space of symmetric polynomials $\mathcal{P}_s(\ell_1)$, then

$$n\xi(G_n) = \xi(F_1)\xi(G_{n-1}) - \xi(F_2)\xi(G_{n-2}) + \cdots + (-1)^{n+1}\xi(F_n). \tag{4.2}$$

Next we point out the limitations of the construction's technique described in Example 4.1.

Remark 4.3. *Let ξ be a complex homomorphism on $\mathcal{P}_s(\ell_1)$ such that $\xi(F_m) = c \neq 0$ for some $m \geq 2$ and $\xi(F_n) = 0$ for $n \neq m$. Then ξ is not continuous.*

Proof. Using formula (4.2) we can see that

$$\xi(G_{km}) = (-1)^{m+1}\frac{\xi(F_m)\xi(G_{(k-1)m})}{km}$$

and $\xi(G_n) = 0$ if $n \neq km$ for some $k \in \mathbb{N}$. By induction we have

$$\xi(G_{km}) = \frac{\left((-1)^{m+1}c/m\right)^k}{k!}$$

and so

$$\mathcal{G}(\xi)(t) = 1 + \sum_{k=1}^{\infty} \frac{\left((-1)^{m+1}c/m\right)^k}{k!} t^{km} = 1 + \sum_{k=1}^{\infty} \frac{((-1)^{m+1}ct^m/m)^k}{k!} = e^{\left((-1)^{m+1}ct^m/m\right)}.$$

Hence

$$\mathcal{G}(\xi)(t) = e^{-(-ct)^m/m} = e^{-(-c)^m/mt^m}.$$

Since $m \geq 2$, $\mathcal{G}(\xi)$ is not of exponential type. So if ξ were continuous, it could be extended to an element in $\mathcal{M}_{bs}(\ell_1)$, leading to a contradiction with Proposition 4.7. □

According to the Hadamard Factorization Theorem (see [31, p. 27]) the function of the exponential type $\mathcal{G}(\varphi)(t)$ is of the form

$$\mathcal{G}(\varphi)(t) = e^{\lambda t}\prod_{k=1}^{\infty}\left(1 - \frac{t}{a_k}\right)e^{t/a_k}, \tag{4.3}$$

where $\{a_k\}$ are the zeros of $\mathcal{G}(\varphi)(t)$. If $\sum |a_k|^{-1} < \infty$, then this representation can be reduced to

$$\mathcal{G}(\varphi)(t) = e^{\lambda t}\prod_{k=1}^{\infty}\left(1 - \frac{t}{a_k}\right). \tag{4.4}$$

Recall how ψ_λ was defined in Example 4.1.

Proposition 4.8. *If $\varphi \in (\mathcal{M}_{bs}(\ell_1), \star)$ is invertible, then $\varphi = \psi_\lambda$ for some λ. In particular, the semigroup $(\mathcal{M}_{bs}(\ell_1), \star)$ is not a group.*

Proof. If φ is invertible then $\mathcal{G}(\varphi)(t)$ is an invertible entire function of exponential type and so has no zeros. By Hadamard's factorization (4.3) we have that $\mathcal{G}(\varphi)(t) = e^{\lambda t}$ for some complex number λ. Hence $\varphi = \psi_\lambda$ by Proposition 4.7.

The evaluation $\delta_{(1,0\ldots,0,\ldots)}$ does not coincide with any ψ_λ since, for instance, $\psi_\lambda(F_2) = 0 \neq 1 = \delta_{(1,0\ldots,0,\ldots)}(F_2)$. □

Another consequence of our analysis is the following remark.

Corollary 4.3. *Let Φ be a homomorphism of $\mathcal{P}_s(\ell_1)$ to itself such that $\Phi(F_k) = -F_k$ for every k. Then Φ is discontinuous.*

Proof. If Φ is continuous it may be extended to continuous homomorphism $\widetilde{\Phi}$ of $\mathcal{H}_{bs}(\ell_1)$. Then for $x = (1, 0 \ldots, 0, \ldots)$, $\delta_x \star (\delta_x \circ \widetilde{\Phi}) = \delta_0$. However, this is impossible since δ_x is not invertible. □

We close this section by analyzing further the relationship established by the mapping $\mathcal{G}$.

It is known from combinatorics (see e.g. [32, pp. 3, 4]) that

$$\mathcal{G}(\delta_x)(t) = \prod_{k=1}^{\infty}(1 + x_k t) \quad \text{and} \quad \mathcal{F}(\delta_x)(t) = \sum_{k=1}^{\infty} \frac{x_k}{1 - x_k t} \tag{4.5}$$

for every $x \in c_{00}$. Formula (4.5) for $\mathcal{G}(\delta_x)$ is true for every $x \in \ell_1$: Indeed, for fixed t, both the infinite product and $\mathcal{G}(\delta_x)(t)$ are analytic functions on ℓ_1.

Taking into account formula (4.5) we can see that the zeros of $\mathcal{G}(\delta_x)(t)$ are $a_k = -1/x_k$ for $x_k \neq 0$. Conversely, if $f(t)$ is an entire function of exponential type which is equal to the right hand side of (4.4) with $\sum |a_k|^{-1} < \infty$, then for $\varphi \in \mathcal{M}_{bs}(\ell_1)$ given by $\varphi = \psi_\lambda \star \delta_x$, where $x \in \ell_1$, $x_k = -1/a_k$ and ψ_λ is defined in Example 4.1, it turns out that $\mathcal{G}(\varphi)(t) = f(t)$. So we have just to examine entire functions of exponential type with Hadamard canonical product

$$f(t) = \prod_{k=1}^{\infty}\left(1 - \frac{t}{a_k}\right) e^{t/a_k} \tag{4.6}$$

with $\sum |a_k|^{-1} = \infty$. Note first that the growth order of $f(t)$ is not greater than 1. According to Borel's theorem [31, p. 30] the series

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^{1+d}}$$

converges for every $d > 0$. Let

$$\Delta_f = \limsup_{n\to\infty} \frac{n}{|a_n|}, \qquad \eta_f = \limsup_{r\to\infty} \left| \sum_{|a_n|<r} \frac{1}{a_n} \right|$$

and $\gamma_f = \max(\Delta_f, \eta_f)$. Due to Lindelöf's theorem [31, p. 33] the type σ_f of f and γ_f simultaneously are equal either to zero, or to infinity, or to positive numbers. Hence $f(t)$ of the form (4.6) is a function of exponential type if and only if $\sum |a_k|^{-1-d}$ converges for every $d > 0$ and γ_f is finite.

Corollary 4.4. *If a sequence* $(x_n) \notin \ell_p$ *for some* $p > 1$, *then there is no* $\varphi \in \mathcal{M}_{bs}(\ell_1)$ *such that*

$$\varphi(F_k) = \sum_{n=1}^{\infty} x_n^k$$

for all k.

Let $x = (x_1, \ldots, x_n, \ldots)$ be a sequence of complex numbers such that $x \in \ell_{1+d}$ for every $d > 0$,

$$\limsup_{n\to\infty} n|x_n| < \infty, \quad \limsup_{r\to 1} \left| \sum_{\frac{1}{|x_n|}<r} x_n \right| < \infty \tag{4.7}$$

and $\lambda \in \mathbb{C}$. Let us denote by $\delta_{(x,\lambda)}$ a homomorphism on the algebra of symmetric polynomials $\mathcal{P}_s(\ell_1)$ of the form

$$\delta_{(x,\lambda)}(F_1) = \lambda, \quad \delta_{(x,\lambda)}(F_k) = \sum_{n=1}^{\infty} x_n^k, \quad k > 1.$$

Proposition 4.9. *Let* $\varphi \in \mathcal{M}_{bs}(\ell_1)$. *Then the restriction of* φ *to* $\mathcal{P}_s(\ell_1)$ *coincides with* $\varphi_{(x,\lambda)}$ *for some* $\lambda \in \mathbb{C}$ *and* x *satisfying (4.7).*

Proof. Consider the exponential type function $\mathcal{G}(\varphi)$ given by (4.3) and the corresponding sequence $x = (\frac{-1}{a_n})$.

If $x \in \ell_1$, then according to (4.4), $\varphi = \psi_\lambda \star \delta_x$. If $x \notin \ell_1$, then $\mathcal{G}(\varphi)(t) = e^{\lambda t} \prod_{n=1}^{\infty}(1+tx_n)e^{-tx_n}$ and, on the other hand, $\mathcal{G}(\varphi)(t) = \sum_{n=0}^{\infty} \varphi(G_n)t^n$.

We have

$$\left(e^{\lambda t}\prod_{n=1}^{\infty}(1+tx_n)e^{-tx_n}\right)'_t = \lambda e^{\lambda t}\prod_{n=1}^{\infty}(1+tx_n)e^{-tx_n} +$$

$$+ e^{\lambda t}\left(-tx_1^2 e^{-tx_1}\prod_{n\neq 1}(1+tx_n)e^{-tx_n} - tx_2^2 e^{-tx_2}\prod_{n\neq 2}(1+tx_n)e^{-tx_n} - \ldots\right) =$$

$$= \lambda e^{\lambda t}\prod_{n=1}^{\infty}(1+tx_n)e^{-tx_n} - te^{\lambda t}\sum_{k=1}^{\infty} x_k^2 e^{-tx_k}\prod_{n\neq k}(1+tx_n)e^{-tx_n}$$

and

$$\left(e^{\lambda t}\prod_{n=1}^{\infty}(1+tx_n)e^{-tx_n}\right)'\Bigg|_{t=0} = \lambda.$$

So by the uniqueness of the Taylor coefficients, $\varphi(G_1) = \varphi(F_1) = \lambda$.

Now

$$\left(e^{\lambda t}\prod_{n=1}^{\infty}(1+tx_n)e^{-tx_n}\right)''_t =$$

$$= \left(\lambda e^{\lambda t}\prod_{n=1}^{\infty}(1+tx_n)e^{-tx_n}\right)'_t - \left(te^{\lambda t}\sum_{k=1}^{\infty}x_k^2e^{-tx_k}\prod_{n\neq k}(1+tx_n)e^{-tx_n}\right)'_t =$$

$$= \lambda^2e^{\lambda t}\prod_{n=1}^{\infty}(1+tx_n)e^{-tx_n} - \lambda te^{\lambda t}\sum_{k=1}^{\infty}x_k^2e^{-tx_k}\prod_{n\neq k}(1+tx_n)e^{-tx_n} -$$

$$-e^{\lambda t}\sum_{k=1}^{\infty}x_k^2e^{-tx_k}\prod_{n\neq k}(1+tx_n)e^{-tx_n} - t\left(e^{\lambda t}\sum_{k=1}^{\infty}x_k^2e^{-tx_k}\prod_{n\neq k}(1+tx_n)e^{-tx_n}\right)'_t$$

and

$$\left.\left(e^{\lambda t}\prod_{n=1}^{\infty}(1+tx_n)e^{-tx_n}\right)''\right|_{t=0} = \lambda^2 - \sum_{k=1}^{\infty}x_k^2.$$

Then

$$\varphi(G_2) = \frac{\lambda^2 - F_2(x)}{2} = \frac{(\varphi(F_1))^2 - F_2(x)}{2}.$$

On the other hand,

$$\varphi(G_2) = \frac{\varphi(F_1^2) - \varphi(F_2)}{2}$$

and we have

$$\varphi(F_2) = F_2(x).$$

Now using induction, we obtain the required result. □

4.2 A multiplicative convolution on the spectra of algebras of symmetric analytic functions

4.2.1 The multiplicative convolution

Definition 4.4. *Let $x, y \in \ell_p$, $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$. We define the multiplicative intertwining of x and y, $x \diamond y$, as the resulting sequence of ordering the set $\{x_iy_j : i, j \in \mathbb{N}\}$ with one single index in some fixed order.*

Note that for further consideration the order of numbering does not matter.

Proposition 4.10. *For arbitrary $x, y \in \ell_p$ we have*

1. $x \diamond y \in \ell_p$ *and* $\|x \diamond y\| = \|x\|\|y\|$;

2. $F_k(x \diamond y) = F_k(x)F_k(y) \;\; \forall k \geq \lceil p \rceil$.

3. *If P is an n-homogeneous symmetric polynomial on ℓ_p and y is fixed, then the function $x \mapsto P(x \diamond y)$ is n-homogeneous.*

Proof. It is clear that $\|x \diamond y\|^p = \sum_{i,j} |x_i y_j|^p = \sum_i |x_i|^p \sum_i |y_j|^p = \|x\|^p \|y\|^p$. Also $F_k(x \diamond y) = \sum_{i,j}(x_i y_j)^k = \sum_i x_i^k \sum_j y_j^k = F_k(x)F_k(y)$. Statement (3) follows from the equality $\lambda(x \diamond y) = (\lambda x) \diamond y$. □

Given $y \in \ell_p$, the mapping $x \in \ell_p \overset{\pi_y}{\to} (x \diamond y) \in \ell_p$ is linear and continuous because of Proposition 4.10. Therefore if $f \in \mathcal{H}_{bs}(\ell_p)$, then $f \circ \pi_y \in \mathcal{H}_{bs}(\ell_p)$ because $f \circ \pi_y$ is analytic and bounded on bounded sets and clearly $f(\sigma(x) \diamond y) = f(x \diamond y)$ for every permutation $\sigma \in \mathcal{G}$. Thus if we denote $M_y(f) = f \circ \pi_y$, M_y is a composition operator on $\mathcal{H}_{bs}(\ell_p)$, that we will call the *multiplicative convolution operator*. Notice as well that $M_y = M_{\sigma(y)}$ for every permutation $\sigma \in \mathcal{G}$ and that $M_y(F_k) = F_k(y)F_k \;\; \forall k \geq \lceil p \rceil$.

Proposition 4.11. *For every $y \in \ell_p$ the multiplicative convolution operator M_y is a continuous homomorphism on $\mathcal{H}_{bs}(\ell_p)$.*

Note that in particular, if f_n is an n-homogeneous continuous polynomial, then $\|M_y(f_n)\| \leq \|f_n\|\|y\|^n$. And also that for $\lambda \in \mathbb{C}$, $M_{\lambda y}(f_n) = \lambda^n M_y(f_n)$, because $\pi_{\lambda y}(x) = \lambda \pi(x)$. Analogously, $M_{y+z}(f_n) = f_n \circ (\pi_y + \pi_z)$, because $\pi_{y+z} = \pi_y + \pi_z$. Therefore the mapping $y \in \ell_p \mapsto M_y(f_n)$ is an n-homogeneous continuous polynomial.

Recall that the *radius function* $R(\phi)$ of a complex homomorphism $\phi \in \mathcal{M}_{bs}(\ell_p)$ is the infimum of all r such that ϕ is continuous with respect to the norm of uniform convergence on the ball rB_{ℓ_p}, that is $|\phi(f)| \leq C_r \|f\|_r$. It is known that

$$R(\phi) = \limsup_{n\to\infty} \|\phi_n\|^{1/n},$$

where ϕ_n is the restriction of ϕ to $\mathcal{P}_s(^n\ell_p)$ and $\|\phi_n\|$ is its corresponding norm (see [11]).

Proposition 4.12. *For every $\theta \in \mathcal{H}_{bs}(\ell_p)'$ and every $y \in \ell_p$ the radius-function of the continuous homomorphism $\theta \circ M_y$ satisfies*

$$R(\theta \circ M_y) \leq R(\theta)\|y\|$$

and for fixed $f \in \mathcal{H}_{bs}(\ell_p)$ the function $y \mapsto \theta \circ M_y(f)$ also belongs to $\mathcal{H}_{bs}(\ell_p)$.

Proof. For a given $y \in \ell_p$, let $(\theta \circ M_y)_n$ (respectively, θ_n) be the restriction of $\theta \circ M_y$ (respectively, θ) to the subspace of n-homogeneous symmetric polynomials. Then we have

$$\|(\theta \circ M_y)_n\| = \sup_{\|f_n\|\leq 1} \left| \theta_n \left(\frac{M_y(f_n)}{\|y\|^n} \right) \right| \|y\|^n \leq \|\theta_n\|\|y\|^n.$$

So

$$R(\theta \circ M_y) \leq \limsup_{n\to\infty}(\|\theta_n\|\|y\|^n)^{1/n} = R(\theta)\|y\|.$$

Since the terms in the Taylor series of the function $y \mapsto \theta \circ M_y(f)$ are $y \mapsto \theta \circ M_y(f_n)$, where (f_n) are the terms in the Taylor series of f, the formula above proves the second statement. □

Using the multiplicative convolution operator we can introduce a multiplicative convolution on $\mathcal{H}_{bs}(\ell_p)'$.

Definition 4.5. *Let $f \in \mathcal{H}_{bs}(\ell_p)$ and $\theta \in \mathcal{H}_{bs}(\ell_p)'$. The multiplicative convolution $\theta\Diamond f$ is defined as*

$$(\theta\Diamond f)(x) = \theta[M_x(f)] \text{ for every } x \in \ell_p .$$

We have by Proposition 4.12, that $\theta\Diamond f \in \mathcal{H}_{bs}(\ell_p)$.

Definition 4.6. *For arbitrary $\varphi, \theta \in \mathcal{H}_{bs}(\ell_p)'$ we define their multiplicative convolution $\varphi\Diamond\theta$ according to*

$$(\varphi\Diamond\theta)(f) = \varphi(\theta\Diamond f) \text{ for every } f \in \mathcal{H}_{bs}(\ell_p).$$

For the evaluation homomorphism at y, δ_y, observe that

$$(\delta_y\Diamond f)(x) = \delta_y(M_x(f)) = (f \circ \pi_x)(y) = f(\pi_x(y)) = f(x \diamond y) = f(\pi_y(x)) = M_y(f)(x).$$

Hence, $\delta_x\Diamond\delta_y = \delta_{x\diamond y}$.

Proposition 4.13. *If $\varphi, \theta \in M_{bs}(\ell_p)$, then $\varphi\Diamond\theta \in \mathcal{M}_{bs}(\ell_p)$.*

Proof. From the multiplicativity of M_y it follows that $\varphi\Diamond\theta$ is a character. Using arguments as in Proposition 4.12, we have that

$$R(\varphi\Diamond\theta) \leq R(\varphi)R(\theta).$$

Hence $\varphi\Diamond\theta \in \mathcal{M}_{bs}(\ell_p)$. □

Theorem 4.4. *1. If $varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$, then $(\varphi\Diamond\theta)(F_k) = \varphi(F_k)\theta(F_k)\ \forall k \geq \lceil p \rceil$.*

2. The semigroup $(\mathcal{M}_{bs}(\ell_p), \Diamond)$ is commutative and the evaluation at $x_0 = (1, 0, 0, \ldots)$, δ_{x_0}, is its identity.

Proof. Let us take firstly $x, y \in \ell_p$ and $\delta_x, \delta_y \in \mathcal{M}_{bs}(\ell_p)$ the corresponding point evaluation homomorphisms. Then $(\delta_x\Diamond\delta_y)(F_k) = F_k(x\diamond y) = \sum x_i^k y_j^k = F_k(x)F_k(y)$.

Now let $\varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$. Then

$$(\theta\Diamond F_k)(x) = \theta(M_x(F_k)) = \theta(F_k(x)F_k) = F_k(x)\theta(F_k).$$

So,

$$(\varphi\Diamond\theta)(F_k) = \varphi(F_k\theta(F_k)) = \varphi(F_k)\theta(F_k).$$

Exchanging φ and θ, we get that

$$(\theta\Diamond\varphi)(F_k) = \theta(F_k)\varphi(F_k) = (\varphi\Diamond\theta)(F_k),$$

whence it follows that the multiplicative convolution is commutative for F_k. Since every symmetric polynomial is an algebraic combination of polynomials F_k and each function of $\mathcal{H}_{bs}(\ell_p)$ is uniformly approximated by symmetric polynomials, then the convolution operation is commutative. Analogously, $\Diamond$ is associative since

$$(\psi\Diamond(\varphi\Diamond\theta))(F_k) = \psi(F_k)\varphi(F_k)\theta(F_k) = ((\psi\Diamond\varphi)\Diamond\theta))(F_k).$$

Also from Theorem 4.4 it follows that the cancelation rule holds and δ_{x_0}, where $x_0 = (1, 0, 0, \ldots)$, is the identity. $\square$

In Section 1 it was constructed a family $\{\psi_\lambda : \lambda \in \mathbb{C}\}$ of elements of the set $\mathcal{M}_{bs}(\ell_p)$ such that $\psi_\lambda(F_p) = \lambda$ and $\psi_\lambda(F_k) = 0$ for $k > p$. Let us recall the construction: Consider for each $n \in \mathbb{N}$, the element $v_n = \left(\frac{\lambda}{n}\right)^{1/p}(e_1 + \cdots + e_n)$ for which $F_p(v_n) = \lambda$, and $\lim_n F_j(v_n) = 0$ for $j > p$. Now, the sequence $\{\delta_{v_n}\}$ has an accumulation point ψ_λ in the spectrum for the pointwise convergence topology for which $\psi_\lambda(F_k) = 0$ for $k > p$ that prevents ψ_λ from being invertible because of Theorem 4.4.

Remark 4.4. *The semigroup $(\mathcal{M}_{bs}(\ell_p), \Diamond)$ is not a group.*

Recall that for any $\varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$ and $f \in \mathcal{H}_{bs}(\ell_p)$, the symmetric convolution $\varphi \star \theta$ was defined in [11] as follows:

$$(\varphi \star \theta)(f) = \varphi(T_y^s(f)),$$

where $T_y^s(f)(x) = f(x \bullet y)$.

Proposition 4.14. *For arbitrary $\theta, \varphi, \psi \in \mathcal{M}_{bs}(\ell_p)$ the following equality holds:*

$$\theta\Diamond(\varphi \star \psi) = (\theta\Diamond\varphi) \star (\theta\Diamond\psi).$$

Proof. Indeed, using Theorem 4.4 and Theorem 4.1, we obtain that

$$((\theta\Diamond\varphi) \star (\theta\Diamond\psi))(F_k) = (\theta\Diamond\varphi)(F_k) + (\theta\Diamond\psi)(F_k) = \theta(F_k)\varphi(F_k) + \theta(F_k)\psi(F_k) =$$
$$= \theta(F_k)(\varphi(F_k) + \psi(F_k)) = \theta(F_k)(\varphi \star \psi)(F_k) = \theta\Diamond(\varphi \star \psi)(F_k).$$

$\square$

Corollary 4.5. *The set $(\mathcal{M}_{bs}(\ell_p), \Diamond, \star)$ is a commutative semi-ring with identity.*

A linear operator $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ is called a *multiplicative convolution operator* if there exists $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta \Diamond f$.

Proposition 4.15. *A continuous homomorphism $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ is a multiplicative convolution operator if and only if it commutes with all multiplicative operators $M_y,\ y \in \ell_p$.*

Proof. Suppose that there exists $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta \Diamond f$. Fix $y \in \ell_p$. Then

$$[T \circ M_y](f)(x) = [T(M_y(f))](x) = [\theta \Diamond M_y(f)](x) = \theta[M_x(M_y(f)] = \theta[M_{x \diamond y}(f)].$$

On the other hand,

$$[M_y \circ T](f)(x) = [M_y(Tf)](x) = Tf(x \diamond y) = (\theta \Diamond f)(x \diamond y) = \theta[M_{x \diamond y}(f)].$$

Conversely, for $x_0 = (1, 0, 0, \ldots)$ we put $\theta = \delta_{x_0} \circ T$. Clearly, $\theta \in \mathcal{M}_{bs}(\ell_p)$. Let us check that $Tf = \theta \Diamond f$. Indeed, $(\theta \Diamond f)(x) = \theta[M_x(f)] = [T(M_x(f))](x_0) = [M_x(T(f))](x_0) = Tf(x_0 \diamond x) = Tf(x)$. □

Theorem 4.5. *A homomorphism $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ such that $T(F_k) = a_k F_k$, $k \geq \lceil p \rceil$, is continuous if and only if there exists $\varphi \in \mathcal{M}_{bs}(\ell_p)$ such that $\varphi(F_k) = a_k,\ \ k \geq \lceil p \rceil$.*

Proof. Let $\varphi \in \mathcal{M}_{bs}(\ell_p)$ with $\varphi(F_k) = a_k$. Then

$$(\varphi \Diamond F_k)(x) = \varphi(M_x(F_k)) = \varphi(F_k F_k(x)) = a_k F_k(x).$$

Thus if $Tf = \varphi \Diamond f$, T defines a continuous homomorphism and $T(F_k) = a_k F_k$.

Conversely, if such homomorphism T is continuous, then clearly T commutes with all M_y. By Proposition 4.15 it has the form $T(f) = \varphi \Diamond f$ for some $\varphi \in \mathcal{M}_{bs}(\ell_p)$. Thus, $T(F_k) = \varphi(F_k) F_k(x) = a_k F_k$, hence $\varphi(F_k) = a_k$. □

Proposition 4.16. *The identity is the only operator on $\mathcal{H}_{bs}(\ell_p)$ that is both a convolution and a multiplicative convolution operator.*

Proof. Let $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ be such an operator. Then there is $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta \star f$ and T commutes with all M_y. In particular we have for all polynomials $F_k,\ \ k \geq \lceil p \rceil$, that

$$M_y(TF_k) = M_y(\theta \star F_k) = M_y(\theta(F_k) + F_k) = \theta(F_k) + M_y(F_k) = \theta(F_k) + F_k(y)F_k$$

and

$$T(M_y(F_k)) = T(F_k(y)F_k) = F_k(y)\theta \star F_k = F_k(y)(\theta(F_k) + F_k)$$

coincide. Hence $\theta(F_k) = F_k(y)\theta(F_k)$, that leads to $\theta(F_k) = 0$, that in turn shows that $\theta = \delta_0$, or in other words, $T = Id$. □

4.2.2 The case of ℓ_1

In this section we consider the algebra $\mathcal{H}_{bs}(\ell_1)$. In addition to the basis $\{F_n\}$, this algebra has a different natural basis that is given by the sequence $\{G_n\}$:

$$G_n(x) = \sum_{k_1<\cdots<k_n}^{\infty} x_{k_1} \cdots x_{k_n}$$

and $G_0 := 1$.

According to Lemma 4.2, $\|G_n\| = \frac{1}{n!}$, so it follows that for every $t \in \mathbb{C}$, the function $\sum_{n=0}^{\infty} t^n G_n \in \mathcal{H}_{bs}(\ell_1)$ and that such series converges uniformly on bounded subsets of ℓ_1. Thus if $\varphi \in \mathcal{M}_{bs}(\ell_1)$,

$$\mathcal{G}(\varphi)(t) = \varphi\left(\sum_{n=0}^{\infty} t^n G_n\right) = \sum_{n=0}^{\infty} t^n \varphi(G_n)$$

is well defined and as it was shown in Proposition 4.7, the mapping

$$\varphi \in \mathcal{M}_{bs}(\ell_1) \overset{\mathcal{G}}{\to} \mathcal{G}(\varphi) \in H(\mathbb{C})$$

is one-to-one and ranges into the subspace of entire functions of exponential (finite) type. Whether $\mathcal{G}$ is an onto mapping was an open question there that we answer negatively here, see Corollary 4.6, using the multiplicative convolution we are dealing with.

Observe that for every $a \in \mathbb{C}$,

$$\left(\delta_{(a,0,0,\dots)} \Diamond \sum_{n=0}^{\infty} t^n G_n\right)(x) = M_x\left(\sum_{n=0}^{\infty} t^n G_n\right)(a,0,0,\dots) =$$
$$= \left(\sum_{n=0}^{\infty} t^n G_n\right)(x \diamond (a,0,0,\dots)) = \sum_{n=0}^{\infty} t^n G_n(ax) = \sum_{n=0}^{\infty} t^n a^n G_n(x).$$

Therefore,

$$\mathcal{G}(\varphi \Diamond \delta_{(a,0,0,\dots)})(t) = \varphi\left(\sum_{n=0}^{\infty} t^n a^n G_n\right) = \sum_{n=0}^{\infty} t^n a^n \varphi(G_n).$$

According to Theorem 4.1(a), $\delta_{(a,0,0,\dots)} \star \delta_{(b,0,0,\dots)} = \delta_{(a,b,0,0,\dots)}$, consequently using Proposition 4.14 and Theorem 4.3(2),

$$\mathcal{G}(\varphi \Diamond \delta_{(a,b,0,0,\dots)})(t) = \mathcal{G}((\varphi \Diamond \delta_{(a,0,0,\dots)}) \star (\varphi \Diamond \delta_{(b,0,0,\dots)}))(t) =$$
$$= \mathcal{G}(\varphi \Diamond \delta_{(a,0,0,\dots)})(t)\mathcal{G}(\varphi \Diamond \delta_{(b,0,0,\dots)})(t) = \sum_{n=0}^{\infty} t^n a^n \varphi(G_n) \cdot \sum_{n=0}^{\infty} t^n b^n \varphi(G_n).$$

Therefore,

$$\mathcal{G}(\varphi\Diamond\delta_{(x_1,x_2,\ldots,x_m,0,\ldots)})(t) = \prod_{k=1}^{m}\sum_{n=0}^{\infty} t^n x_k^n \varphi(G_n).$$

Further since the sequence $(\delta_{(x_1,x_2,\ldots,x_m,0,\ldots)})_m$ is pointwise convergent to $\delta_{(x_1,x_2,\ldots,x_m\ldots)}$ in $\mathcal{M}_{bs}(\ell_1)$ we have, bearing in mind the commutativity of $\Diamond$, that the sequence $(\varphi\Diamond\delta_{(x_1,x_2,\ldots,x_m,0,\ldots)})_m$ is pointwise convergent to $\varphi\Diamond\delta_{(x_1,x_2,\ldots,x_m\ldots)}$. Thus

$$\mathcal{G}(\varphi\Diamond\delta_x)(t) = \prod_{k=1}^{\infty}\sum_{n=0}^{\infty} t^n x_k^n \varphi(G_n) \quad \text{for } x = (x_1, x_2, \ldots, x_m \ldots) \in \ell_1. \tag{4.8}$$

For the mentioned above family $\{\psi_\lambda : \lambda \in \mathbb{C}\}$, it was shown in Section 1 that $\mathcal{G}(\psi_\lambda)(t) = e^{\lambda t}$. Further, it is easy to see that

1. $\psi_\lambda\Diamond\varphi(F_1) = \lambda\varphi(F_1)$.
2. $\psi_\lambda\Diamond\varphi(F_k) = 0, \quad k > 1$.
3. $\mathcal{G}(\psi_\lambda\Diamond\varphi) = e^{\lambda\varphi(F_1)t}$.

The following theorem might be of interest in Function Theory.

Theorem 4.6. *Let $g(t)$ and $h(t)$ be entire functions of exponential type of one complex variable such that $g(0) = h(0) = 1$. Let $\{a_n\}$ be zeros of $g(t)$ with $\sum_{n=1}^{\infty}\frac{1}{|a_n|} < \infty$ and let $\{b_n\}$ be zeros of $h(t)$ with $\sum_{n=1}^{\infty}\frac{1}{|b_n|} < \infty$. Then there exists a function of exponential type $u(t)$ with zeros $\{a_n b_m\}_{n,m}$, which can be represented as*

$$u(t) = \prod_{k=1}^{\infty}\sum_{n=0}^{\infty}\left(-\frac{1}{a_k}\right)^n h_n(t) = \prod_{k=1}^{\infty}\sum_{n=0}^{\infty}\left(-\frac{1}{b_k}\right)^n g_n(t).$$

Proof. According to Section 1 (text after formula (4.5)), $g(t) = \mathcal{G}(\delta_x)(t)$ and $h(t) = \mathcal{G}(\delta_y)(t)$, where $x, y \in \ell_1$, $x_n = -\frac{1}{a_n}$, $y_n = -\frac{1}{b_n}$. So $u(t) = \mathcal{G}(\delta_x\Diamond\delta_y)(t)$ and using (4.8) we obtain the statement of the theorem. □

We know that for a given sequence of complex numbers $x = (x_1, \ldots, x_n, \ldots)$ such that $x \in \ell_{1+d}$ for every $d > 0$,

$$\limsup_{n\to\infty} n|x_n| < \infty, \quad \limsup_{r\to\infty}\left|\sum_{|x_n|^{-1}<r} x_n\right| < \infty \tag{4.9}$$

(think for instance of $x_n = \frac{(-1)^n}{n}$) and $\lambda \in \mathbb{C}$. Let us denote by $\delta_{(x,\lambda)}$ a homomorphism on the algebra of symmetric polynomials $\mathcal{P}_s(\ell_1)$ of the form

$$\delta_{(x,\lambda)}(F_1) = \lambda, \quad \delta_{(x,\lambda)}(F_k) = \sum_{n=1}^{\infty} x_n^k, \quad k > 1.$$

Recall that according to [31] p.17, $\limsup_{n\to\infty} n|x_n|$ coincides with the so-called *upper density* of the sequence $(\frac{1}{x_n})$ that is defined by $\limsup_{r\to\infty} \frac{\mathbf{n}(r)}{r}$, where $\mathbf{n}(r)$ denotes the *counting number* of $(\frac{1}{x_n})$, that is, the number of terms of the sequence with absolute value not greater than r.

Proposition 4.17. *(4.9) Let $\varphi \in \mathcal{M}_{bs}(\ell_1)$. Then the restriction of φ to $\mathcal{P}_s(\ell_1)$ coincides with $\delta_{(x,\lambda)}$ for some $\lambda \in \mathbb{C}$ and x satisfying (4.9).*

Actually, due to [1, Theorem 1.3] such sequence x is unique up to permutation.

Theorem 4.7. *There is no continuous character of the form $\delta_{(v,\lambda)}$ in the space $\mathcal{M}_{bs}(\ell_1)$, where*

$$v = \left\{c_1, \frac{c_2}{2}, \dots, \frac{c_n}{n}, \dots\right\},$$

and $|c_k| = 1$ for each k.

Proof. Assume otherwise, i.e., $\delta_{(v,\lambda)}$ is the restriction of some $\varphi \in \mathcal{M}_{bs}(\ell_1)$. Then by Theorem 4.4,

$$(\varphi\Diamond\varphi)(F_k) = \varphi(F_k)^2 = \left(\sum_{n=1}^{\infty} v_n^k\right)^2 = \left(\sum_{n=1}^{\infty} v_n^k\right)\left(\sum_{m=1}^{\infty} v_m^k\right) = \sum_{n,m=1}^{\infty} (v_n v_m)^k.$$

Therefore the sequence $(v_n v_m)_{n,m} = v \diamond v := s$, is, up to permutation, the one appearing in Proposition 4.9, so it must satisfy condition (4.9), that is, the sequence of the inverses has finite upper density.

Denote by $d(m)$ the number of divisors of a positive integer m. Then in the sequence $|s|$ of absolute values each number with absolute value $1/m$ can be found $d(m)$ times. So $|s|$ can be rearranged, if necessary, in the form

$$\Big(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \dots, \underbrace{\frac{1}{m}, \dots \frac{1}{m}}_{d(m)}, \dots\Big).$$

In particular, the index of the last entry of the element with absolute value $\frac{1}{m}$ is $\sum_{n=1}^{m} d(n)$. Hence for the sequence of the inverses and their counting number $\mathbf{n}(m)$, we have $\mathbf{n}(m) = \sum_{n=1}^{m} d(n)$. From Number Theory [2, Theorem 3.3] it is known that

$$\sum_{n=1}^{m} d(n) = m \ln m + 2(\gamma - 1)m + O(\sqrt{m}),$$

where γ is the Euler constant. So we are led to a contradiction because

$$\limsup_{m\to\infty} \frac{\mathbf{n}(m)}{m} \geq \limsup_{m\to\infty} \frac{m \ln m}{m} = \limsup_{m\to\infty} \ln m = \infty.$$

□

Corollary 4.6. *There is a function of exponential type $g(t)$ for which there is no character $\varphi \in \mathcal{M}_{bs}(\ell_1)$ such that $\mathcal{G}(\varphi)(t) = g(t)$.*

Proof. It is enough to take a function of exponential (finite) type whose zeros are the elements of the sequence

$$\left\{\frac{1}{v_n}\right\} = \{-1, 2, \dots (-1)^n n, \dots\}.$$

Such is, for example, the function $g(t) = \prod_{n=1}^{\infty} \left(1 + (-1)^n \frac{t}{n}\right) \exp\left((-1)^n \frac{t}{n}\right)$. □

Every $\varphi \in \mathcal{M}_{bs}(\ell_1)$ is determined by the sequence $(\varphi(F_m))$, that verifies the inequality $\limsup_n |\varphi(F_m)|^{1/m} \le R(\varphi)$ because $\|F_m\| \le 1$. As a byproduct of Theorem 4.7, we notice that the condition $\limsup_m |a_m|^{1/m} < +\infty$, does not guarantee that there is $\varphi \in \mathcal{M}_{bs}(\ell_1)$ such that $\varphi(F_m) = a_m$: Indeed, let $a_m = \sum_n \frac{1}{n^m}$ for $m > 1$ and arbitrary a_1. Then the sequence (a_m) is bounded, so $\limsup_m |a_m|^{1/m} \le 1$, and if there existed $\varphi \in \mathcal{M}_{bs}(\ell_1)$ such that $\varphi(F_m) = a_m$, it would mean that for the sequence $x := (\frac{1}{n})$, $\varphi(F_m) = \sum_n \frac{1}{n^m}$, so $\delta_{(x,a_1)} = \varphi_{|_{P_s(\ell_1)}}$.

Question 4.1. *Can each element of $\mathcal{M}_{bs}(\ell_1)$ be represented as an entire function of exponential type with zeros $\{a_n\}_{n=1}^{\infty}$ such that either $\{a_n\} = \emptyset$ or $\sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty$?*

4.3 Description of the spectrum of $\mathcal{H}_{bs}(\ell_1)$

Let us remaind that in Corollary 4.4 and Proposition 4.9 is obtained the following result:

Theorem 4.8. *Let $\varphi \in \mathcal{M}_{bs}(\ell_1)$, then there is a number $\lambda \in \mathbb{C}$ and an element $x = (x_1, \dots, x_n, \dots) \in \ell_p$ for all $p > 1$ such that*

$$\mathcal{G}(\varphi)(t) = \begin{cases} e^{\lambda t} \prod_{n=1}^{\infty} (1 + t x_n), & \text{if } x \in \ell_1, \\ e^{\lambda t} \prod_{n=1}^{\infty} (1 + t x_n) e^{-t x_n}, & \text{if } x \notin \ell_1. \end{cases}$$

If $x \in \ell_1$, then $\varphi = \psi_\lambda \star \delta_x$. If $x \notin \ell_1$, then $\varphi = \delta_{(x,\lambda)}$, where for every $k > 1$

$$\delta_{(x,\lambda)}(F_1) = \lambda \quad \text{and} \quad \delta_{(x,\lambda)}(F_k) = \sum_{n=1}^{\infty} x_n^k.$$

Note that no characters of the form $\delta_{(x,\lambda)}$ were found for $x \notin \ell_1$. The goal of the current section is to show that any $\varphi \in \mathcal{M}_{bs}(\ell_1)$ is of the form $\varphi = \psi_\lambda \star \delta_x$. Thus providing a complete description of the spectrum $\mathcal{M}_{bs}(\ell_1)$.

It is well known that every linear continuous functional φ on the space of all entire functions of one complex variable $H(\mathbb{C})$ can be represented by a sequence $\varphi = (\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots), \quad \varphi_j \in \mathbb{C}$ such that

$$\limsup_{n\to\infty} |\varphi_n|^{\frac{1}{n}} =: R(\varphi) < \infty \tag{4.10}$$

and that every sequence of complex numbers satisfying (4.10) generates a linear continuous functional φ such that

$$\varphi(f) = \sum_{n=0}^{\infty} \varphi_n c_n,$$

where $f = \sum_{n=0}^{\infty} c_n t^n \in H(\mathbb{C})$. Equivalently, using the Borel transform $\varphi \rightsquigarrow \varphi(e^{\lambda t}) = \sum_{n=0}^{\infty} \frac{\varphi_n \lambda^n}{n!}$ we can identify the functional $\varphi \in H(\mathbb{C})'$ with the function of exponential type $\gamma_\varphi(\lambda) := \varphi(e^{\lambda t})$ and the map $\varphi \rightsquigarrow \gamma_\varphi$ is a topological isomorphism of $H(\mathbb{C})'$ onto the space of analytic functions of exponential type $Exp(\mathbb{C})$. Note that $R(\varphi)$ is the exponential type of γ_φ. This approach works for entire functions of several variables and even for nuclear entire functions on a Banach space (see [23]).

For the case of all entire functions of bounded type we have the following situation. Accordingly to [3] if $\varphi \in H_b(X)'$ and φ_n is its restriction to the Banach space of all n-homogeneous polynomials $\mathcal{P}(^nX)$, then

$$\limsup_{n\to\infty} \|\varphi_n\|^{\frac{1}{n}} < \infty. \tag{4.11}$$

Conversely, if a sequence of functionals (φ_n), $\varphi_n \in \mathcal{P}(^nX)'$, satisfies (4.11), then it generates a functional φ such that

$$\varphi(f) = \sum_{n=0}^{\infty} \varphi_n(f_n),$$

where $f = \sum_{n=0}^{\infty} f_n$ is the Taylor expansion series of $f \in H_b(X)$.

The limit appearing in (4.11) coincides [3] with $R(\varphi)$, the so called *radius function* of φ, which is defined to be to infimum of $r > 0$ such that φ is continuous with respect to the norm of uniform convergence in $H_b(X)$ on the ball of radius r centered at the origin.

In [11] it is shown that all mentioned results about $H_b(X)$ are also true for the algebra $\mathcal{H}_{bs}(\ell_p)$. By $R_p(\varphi)$ we denote the radius of the element $\varphi \in \mathcal{M}_{bs}(\ell_p)$.

Lemma 4.3. *Let $\delta_\lambda = \delta_{(\lambda,0,0,\ldots)}$, $\lambda \in \mathbb{C}$ and $\phi \in \mathcal{M}_{bs}(\ell_1)$. Then*

$$R_1(\delta_\lambda \Diamond \phi) = |\lambda| R_1(\phi).$$

Proof. Clearly, $F_k(\delta_\lambda \Diamond \phi) = F_k(\delta_\lambda)F_k(\phi) = \lambda^k F_k(\phi)$. Let now P_n be an arbitrary n-homogeneous symmetric polynomial on ℓ_1. Then

$$P_n(x) = \sum_{k_1+2k_2+\ldots+nk_n=n} a_{k_1k_2\ldots k_n} F_1^{k_1}(x)F_2^{k_2}(x)\ldots F_n^{k_n}(x)$$

and so $P_n(\delta_\lambda \Diamond \phi) = \lambda^n P_n(\phi)$. Hence,

$$R_1(\delta_\lambda \Diamond \phi) = \limsup_{\|P_n\|\le 1} |\lambda^n \phi(P_n)|^{1/n} = |\lambda| R_1(\phi).$$

□

The following lemma is probably known.

Lemma 4.4. *For every* $x = (x_1, x_2, \ldots) \in \ell_1$,

$$\lim_{p\to\infty} \|x\|_p = \|x\|_\infty = \max_n |x_n|.$$

Proof. In [25, pp. 28-29, Theorem 20] it is proved that for any finite set of positive numbers $\{a_1, \ldots, a_m\}$ one has

$$\lim_{p\to\infty} \left(\sum_{n=1}^m a_n^p\right)^{1/p} = \max_n a_n.$$

Given $\varepsilon > 0$, let us take a number m so that $\max_{n\in\mathbb{N}} |x_n| = \max_{n\le m} |x_n|$ and $\sum_{n=m+1}^\infty |x_n| < \varepsilon$. Then

$$\|x\|_p \le \left(\sum_{n=1}^m |x_n|^p\right)^{1/p} + \left(\sum_{n=m+1}^\infty |x_n|^p\right)^{1/p} \le$$
$$\le \left(\sum_{n=1}^m |x_n|^p\right)^{1/p} + \sum_{n=m+1}^\infty |x_n| \le \left(\sum_{n=1}^m |x_n|^p\right)^{1/p} + \varepsilon.$$

Hence $\limsup_p \|x\|_p \le \limsup_p \|(x_1, \ldots, x_m)\|_p + \epsilon \le \|x\|_\infty + \varepsilon$, and therefore, $\limsup_p \|x\|_p \le \|x\|_\infty + \varepsilon$. So, $\limsup_p \|x\|_p \le \|x\|_\infty$.

On the other hand, $\|x\|_p \ge \|x\|_\infty$ and so $\liminf_p \|x\|_p \ge \|x\|_\infty$. Thus, the seeked limit exists and $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$. □

Theorem 4.9. *For every* $x \in \ell_1$,

$$R_1(\delta_x) = \|x\|_1.$$

Proof. Clearly, $R_1(\delta_x) \leq \|x\|_1$. Suppose that there exists $x \in \ell_1$ with $\|x\|_1 = 1$ and $R_1(\delta_x) = c < 1$. If x has just one nonzero coordinate x_m, then $|x_m| = 1$ and $R_1(\delta_x) \geq |F_m(x)|^{1/m} = 1$. Thus the number of nonzero coordinates of x is greater than 1 and the modulus of all coordinates of x are less than 1. Let $b = \max_n |x_n|$. Clearly, $0 < b < 1$. Let λ be a number such that

$$1 < \lambda < \min\left(\frac{1}{c}, \frac{1}{b}\right).$$

Such a number always exists because both $1/c$ and $1/b$ are greater than 1. Let $z = \lambda x$. Then z has the following properties:

1. $\|z\|_1 > 1$ because $\|x\|_1 = 1$ and $\lambda > 1$.
2. $R_1(\delta_z) = R_1(\delta_{\lambda x}) = R_1(\delta_\lambda \Diamond \delta_x) = \lambda c < 1$ by Lemma 4.3.
3. $|z_n| < 1$ for every coordinate z_n of z because $\lambda b < 1$.

So there exists a number n such that $\sum_{k=1}^n |z_k| > 1 + \varepsilon$ for some $\varepsilon > 0$. Since

$$\sum_{k=1}^n |z_k|^t, \quad t > 1$$

is continuous, there is some $t = q > 1$ such that $\|z\|_q > 1$. On the other hand, there is $r > q$ such that $\|z\|_r < 1$. Indeed, since $\lim_{p\to\infty} \|z\|_p = \|z\|_\infty < 1$ it follows that $\|z\|_r < 1$ for some $q < r < \infty$.

Let

$$w_n(z) = \delta_z \star (\delta_z \Diamond \delta_z) \star \ldots \star (\delta_z \Diamond \overset{n}{\ldots} \Diamond \delta_z) = \bigstar_{i=1}^{n} \delta_z^{\Diamond i} \in \mathcal{M}_{bs}(\ell_1).$$

From properties of the operations "$\star$" and "$\Diamond$" and properties of the radius function (Section 1, 2) it follows that

$$R_1(w_n) \leq \sum_{k=1}^n (R_1(\delta_z))^k \leq \frac{R_1(\delta_z)}{1 - R_1(\delta_z)} < \infty.$$

Since the set of characters $\{\phi \in \mathcal{M}_{bs}(\ell_1) \colon R_1(\phi) \leq C\}$ for some $C > 0$ is pointwise bounded and closed, it is a compact subset in $\mathcal{M}_{bs}(\ell_1)$ for the Gelfand topology, i. e., the weak* topology. Thus the sequence $\{w_n\}$ has a limit-point $w_z \in \mathcal{M}_{bs}(\ell_1)$. According to Section 1, the generating function, associated with the basis $\{G_n\}$

$$\mathcal{G}(w_z)(t) = \sum_{n=0}^{\infty} w_z(G_n) t^n, \quad t \in \mathbb{C}$$

is a function of exponential type. Let $\{\xi_n\}$ be the zero-set of the function. By Section 1, $\xi_n \neq 0$, $n \in \mathbb{N}$ and if $y_n = -1/\xi_n$, then $(y_n) \in \ell_p$ for every $p > 1$ and

$$w_z(F_k) = \sum_{n=1}^{\infty} y_n^k, \quad k > 1$$

(see Theorem 4.8). On the other hand,

$$w_z(F_k) = \lim_{j\to\infty} \mathop{\bigstar}_{m=1}^{n_j} \delta_z^{\diamond m}(F_k) = \lim_{j\to\infty} \sum_{m=1}^{n_j} F_k(z\diamond \overset{m}{\cdots} \diamond z).$$

Consider the sequence $u = (u_1, \ldots, u_n, \ldots)$ arising according to the usual principal diagonals arrangement of the $\infty \times \infty$ matrix

$$\begin{pmatrix} z \\ z \diamond z \\ \cdots \\ z\diamond \overset{m}{\cdots} \diamond z \\ \cdots \end{pmatrix}.$$

To see that u belongs to ℓ_r, consider any partial sum $\sum_1^s |u_i|^r$; such sum arises from elements in the $s \times s$ principal minor, thus

$$\|(u_1, \ldots, u_s)\|_r \leq \sum_{i=1}^{s} \|z\diamond \overset{i}{\cdots} \diamond z\|_r = \sum_{i=1}^{s} \|z\|_r^i \leq \frac{\|z\|_r}{1 - \|z\|_r}.$$

Hence $u \in \ell_r$, as wanted.

Moreover, for every $k > r$,

$$F_k(u) = \sum_{n=1}^{\infty} u_n^k = \sum_{m=1}^{\infty} F_k(z\diamond \overset{m}{\cdots} \diamond z) = w_z(F_k).$$

Consequently, for every $k > r$, $\sum_{n=1}^{\infty} y_n^k = \sum_{n=1}^{\infty} u_n^k$. Hence the sequence $y = (y_1, y_2, \ldots)$ is equivalent to u in ℓ_r, and to the sequence w resulting from discarding in u all vanishing terms. That is, the sequences y and w are equal up to permutation. In addition,

$$\|z \bullet z \diamond z \bullet \ldots \bullet z\diamond \overset{m}{\cdots} \diamond z \bullet \|_q^q = \sum_{n=1}^{m} \|z\|_q^{nq},$$

thus neither w nor u may belong to ℓ_q because $\|z\|_q > 1$. So we arrive at a contradiction since y does belong to ℓ_q. □

Corollary 4.7. *For every $x \in \ell_1$ there exists a sequence of n_j-homogeneous polynomials $P_{n_j} \in \mathcal{P}_s(\ell_1)$ such that $\|P_{n_j}\| = 1$ and*

$$\lim_{j\to\infty} |P_{n_j}(x))|^{1/n_j} = \|x\|_{\ell_1}.$$

We say that a subset $\mathcal{P}_0(X)$ of $\mathcal{P}(X)$ is *norming* for X if there exists a sequence of n_j-homogeneous polynomials $P_{n_j} \in \mathcal{P}_0(X)$ such that $\|P_{n_j}\| = 1$ and

$$\lim_{j\to\infty} |P_{n_j}(x))|^{1/n_j} = \|x\|_X.$$

Corollary 4.7 asserts that $\mathcal{P}_s(\ell_1)$ is a norming subset in ℓ_1.

Let us denote by $\mathcal{P}_{ks}(\ell_1)$ the subalgebra in $\mathcal{P}_s(\ell_1)$ generated by the polynomials $\{F_{nk}\}_{n=1}^{\infty}$.

Proposition 4.18. *The algebra $\mathcal{P}_{ks}(\ell_1)$ is norming in ℓ_1 for every positive integer k.*

Proof. Let $x \in \ell_1$ and $k \in \mathbb{N}$. Let us denote by $\alpha_0 = 1, \alpha_1, \ldots, \alpha_{k-1}$ the k-th roots of unity. Set

$$y_x = \left(\frac{x}{k} \bullet \frac{\alpha_1 x}{k} \bullet \cdots \bullet \frac{\alpha_{k-1} x}{k}\right).$$

Then $\|y_x\|_1 = \|x\|_1$. Let $\varepsilon > 0$ and P be a homogeneous polynomial in $\mathcal{P}_s(\ell_1)$ such that $\|P\| = 1$ and $||P(y_x)||^{1/\deg P} - \|y_x\|_1| < \varepsilon$. Such a polynomial must exist by Corollary 4.7. Consider the polynomial

$$Q(x) = P\left(\frac{x}{k} \bullet \frac{\alpha_1 x}{k} \bullet \cdots \bullet \frac{\alpha_{k-1} x}{k}\right).$$

It is easy to check that Q is a homogeneous polynomial in $\mathcal{P}_s(\ell_1)$, with $Q(x) = P(y_x)$, $\deg Q = \deg P$, and $\|Q\| = 1$. On the other hand, if $m = nk + r$ for $r < k$, then

$$F_{nk+r}(y_x) = F_m(y_x) = \sum_{j=0}^{k-1} F_m\left(\frac{\alpha_j x}{k}\right) = F_m\left(\frac{x}{k}\right)\sum_{j=0}^{k-1} \alpha_j^r = \begin{cases} 0, & \text{if } r \neq 0 \\ \frac{F_m(x)}{k^{m-1}}, & \text{if } r = 0. \end{cases}$$

So $Q \in \mathcal{P}_{ks}(\ell_1)$. Hence,

$$||Q(x)||^{1/\deg Q} - \|x\|_1| = ||P(y_x)||^{1/\deg P} - \|y_x\|_1| < \varepsilon.$$

□

Proposition 4.19. *For every $\varphi \in \mathcal{M}_{bs}(\ell_p)$ and $k \in \mathbb{N}$, one has*

$$R_p(\varphi) = \sup_n \|\varphi_n\|^{1/n} = \sup_n \|\varphi_{nk}\|^{1/nk}.$$

Proof. It suffices to observe that since $\mathcal{P}_s({}^n\ell_p)\cdot\mathcal{P}_s({}^m\ell_p)\subset\mathcal{P}_s({}^{n+m}\ell_p)$, we have $\|\varphi_n\|\|\varphi_m\|\leq\|\varphi_{n+m}\|$. In particular, $\|\varphi_{kn}\|=\|\varphi_{k(n-1)+k}\|\geq\|\varphi_{k(n-1)}\|\cdot\|\varphi_k\|\geq\cdots\geq\|\varphi_k\|^n$. Hence

$$\|\varphi_{kn}\|^{1/kn}\geq\|\varphi_k\|^{1/k}. \tag{4.12}$$

Clearly, $R_p(\varphi)\leq\sup_n\|\varphi_n\|^{1/n}$. If for some λ, $R_p(\varphi)<\lambda<\sup_n\|\varphi_n\|^{1/n}$, then there would be some $k\in\mathbb{N}$ such that $\lambda<\|\varphi_k\|^{1/k}$, and hence $\lambda<\|\varphi_{kn}\|^{1/kn}\ \forall n$. Thus, we would get the contradiction $\lambda<\limsup\|\varphi_{nk}\|^{1/nk}=R_p(\varphi)$.

For the second equality, appeal to (4.12). □

Let $1\leq p_1<p_2<\infty$. We know that the canonical embedding $\ell_{p_1}\overset{i}{\to}\ell_{p_2}$ has norm 1. By composition with i (or restriction) we have the mapping $f\in\mathcal{H}_{bs}(\ell_{p_2})\mapsto f\circ i\in\mathcal{H}_{bs}(\ell_{p_1})$ and the corresponding adjoint mapping $\varphi\in\mathcal{M}_{bs}(\ell_{p_1})\mapsto\tilde{\varphi}\in\mathcal{M}_{bs}(\ell_{p_2})$.

Lemma 4.5. *Let $1\leq p_1<p_2<\infty$. For each $\varphi\in\mathcal{M}_{bs}(\ell_{p_1})$, we have $\|\tilde{\varphi}_n\|_{p_2}\leq\|\varphi_n\|_{p_1}$. Consequently, $R_{p_2}(\tilde{\varphi})\leq R_{p_1}(\varphi)$.*

Proof. Notice that $\|Q\circ i\|_{p_1}\leq\|Q\|_{p_2}$, and that $\{Q\circ i:Q\in P_s({}^n\ell_{p_2})\text{ with }\|Q\|_{p_2}\leq 1\}\subset\{P\in P_s({}^n\ell_{p_1}):\|P\|_{p_1}\leq 1\}$. Thus,

$$\begin{aligned}\|\tilde{\varphi}_n\|_{p_2}&=\sup_{\{Q\in\mathcal{P}_s({}^n\ell_{p_2}):\|Q\|_{p_2}\leq 1\}}|\tilde{\varphi}(Q)|=\sup_{\{Q\in\mathcal{P}_s({}^n\ell_{p_2}):\|Q\|_{p_2}\leq 1\}}|\varphi(Q\circ i)|\\&\leq\sup_{\{P\in\mathcal{P}_s({}^n\ell_{p_1}):\|P\|_{p_1}\leq 1\}}|\varphi(P)|=\|\varphi_n\|_{p_1}.\end{aligned}$$

□

Theorem 4.10. *For each $x\in\ell_p$, $1\leq p<\infty$, one has $R_p(\delta_x)=\|x\|_p$.*

Proof. Let us suppose first that $p=m/k\geq 1$, $m,k\in\mathbb{N}$. For a given $x=(x_n)\in\ell_p$ we denote by x^p the sequence $(x_n^p)=(x_n^{m/k})\in\ell_1$, where we take the principal branch of the argument. Notice that

$$\|x\|_p^p=\|x^p\|_1. \tag{4.13}$$

From Proposition 4.18 it follows that for every $\varepsilon>0$ there exists a homogeneous polynomial $Q\in\mathcal{P}_{ks}(\ell_1)$, $\|Q\|=1$, such that $|\|x^p\|_1-|Q(x^p)|^{1/\deg Q}|<\varepsilon$. Since $Q\in\mathcal{P}_{ks}(\ell_1)$, we can write

$$Q=\sum_{km_1+\cdots+skm_s=\deg Q}a_{1,\ldots,s}F_k^{m_1}F_{2k}^{m_2}\cdots F_{sk}^{m_s},$$

and $\deg Q = nk$ for some $n \in \mathbb{N}$. Note that

$$F_{sk}(x^p) = F_{sk}(x^{m/k}) = \sum_{j=1}^{\infty} x_j^{ms} = F_{ms}(x).$$

Thus,

$$Q(x^p) = \sum_{mm_1+\cdots+mkm_s=nm} a_{1,\ldots,s} F_1^{m_1}(x) F_2^{m_2}(x) \cdots F_s^{m_s}(x).$$

Set

$$T_Q(x) = Q(x^p) = \sum_{mm_1+\cdots+mkm_s=nm} a_{1,\ldots,s} F_1^{m_1}(x) F_2^{m_2}(x) \cdots F_s^{m_s}(x).$$

Then T_Q is a homogeneous polynomial on ℓ_p and $\deg T_Q = nm$.

For every nk-homogeneous polynomial $Q \in \mathcal{P}_{ks}({}^{nk}\ell_1)$ we have that $T_Q \in \mathcal{P}_s({}^{mn}\ell_p)$ and

$$\|T_Q\|_p = \sup_{\|z\|_p=1} |Q(z^p)| \leq \sup_{\|y\|_1=1} |Q(y)| = \|Q\|_1.$$

As a consequence,

$$\{T_Q \colon Q \in \mathcal{P}_{ks}({}^{nk}\ell_1) \text{ with } \|Q\|_1 \leq 1\} \subset \{P \in \mathcal{P}_s({}^{mn}\ell_p) \colon \|P\|_p \leq 1\}.$$

Now,

$$\|(\delta_{x^p})_{nk}\|_1 = \sup_{\{Q \in \mathcal{P}_{ks}({}^{nk}\ell_1) \colon \|Q\|_1 \leq 1\}} |Q(x^p)| \leq \sup_{\{P \in \mathcal{P}_s({}^{mn}\ell_p) \colon \|P\|_p \leq 1\}} |P(x)| = \|(\delta_x)_{mn}\|_p.$$

Therefore, $\|(\delta_{x^p})_{nk}\|_1^{k/nk} \leq \|(\delta_x)_{mn}\|_p^{m/mn} \leq R_p(\delta_x)^m$, from where $R_1(\delta_{x^p})^k \leq R_p(\delta_x)^m$. From Proposition 4.19 and Theorem 4.9 we have

$$\|x^p\|_1^k = R_1(\delta_{x^p})^k \leq R_p(\delta_x)^m \leq \|x\|_p^m$$

and so

$$\|x^p\|_1 = R_1(\delta_{x^p}) \leq R_p(\delta_x)^p \leq \|x\|_p^p.$$

Taking into account (4.13), we have $R_p(\delta_x) = \|x\|_p$.

Let now $1 \leq p < \infty$ be an arbitrary irrational number. For every $\epsilon > 0$ we can take a number $r > 0$ such that $p + r \in \mathbb{Q}$, $r < \epsilon$ and using Lemma 4.5 one has

$$R_p(\delta_x) \geq R_{p+r}(\delta_x) = \|x\|_{p+r}.$$

Since it is true for every $\epsilon > 0$, $R_p(\delta_x) \geq \|x\|_p$. So $R_p(\delta_x) = \|x\|_p$ as claimed. $\square$

From Theorem 4.10 we can get an analogue to Proposition 4.18 for the general case ℓ_p.

Proposition 4.20. *The algebra $\mathcal{P}_{ks}(\ell_p)$ is norming in ℓ_p, $1 \le p < \infty$ for every positive integer k.*

Proof. From Theorem 4.10 it follows that $\mathcal{P}_s(\ell_p)$ is norming in ℓ_p.

Let $x \in \ell_p$, $k \in \mathbb{N}$, and $\alpha_0 = 1, \alpha_1, \ldots, \alpha_{k-1}$ be the k-th roots of unity. Denote

$$y_x = \left(\frac{x}{k^{1/p}} \bullet \frac{\alpha_1 x}{k^{1/p}} \bullet \cdots \bullet \frac{\alpha_{k-1} x}{k^{1/p}}\right).$$

Then $\|y_x\|_p = \|x\|_p$. Let $\varepsilon > 0$ and P be a homogeneous polynomial in $\mathcal{P}_s(\ell_p)$ such that $\|P\| = 1$ and $||P(y_x)||^{1/\deg P} - \|y_x\|_1| < \varepsilon$. Consider the polynomial

$$Q(x) = P\left(\frac{x}{k} \bullet \frac{\alpha_1 x}{k} \bullet \cdots \bullet \frac{\alpha_{k-1} x}{k}\right).$$

As in Proposition 4.18, one can check that $Q \in \mathcal{P}_{ks}(\ell_p)$, $Q(x) = P(y_x)$, $\|Q\| = 1$, $\deg Q = \deg P$, and

$$||Q(x)||^{1/\deg Q} - \|x\|_1| = ||P(y_x)||^{1/\deg P} - \|y_x\|_1| < \varepsilon.$$

□

Theorem 4.11. *Let $x \in \ell_p$, $1 \le p < \infty$ and $\|x\| < 1$. Then there exists a sequence $\{z_n\} \subset \ell_p$, $\|z_n\| = 1$ such that for every $f \in \mathcal{H}_{bs}(\ell_p)$,*

$$f(z_n) \to f(x) \quad \text{as} \quad n \to \infty.$$

In other words, point evaluation functionals at points in the unit sphere of ℓ_1 are dense in the set of point evaluation functionals at points in the unit ball in the Gelfand topology.

Proof. Let $y \in \ell_p$ such that $\|y\|^p = 1 - \|x\|^p$. In addition, if p is integer, we assume that $F_p(y) = 0$. Let us consider the following sequence

$$z_1 = x \bullet y$$
$$z_2 = x \bullet \frac{y}{2^{1/p}} \bullet \frac{y}{2^{1/p}}$$
$$\cdots$$
$$z_n = x \bullet \frac{y}{n^{1/p}} \bullet \ldots \bullet \frac{y}{n^{1/p}}$$
$$\cdots$$

Then $\|z_n\| = 1$ and $F_p(z_n) = F_p(x) + F_p(y) = F_p(x)$ if p is integer. Also, for every $k > p$,

$$F_k(z_n) = F_k(x) + \frac{nF_k(y)}{n^{p/k}} \to F_k(x) \quad \text{as} \quad n \to \infty.$$

So, for every polynomial $P \in \mathcal{H}_{bs}(\ell_1)$, $P(z_n) \to P(x)$. Since $\{z_n\}$ is bounded and every function $f \in \mathcal{H}_{bs}(\ell_1)$ is approximated by polynomials uniformly on bounded sets, it follows $f(z_n) \to f(x)$ as $n \to \infty$. □

Corollary 4.8. *The radius function is discontinuous in the Gelfand topology as a function from $\mathcal{M}_{bs}(\ell_1)$ to $\mathbb{C}$.*

Proof. Let $x = 0$ and $\{z_n\}$ as in Theorem 4.11. According to Theorem 4.11, $\delta_{z_n} \to \delta_0$ in the Gelfand topology of $\mathcal{M}_{bs}(\ell_1)$. By Theorem 4.9, $R_1(\delta_{z_n}) = \|z_n\| = 1$ but $R_1(\delta_0) = 0$. □

Theorem 4.12. *Let $\varphi \in \mathcal{M}_{bs}(\ell_1)$. Then $\varphi = \psi_\lambda \star \delta_x$ for some $x \in \ell_1$, $\lambda \in \mathbb{C}$.*

Proof. If it is not true, then from Theorem 4.8 it follows that there is a number $\lambda \in \mathbb{C}$ and an element $y = (y_1, \ldots, y_n, \ldots)$, $y \in \ell_p$ for all $p > 1$ and $y \notin \ell_1$ such that $\varphi(F_1) = \lambda$ and $\varphi(F_k) = F_k(y)$, $k > 1$. Let $c = R_1(\varphi) < \infty$. By Theorem 4.10 $R_p(\varphi) = \|y\|_p$, $p > 1$. Since $y \notin \ell_1$, there is $q > 1$ such that $\|y\|_q > c$. So $R_q(\varphi) > R_1(\varphi)$ that contradicts Lemma 4.5. □

Clearly, $\mathcal{M}_{bs}(\ell_1)$ contains a copy of $\mathbb{C}$: $\lambda \mapsto \psi_\lambda$. But we do not know whether $\mathcal{M}_{bs}(\ell_1)$ contains a copy of $\mathbb{C}^n$ for some $n > 1$?

4.4 Applications of symmetric polynomials in statistical quantum physics

Symmetric polynomial variables and relations between the bases of the algebra of symmetric polynomials are widely used in algebra, combinatorics (see [32]), and, in particular, in statistical quantum mechanics. In [39, 40], Schmidt and Schnack proposed some correspondence between the relations in the algebra of symmetric polynomials and partition functions of bosons and fermions. Under this correspondence, one basis of symmetric polynomials is responsible for bosons and another for fermions. Such an approach was applied and developed for different cases by many authors (see, e.g., [21, 34, 36, 37, 48]). On the other hand, recently, some new results for the algebras of symmetric analytic functions on infinite-dimensional Banach spaces were obtained [1, 11, 22]. The infinite number of variables of the underlying space allows us to introduce some interesting algebraic operations on the spectra of such algebras that may have a physical meaning. In addition, in the infinite-dimensional case, we can consider the behavior of the ideal gas "at infinity" if, for example, the number of particles grows to infinity while the total energy of the system is bounded.

Let us recall the bases of symmetric polynomials on ℓ_1 which we defined above. The basis of power symmetric polynomials (F_n) is

$$F_n((x_1, x_2, \ldots)) = \sum_{i=1}^{\infty} x_i^n, \quad n \in \mathbb{N}. \tag{4.14}$$

The basis of elementary symmetric polynomials (G_n) is

$$G_n((x_1, x_2, \ldots)) = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}, \tag{4.15}$$

Also, in this section, we consider the basis of complete symmetric polynomials

$$B_n((x_1, x_2, \ldots)) = \sum_{i_1 \le \cdots \le i_n} x_{i_1} \cdots x_{i_n}, \quad n \in \mathbb{N}. \tag{4.16}$$

There are so-called Newton recurrent formulas connecting different algebraic bases:

$$mG_m = \sum_{k=1}^{m} (-1)^{k-1} G_{m-k} F_k, \quad m \in \mathbb{N}, \tag{4.17}$$

$$mB_m = \sum_{k=1}^{m} B_{m-k} F_k, \quad m \in \mathbb{N}, \tag{4.18}$$

$$G_m = \sum_{k=1}^{m} (-1)^{k-1} G_{m-k} B_k, \quad m \in \mathbb{N}, \tag{4.19}$$

and

$$B_m = \sum_{k=1}^{m} (-1)^{k-1} B_{m-k} G_k, \quad m \in \mathbb{N}. \tag{4.20}$$

Let $\mathcal{B}(x)(t)$ and $\mathcal{G}(x)(t)$ be the generating functions for polynomials B_n and G_n, respectively, defined as the following formal series:

$$\mathcal{B}(x)(t) = \sum_{n=0}^{\infty} t^n B_n(x), \quad B_0 = 1, \tag{4.21}$$

$$\mathcal{G}(x)(t) = \sum_{n=0}^{\infty} t^n G_n(x), \quad G_0 = 1. \tag{4.22}$$

The following relations are well-known ([32], p. 3):

$$\mathcal{G}(x)(t) = \exp\left(- \sum_{n=1}^{\infty} t^n \frac{F_n(-x)}{n} \right) \quad \text{and} \quad \mathcal{B}(x)(t) = \exp\left(\sum_{n=1}^{\infty} t^n \frac{F_n(x)}{n} \right),$$

and they immediately imply that

$$\mathcal{G}(x)(t)\mathcal{B}(-x)(t) = 1. \tag{4.23}$$

Here, the equality holds for every $x \in \ell_1$ and for every t in the common domain of convergence. Note that $\mathcal{G}(x)(t)$ is a well-defined analytic function of $x \in \ell_1$ for every fixed $t \in \mathbb{C}$ and an exponential-type function of t for every fixed x (Section 4.1).

4.4.1 Partition functions

The canonical partition function plays a fundamental role in statistical mechanics since most thermodynamic functions can be derived from it [40]. It is defined by

$$Z_N(\beta) = \operatorname{Tr} \exp(-\beta H),$$

where H denotes the Hamiltonian of the system, N is the number of particles, and $\beta = \frac{1}{k_B T}$ denotes the inverse temperature (k_B is the Boltzmann constant, and T is the temperature). In other words, H is a self-adjoined operator such that $\exp(-\beta H)$ is a trace class operator for $\beta \in \mathbb{R}$.

The grand canonical partition function is defined by

$$Z(z, \beta) = \sum_{N=0}^{\infty} Z_N(\beta) z^N, \tag{4.24}$$

where the variable z is physically interpreted as the fugacity of the system, i.e., $z = e^{\mu/(k_B T)}$ (μ is the chemical potential). It describes the system in which the number of particles can be changed. The physical interpretation implies that z must be non-negative.

Note that the partition function completely defines all possible states of the system. Moreover, it can be used for deriving the likelihood of states.

Consider the ideal gas consisting of non-interacting identical particles (bosons or fermions). In this case, the Hamiltonian H is the sum of N identical single-particle Hamiltonians:

$$H = \sum_{n=1}^{N} h_n.$$

Let E_i be single-particle energy eigenvalues counted in such a way that several E_i have the same value in the case of degeneracy. In [8], it is shown that

$$Z_N(\beta) = B_N((x_1, x_2, \ldots)) \tag{4.25}$$

for the system of bosons and

$$Z_N(\beta) = G_N((x_1, x_2, \ldots)) \tag{4.26}$$

for the system of fermions, where B_N is defined by (4.16), G_N is defined by (4.15), and

$$x_i = \exp(-\beta E_i). \tag{4.27}$$

Note that Z_N is a symmetric function between energy levels, not between particles. The ordering of levels needed for (4.25) and (4.26) is simple for one-dimensional systems, but this is potentially difficult in higher dimensions due to the degeneracies of energy levels and the use of multi-indices to characterize them.

According to (4.21), (4.22), (4.24), (4.25), and (4.26), the grand canonical partition function can be represented in the form

$$Z(z,\beta) = \mathcal{B}((x_1, x_2, \ldots))(z)$$

for bosons and

$$Z(z,\beta) = \mathcal{G}((x_1, x_2, \ldots))(z)$$

for fermions, where x_i are defined by (4.27). In addition, according to [39], the co-ordinates $(x_1, x_2, \ldots)$ of $x \in \ell_1$ correspond to the abstract energy levels of the system; a monomial $x_1^{n_1} \cdots x_m^{n_m}$, $n_1 + \cdots + n_m = N$ in a partition function corresponds to the possible occupation of levels $x_1, \ldots, x_m$ by N particles. Moreover, there exists a so-called *fundamental symmetry* ω of $\mathcal{P}_s(\ell_1)$, which is defined as an algebra homomorphism from $\mathcal{P}_s(\ell_1)$ to itself such that $\omega(F_n) = (-1)^{n-1} F_n\ n \in \mathbb{N}$. In other words, for every n, $\big(\omega(F_n)\big)(x) = -F_n(x)$. Note that ω is an involution in the sense that ω^2 is the unity operator. It is known that $\omega G_n = B_n$ and $\omega B_n = G_n$ for every $n \in \mathbb{N}$ ([32], p. 4). In [39], it was observed that Newton's identity (4.17) corresponds to Landsberg's identity in physics [29], and gather (4.23) is related to a Bose-Fermi symmetry. Some specific examples for the mentioned Bose-Fermi symmetry can be found in [30, 41, 42].

4.4.2 Supersymmetric polynomials and partition functions for mixed systems of bosons and fermions

Let $\mathbb{Z}$ be the set of all integers and $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$. By $\ell_1(\mathbb{Z}_0)$, we denote the Banach space of all absolutely summing complex sequences indexed by the elements of $\mathbb{Z}_0$ (two-sided sequences). Every element of $\ell_1(\mathbb{Z}_0)$ can be represented in the form

$$(y|x) = (\ldots, y_2, y_1 | x_1, x_2, \ldots)$$

with

$$\|(y|x)\| = \sum_{i=1}^{\infty} (|x_i| + |y_i|),$$

where $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ belong to ℓ_1.

For every $n \in \mathbb{N}$, we define the polynomials T_n, $n \in \mathbb{N}$ on $\ell_1(\mathbb{Z}_0)$ by

$$T_n((y|x)) = F_n(x) - F_n(y),$$

where F_n is defined by (4.14).

A polynomial on $\ell_1(\mathbb{Z}_0)$ is called *supersymmetric* (see [27]) if it can be represented as an algebraic combination of elements of the set $\{T_n\colon n \in \mathbb{N}\}$. Let us denote $\mathcal{P}_{sup}(\ell_1(\mathbb{Z}_0))$ as the algebra of all supersymmetric polynomials on $\ell_1(\mathbb{Z}_0)$. Note that the set $\{T_n\colon n \in \mathbb{N}\}$ is the algebraic basis of the algebra $\mathcal{P}_{sup}(\ell_1(\mathbb{Z}_0))$. Let us define another important supersymmetric polynomial on $\ell_1(\mathbb{Z}_0)$, which also forms the algebraic basis of the algebra $\mathcal{P}_{sup}(\ell_1(\mathbb{Z}_0))$. For $n \in \mathbb{N}$, let $W_n\colon \ell_1(\mathbb{Z}_0) \to \mathbb{C}$ be defined by

$$W_n((y|x)) = \sum_{k=0}^{n} G_k(x) B_{n-k}(-y). \tag{4.28}$$

Note that polynomial W_n can be obtained if we substitute in Newton's formula (4.17) for polynomials T_n instead of F_n [27]. In other words,

$$mW_m((y|x)) = \sum_{k=1}^{m} (-1)^{k-1} W_{m-k}((y|x)) T_k((y|x)), \quad m \in \mathbb{N}. \tag{4.29}$$

From (4.29), in particular, it follows that all polynomials, W_n, are supersymmetric and form the algebraic basis in $\mathcal{P}_{sup}(\ell_1(\mathbb{Z}_0))$.

Let $\mathcal{W}((y|x))(t)$ be the formal series

$$\mathcal{W}((y|x))(t) = \sum_{n=0}^{\infty} t^n W_n((y|x)), \quad W_0 = 1, \tag{4.30}$$

that is, $\mathcal{W}$ is the generating function for polynomial W_n. According to ([27], Theorem 2),

$$\mathcal{W}((y|x))(t) = \frac{\mathcal{G}(x)(t)}{\mathcal{G}(y)(t)}, \tag{4.31}$$

the equality is true on the common domain of convergence.

Consider a mixed system of bosons and fermions. In [8], it is shown that the partition function for the system, where the total number, N, of bosons and fermions is fixed, can be represented in the form

$$Z_N(\beta) = \sum_{k=0}^{N} G_k((x_1^{(F)}, x_2^{(F)}, \ldots)) B_{N-k}((x_1^{(B)}, x_2^{(B)}, \ldots)), \tag{4.32}$$

where $x_i^{(F)} = \exp(-\beta E_i^{(F)})$ and $x_i^{(B)} = \exp(-\beta E_i^{(B)})$, $E_i^{(F)}$ and $E_i^{(B)}$ are the single-particle energies of fermions and bosons, respectively.

Let $\widetilde{W}_n : \ell_1(\mathbb{Z}_0) \to \mathbb{C}$ be defined by

$$\widetilde{W}_n((y|x)) = W_n((-x| - y)), \tag{4.33}$$

where W_n is defined by (4.28). According to (4.28) and (4.33),

$$\widetilde{W}_n((y|x)) = \sum_{k=0}^{n} G_k(-y)B_{n-k}(x). \tag{4.34}$$

According to (4.32) and (4.34),

$$Z_N(\beta) = \widetilde{W}_N((\tilde{y}|\tilde{x})), \tag{4.35}$$

where

$$\tilde{y} = (-x_1^{(F)}, -x_2^{(F)}, \ldots), \quad \tilde{x} = (x_1^{(B)}, x_2^{(B)}, \ldots). \tag{4.36}$$

If the sequences are finite, we complete them with an infinite number of zeros. Note that the equality (4.35) makes sense only if $\tilde{x}$ and $\tilde{y}$ belong to ℓ_1. Otherwise, we can only consider (4.35) as the formal equality.

Let us consider the grand canonical partition function. According to (4.24) and (4.35),

$$Z(z,\beta) = \sum_{N=0}^{\infty} z^N \widetilde{W}_N((\tilde{y}|\tilde{x})), \quad \widetilde{W}_0 = 1.$$

For $(y|x) \in \ell_1(\mathbb{Z}_0)$ and $t \in \mathbb{C}$, let $\widetilde{\mathcal{W}}((y|x))(t)$ be the formal series

$$\widetilde{\mathcal{W}}((y|x))(t) = \sum_{n=0}^{\infty} t^n \widetilde{W}_n((y|x)). \tag{4.37}$$

Evidently,

$$Z(z,\beta) = \widetilde{\mathcal{W}}((\tilde{y}|\tilde{x}))(z). \tag{4.38}$$

On the other hand, according to (4.37), (4.33), (4.31), and (4.30),

$$\widetilde{\mathcal{W}}((y|x))(t) = \sum_{n=0}^{\infty} t^n \widetilde{W}_n((-x|-y)) = \mathcal{W}((-x|-y))(t) = \frac{\mathcal{G}(-y)(t)}{\mathcal{G}(-x)(t)} = \frac{\mathcal{B}(x)}{\mathcal{B}(y)}. \tag{4.39}$$

Therefore, according to (4.38), (4.39), and (4.23),

$$Z(z,\beta) = \frac{\mathcal{G}(-\tilde{y})(t)}{\mathcal{G}(-\tilde{x})(t)} = \frac{\mathcal{B}(\tilde{x})}{\mathcal{B}(\tilde{y})},$$

where $\tilde{y}$ and $\tilde{x}$ are defined by (4.36).

Thus, we have represented the grand canonical partition function of the mixed system of bosons and fermions via the generating functions $\mathcal{G}$ and $\mathcal{B}$ for elementary symmetric polynomials.

Let us observe that, if we apply the transformation $(y|x) \mapsto (-x|-y)$ to T_n for the case $y = 0$, we will obtain

$$F_n(x) = T_n((0|x)) \mapsto T_n((-x|0)) = \\ = (-1)^{n-1}T_n((0|x)) = (-1)^{n-1}F_n(x) = (\omega(F_n))(x).$$

In other words, the involution ω on $\mathcal{P}_s(\ell_1)$ can be extended to $\mathcal{P}_{sup}(\ell_1(\mathbb{Z}_0))$, setting

$$(\omega(P))((y|x)) = P((-x|-y)).$$

In particular, $\omega(W_n) = \widetilde{W}_n$. Applying the homomorphism ω to (4.29), we obtain

$$m\widetilde{W}_m((y|x)) = \sum_{k=1}^{m} \widetilde{W}_{m-k}((y|x))T_k((y|x)), \quad m \in \mathbb{N},$$

that is, $\widetilde{W}_n$ can be obtained if we substitute T_n instead of F_n into the Newton formula (4.18); therefore, we have another representation for $\widetilde{W}_n$, which can be interpreted as another realization of Landsberg's identity. In addition, from (4.19), (4.20), we can obtain

$$\widetilde{W}_m = \sum_{k=1}^{m} (-1)^{k-1}\widetilde{W}_{m-k}W_k, \quad m \in \mathbb{N}.$$

Example 4.3. *Let us compute* $Z_N(\beta) = \widetilde{W}_N(\widetilde{y}, \widetilde{x})$ *for* $N = 4$, $\widetilde{x} = (x_1, x_2)$, $\widetilde{y} = (-y_1, -y_2, -y_3)$. *According to* (4.34)*,*

$$\widetilde{W}_N(\widetilde{y}, \widetilde{x}) = B_4(x) + G_1(-y)B_3(x) + G_2(-y)B_2(x) + G_3(-y)B_1(x) + G_4(-y) = \\ = x_1^4 + x_2^4 + x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + (y_1 + y_2 + y_3)(x_1^3 + x_2^3 + x_1^2x_2 + x_1x_2^2) + \\ +(y_1y_2 + y_1y_3 + y_2y_3)(x_1^2 + x_2^2 + x_1x_2) + y_1y_2y_3(x_1 + x_2).$$

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Roman DMYTRYSHYN

5 Representations of Appell hypergeometric functions by branched continued fractions

Hypergeometric functions are among the most interesting in function theory due to their, importance in mathematics, physics, engineering, chemistry, economics, and other applications. Significant contribution to the theory of hypergeometric functions was made by such prominent mathematicians as J. Wallis, L. Euler, C. Gauss, E. Kummer, B. Riemann, P. Appell, and many others.

In 1812, C. Gauss [35] considered the hypergeometric series,

$$F(a,b;c;z)=1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+\ldots,\quad |z|<1, \tag{5.1}$$

and obtained the continued fraction expansion,

$$\frac{F(a,b+1;c+1;z)}{F(a,b;c;z)}=\cfrac{1}{1-\cfrac{\frac{a(c-b)}{c(c+1)}z}{1-\cfrac{\frac{(b+1)(c-a+1)}{(c+1)(c+2)}z}{1-\ddots}}}, \tag{5.2}$$

which was called the Gauss continued fraction, where $a,b,c\in\mathbb{C}$ herewith $c\notin\{0,-1,-2,\ldots\}$. Later, B. Riemann [50], L. Thomé [54], E. Van Vleck [55] proved the following: provided that all coefficients of the continued fraction (5.2) are nonzero, it converges in the domain

$$\mathcal{H}=\{z\in\mathbb{C}:\ z\notin[1,+\infty)\},$$

excepting possibly at certain isolated points, and is equal to the function on the left-hand side in (5.2) in the neighborhood of the origin, and provides the analytic continuation of this function in $\mathcal{H}$. In addition, the convergence is uniform on every compact subset of the domain $\mathcal{H}$, that does not contain any of the above-mentioned isolated points. These points, if they exist, are poles of the function represented by the Gauss continued fraction.

Roman DMYTRYSHYN, Vasyl Stefanyk Carpathian National University, 57 Shevchenko Str., 76018, Ivano-Frankivsk, Ukraine, e-mail: roman.dmytryshyn@cnu.edu.ua

The wider convergence sets of continued fractions than the convergence sets of the corresponding power series, the much higher rate of convergence of their rational approximations then rate of convergence of polynomial approximations, and, in addition, the property of not to accumulate errors during their computations have made continued fractions a simple and effective mathematical tool in applications (see books [20, 42, 43, 57]).

The natural two-variable extension of the series (5.1) is the double hypergeometric series,

$$F_1(a,b_1,b_2;c;\mathbf{z})=\sum_{r,s=0}^{+\infty}\frac{(a)_{r+s}(b_1)_r(b_2)_s}{(c)_{r+s}}\frac{z_1^r z_2^s}{r!s!},\ |z_1|<1,\ |z_2|<1, \tag{5.3}$$

$$F_2(a,b_1,b_2;c_1,c_2;\mathbf{z})=\sum_{r,s=0}^{+\infty}\frac{(a)_{r+s}(b_1)_r(b_2)_s}{(c_1)_r(c_2)_s}\frac{z_1^r z_2^s}{r!s!},\ |z_1|+|z_2|<1, \tag{5.4}$$

$$F_3(a_1,a_2,b_1,b_2;c;\mathbf{z})=\sum_{r,s=0}^{+\infty}\frac{(a_1)_r(a_2)_s(b_1)_r(b_2)_s}{(c)_{r+s}}\frac{z_1^r z_2^s}{r!s!},\ |z_1|<1,\ |z_2|<1, \tag{5.5}$$

$$F_4(a,b;c_1,c_2;\mathbf{z})=\sum_{r,s=0}^{+\infty}\frac{(a)_{r+s}(b)_{r+s}}{(c_1)_r(c_2)_s}\frac{z_1^r z_2^s}{r!s!},\ |z_1|^{1/2}+|z_2|^{1/2}<1, \tag{5.6}$$

which were introduced by P. Appell as early as 1880 [14], where $a,a_1,a_2,b,b_1,b_2\in\mathbb{C}$ and $c,c_1,c_2\in\mathbb{C}$ herewith $c,c_1,c_2\notin\{0,-1,-2,\ldots\}$, $(\mathbf{z})=(z_1,z_2)\in\mathbb{C}^2$, and $(\alpha)_k$ is the rising factorial Pochhammer symbol.

Expansions of Appell hypergeometric functions F_1–F_4 into power series by products of two Gaussian functions, systems of second-order differential equations whose solutions are these functions, representations functions (5.3)-(5.6) in terms of double integrals involving elementary functions, and much more can be found in [13, 19, 33]. The applications of functions F_1–F_4 in science and technology is discussed, in particular, in the book [34].

A well-known generalization of continued fractions is branched continued fractions,

$$\sum_{i_1=1}^{N}\cfrac{a_{i_1}}{b_{i_1}+\sum\limits_{i_2=1}^{N}\cfrac{a_{i_1,i_2}}{b_{i_{1,2}}+\sum\limits_{i_3=1}^{N}\cfrac{a_{i_1,i_2,i_3}}{b_{i_1,i_2,i_3}+\ddots}}},$$

proposed by V. Skorobohatko in 1966 together with N. Droniuk, O. Bobyk and B. Ptashnyk [52], where N is a fixed natural number, $a_{i_1},b_{i_1},a_{i_1,i_2},b_{i_1,i_2},\ldots$ called elements (this can be numbers, functions, etc.). At the same time, the first expansions into branched continued fractions were considered for ratios of Appell hypergeometric functions F_1 by N. Dronyuk [32]. The expansions of various generalizations of the hypergeometric series into continued fractions or branched continued fractions were considered in [2–9, 16, 18, 39, 40, 44, 45, 48, 49]. In addition, their convergence, rate of convergence and numerical stability were discussed in [11, 21–31, 36–38] and others.

The chapter is devoted to the expansions of the Appell hypergeometric functions F_1–F_4 into branched continued fractions, their study and application to approximation of some special functions.

5.1 Branched continued fractions: definition and their properties

The analytic theory of branched continued fractions is presented in the book [15]. Let us give its briefly review.

Definition of branched continued fraction. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $i(0) = 0$ and $\mathcal{I}_0 = \{0\}$. Let us introduce the following sets of multiindices

$$\mathcal{I}_k = \{i(k) : \; i(k) = (i_1, i_2, \dots, i_k), \; 1 \le i_r \le 2, \; 1 \le r \le k\}, \quad k \in \mathbb{N}.$$

By D. Bodnar [15, p. 15] for each $r \ge 1$ the symbol $\mathbf{u}^{(r)}$ denotes a vector in $\mathbb{C}^{2^r}$ with components $u_{j(r)}$, $j(r) \in \mathcal{I}_r$; for each $r \ge 1$, $k \in \mathbb{N}$ and for each multiindex $i(k) \in \mathcal{I}_k$ the symbol $\mathbf{u}^{(r)}_{i(k)}$ is a vector in $\mathbb{C}^{2^r}$ with components

$$u_{i(k),j(r)}, \quad i(k) \in \mathcal{I}_k, \quad 1 \le j_s \le 2, \quad 1 \le s \le r, \quad j_0 = i_k,$$

with the following order of components:

(a) $u_{n(r)} \prec u_{m(r)}$ ($u_{i(k),n(r)} \prec u_{i(k),m(r)}$), if $n(r) \prec m(r)$;

(b) $n(r) \prec m(r)$, if $n_1 < m_1$ or there exists index s, $1 \le s < r$, such that $n_p = m_p$, $1 \le p \le s$, and $n_{s+1} < m_{s+1}$.

Let

$$\langle \{a_{i(k)}\}_{i(k) \in \mathcal{I}_k,\, k \in \mathbb{N}}, \{b_{i(k)}\}_{i(k) \in \mathcal{I}_k,\, k \in \mathbb{N}_0} \rangle$$

denote the ordered pair of sequences of complex numbers satisfy the following conditions:

(c) $a_{i(k)} \ne 0$ for all $i(k) \in \mathcal{I}_k$, $k \in \mathbb{N}$;

(d) if for $k \in \mathbb{N}$ there exists a multiindex $i(k) \in \mathcal{I}_k$ such that $b_{i(k)} = 0$, then $b_{i(k-1),j} \ne 0$ for $1 \le j \le i_{k-1}$ and $j \ne i_k$, gives rise to sequence

$$\{s_{i(k)}(\mathbf{w}^{(1)}_{i(k)})\}_{i(k)\in\mathcal{I}_k,\, k\in\mathbb{N}_0}$$

herewith $\mathbf{w}^{(1)}_0 = \mathbf{w}^{(1)}$ and

$$\{S_k(\mathbf{w}^{(k+1)})\}_{k\in\mathbb{N}_0}$$

of two-dimensional linear fractional transformations

$$s_0(\mathbf{w}^{(1)}) = b_0 + w_1 + w_2,$$

$$v_{i(k)} = s_{i(k)}(\mathbf{w}^{(1)}_{i(k)}) = \frac{a_{i(k)}}{b_{i(k)} + w_{i(k),1} + w_{i(k),2}}, \quad i(k) \in \mathcal{I}_k,\ k \in \mathbb{N},$$

and

$$S_0(\mathbf{w}^{(1)}) = s_0(\mathbf{w}^{(1)}), \quad S_k(\mathbf{w}^{(k+1)}) = S_{k-1}(\mathbf{v}^{(k)}), \quad k \in \mathbb{N},$$

and to a sequence $\{f_k\}_{k\in\mathbb{N}_0 0}$, given by

$$f_k = S_k(\mathbf{0}^{(k+1)}) \in \widehat{\mathbb{C}}, \quad k \in \mathbb{N}_0,$$

where $\mathbf{0}^{(k+1)} = (0, 0, \ldots, 0)$ is a vector in $\mathbb{C}^{2^{k+1}}$.

Definition 5.1. *The ordered pair*

$$\langle\langle\{a_{i(k)}\}_{i(k)\in\mathcal{I}_k,k\in\mathbb{N}}, \{b_{i(k)}\}_{i(k)\in\mathcal{I}_k,k\in\mathbb{N}_0}\rangle, \{f_k\}_{k\in\mathbb{N}_0}\rangle$$

is the branched continued fraction denoted by symbols

$$b_0 + \sum_{i_1=1}^{2} \cfrac{a_{i(1)}}{b_{i(1)} + \sum_{i_2=1}^{2} \cfrac{a_{i(2)}}{b_{i(2)} + \sum_{i_3=1}^{2} \cfrac{a_{i(3)}}{b_{i(3)} + \ddots}}}. \tag{5.7}$$

The elements of (5.7) b_0 is called the free term, $a_{i(k)}$, $i(k) \in \mathcal{I}_k$, are called the kth partial numerators, and $b_{i(k)}$, $i(k) \in \mathcal{I}_k$, are called the kth partial denominators, $k \in \mathbb{N}$. The value

$$f_k = b_0 + \sum_{i_1=1}^{2} \cfrac{a_{i(1)}}{b_{i(1)} + \sum_{i_2=1}^{2} \cfrac{a_{i(2)}}{b_{i(2)} + \ddots + \sum_{i_k=1}^{2} \cfrac{a_{i(k)}}{b_{i(k)}}}} \tag{5.8}$$

is called the kth approximant of the branched continued fraction, $k \in \mathbb{N}_0$.

In fact, the branched continued fraction is the mapping of the ordered pair of sequences

$$\langle\{a_{i(k)}\}_{i(k)\in\mathcal{I}_k,k\in\mathbb{N}}, \{b_{i(k)}\}_{i(k)\in\mathcal{I}_k,k\in\mathbb{N}_0}\rangle$$

onto the sequence $\{f_k\}_{k\in\mathbb{N}_0}$.

Convergence. The crucial problem with branched continued fractions is their convergence. When considering branched continued fractions, we will also allow confluent cases, where the restriction on the difference from zero of partial numerators and, under certain conditions, partial denominators, is lifted. In this regard, we will assume that $1/0 = \infty$, $1/\infty = 0$, and $a/0 + b/0 = 0/0$ for all $a \in \mathbb{C}$ and $b \in \mathbb{C}$, and that the approximant (5.8) makes sense if, when it is collapsed from bottom to top without any reductions in intermediate operations, we do not obtain an uncertainty of $0/0$.

Without loss of generality, let us set $b_0 = 0$.

Definition 5.2. *A branched continued fraction*

$$\sum_{i_1=1}^{2} \cfrac{a_{i(1)}}{b_{i(1)} + \sum\limits_{i_2=1}^{2} \cfrac{a_{i(2)}}{b_{i(2)} + \sum\limits_{i_3=1}^{2} \cfrac{a_{i(3)}}{b_{i(3)} + \ddots}}} \tag{5.9}$$

converges if, at most, a finite number of its approximants don't make sense and if there exists a finite limit of its sequence of approximants

$$\lim_{n \to +\infty} f_n$$

.

Definition 5.3. *A branched continued fraction (5.9) converges absolutely if its sequence of approximants such that satisfies*

$$\sum_{n=1}^{+\infty} |f_{n+1} - f_n| < +\infty.$$

Many convergence criteria for branched continued fractions concern convergence sets and sets of uniform convergence.

Definition 5.4. *A convergence set Ω is a set $\Omega \neq \emptyset$ and $\Omega \subseteq \mathbb{C} \times \mathbb{C}$ such that: if $\langle a_{i(k)}, b_{i(k)} \rangle \in \Omega$ for all $i(k) \in \mathcal{I}_k$, $k \in \mathbb{N}$, then a branched continued fraction (5.9) converges.*

Definition 5.5. *A uniform convergence set Ω is a convergence set to which there corresponds a sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ of positive numbers depending only on Ω and converging to 0 such that*

$$|f_{n+k} - f_n| \leq \varepsilon_n, \quad n \in \mathbb{N},\ k \in \mathbb{N},$$

for every branched continued fraction (5.9) with all $\langle a_{i(k)}, b_{i(k)} \rangle \in \Omega$.

Formula of difference of two approximants. Fork property. Formula of difference of two approximants of branched continued fractions is a widely used tool that has yielded various numerous results, including those related to hypergeometric functions.

Let $Q_{i(k)}^{(n)}(\mathbf{z}), i(k) \in \mathcal{I}, 1 \leq k \leq n, n \in \mathbb{N}$, denote the so-called tails of branched continued fraction (5.7), that is

$$Q_{i(n)}^{(n)} = b_{i(n)}, \quad i(n) \in \mathcal{I},\ n \in \mathbb{N},$$

$$Q_{i(k)}^{(n)} = b_{i(k)} + \sum_{i_{k+1}=1}^{2} \cfrac{a_{i(k+1)}}{b_{i(k+1)} + \sum\limits_{i_{k+2}=1}^{2} \cfrac{a_{i(k+2)}}{b_{i(k+2)} + \ddots + \sum\limits_{i_n=1}^{2} \cfrac{a_{i(n)}}{b_{i(n)}}}},$$

where $i(k) \in \mathcal{I}$, $1 \leq k \leq n-1$, $n \geq 2$. Then

$$Q_{i(k)}^{(n)} = b_{i(k)} + \sum_{i_{k+1}=1}^{2} \frac{a_{i(k+1)}}{Q_{i(k+1)}^{(n)}}, \quad i(k) \in \mathcal{I},\ 1 \leq k \leq n-1,\ n \geq 2,$$

and

$$f_n = b_0 + \sum_{i_1=1}^{2} \frac{a_{i(1)}}{Q_{i(1)}^{(n)}}, \quad n \in \mathbb{N}.$$

If $Q_{i(k)}^{(n)} \neq 0$ for all $i(k) \in \mathcal{I}$, $1 \leq k \leq n$, $n \in \mathbb{N}$, then for each $m > n \geq 1$ the following formula is valid (see, [15, p. 28])

$$f_m - f_n = (-1)^n \sum_{i_1=1}^{2} \frac{a_{i(1)}}{Q_{i(1)}^{(m)} Q_{i(1)}^{(n)}} \cdots \sum_{i_n=1}^{2} \frac{a_{i(n)}}{Q_{i(n)}^{(m)} Q_{i(n)}^{(n)}} \sum_{i_{n+1}=1}^{2} \frac{a_{i(n+1)}}{Q_{i(n+1)}^{(m)}}.$$

A direct application of this formula yields the so-called fork property for branched continued fraction.

Proposition 5.1 ([15, Proposition 1.4]). *Let (5.7) be a branched continued fraction with positive elements. Then*

$$f_{2n-2} < f_{2n} < f_{2n+1} < f_{2n-1}, \quad n \in \mathbb{N}.$$

Equivalence transformations. Even (odd) part. A transformation of a branched continued fraction that does not change the values of the approximants is called equivalent. Branched continued fractions in which all corresponding approximants coincide are called equivalent.

Proposition 5.2. *A branched continued fraction*

$$\sum_{i_1=1}^{2}\cfrac{c_{i(1)}}{d_{i(1)}+\sum\limits_{i_2=1}^{2}\cfrac{c_{i(2)}}{d_{i(2)}+\sum\limits_{i_3=1}^{2}\cfrac{c_{i(3)}}{d_{i(3)}+\ddots}}}$$

is equivalent to (5.9) if there exist non-zero complex numbers $\varrho_{i(k)}$, $i(k)\in\mathcal{I}_k$, $k\geq 1$, such that with $\varrho_0=1$

$$c_{i(k)}=\varrho_{i(k-1)}\varrho_{i(k)}a_{i(k)},\quad d_{i(k)}=\varrho_{i(k)}b_{i(k)},\quad i(k)\in\mathcal{I}_k,\quad k\geq 1.$$

If $b_{i(k)}\neq 0$, $i(k)\in\mathcal{I}_k$, $k\geq 1$, then the branched continued fraction (5.9) is equivalent to

$$\sum_{i_1=1}^{2}\cfrac{c_{i(1)}}{1+\sum\limits_{i_2=1}^{2}\cfrac{c_{i(2)}}{1+\sum\limits_{i_3=1}^{2}\cfrac{c_{i(3)}}{1+\ddots}}},$$

where

$$c_{i(1)}=\frac{a_{i(1)}}{b_{i(1)}},\quad i(1)\in\mathcal{I}_1,\quad c_{i(k)}=\frac{a_{i(k)}}{b_{i(k-1)}b_{i(k)}},\quad i(k)\in\mathcal{I}_k,\quad k\geq 2.$$

Proposition 5.3. *A branched continued fraction (5.9) with the sequence of approximants $\{f_n\}_{n\in\mathbb{N}}$ is called the even (odd) part of a branched continued fraction with unequal variables*

$$\sum_{i_1=1}^{2}\cfrac{\widetilde{a}_{i(1)}}{\widetilde{b}_{i(1)}+\sum\limits_{i_2=1}^{2}\cfrac{\widetilde{a}_{i(2)}}{\widetilde{b}_{i(2)}+\sum\limits_{i_3=1}^{2}\cfrac{\widetilde{a}_{i(3)}}{\widetilde{b}_{i(3)}+\ddots}}}$$

with the sequence of approximants $\{\widetilde{f}_n\}_{n\in\mathbb{N}}$, if $f_n=\widetilde{f}_{2n}$ ($f_n=\widetilde{f}_{2n+1}$) for $n\geq 1$.

Uniform convergence. Let $\mathcal{D}$ be some domain in $\mathbb{C}^2$. Here, the domain is an open connected subset of $\mathbb{C}^2$.

Definition 5.6. *A sequence $\{f_n(\mathbf{z})\}_{n\in\mathbb{N}}$ of functions holomorphic in the domain $\mathcal{D}$ is said to converge uniformly on a compact subset $\mathcal{K}$ of $\mathcal{D}$ if:*

(i) there exists $N(\mathcal{K})$ *such that* $f_n(\mathbf{z})$ *is holomorphic in the domain* $\mathcal{K}$ *for all* $n \geq N(\mathcal{K})$, *and*

(ii) given $\varepsilon > 0$ *there exists* $N_\varepsilon > N(\mathcal{K})$ *such that*

$$\sup_{\mathbf{z}\in\mathcal{K}} |f_{n+k}(\mathbf{z}) - f_n(\mathbf{z})| < \varepsilon, \quad n \geq N_\varepsilon,\ k \in \mathbb{N}_0.$$

Definition 5.7. *The sequence* $\{f_n(\mathbf{z})\}_{n\in\mathbb{N}}$ *of functions holomorphic in the domain* $\mathcal{D}$ *is said to be uniformly bounded on a cony¿act subset* $\mathcal{K}$ *of* $\mathcal{D}$ *if there exist* $N(\mathcal{K})$ *and* $M(\mathcal{K})$ *such that*

$$\sup_{\mathbf{z}\in\mathcal{K}} |f_n(\mathbf{z})| < M(\mathcal{K}), \quad n \geq N(\mathcal{K}).$$

Definition 5.8 (Pointwise Convergence)**.** *A branched continued fraction*

$$\sum_{i_1=1}^{2} \cfrac{a_{i(1)}(\mathbf{z})}{b_{i(1)}(\mathbf{z}) + \sum_{i_2=1}^{2} \cfrac{a_{i(2)}(\mathbf{z})}{b_{i(2)}(\mathbf{z}) + \sum_{i_3=1}^{2} \cfrac{a_{i(3)}(\mathbf{z})}{b_{i(3)}(\mathbf{z}) + \ddots}}}. \tag{5.10}$$

is said to converge at $\mathbf{z} = \mathbf{z}^0$ *if its sequence of approximants* $\{f_n(\mathbf{z}^0)\}_{k\in\mathbb{N}}$ *converges, and*

$$\lim_{n\to+\infty} f_n(\mathbf{z}^0)$$

is called its value.

Definition 5.9. *A branched continued fraction (5.10) is said to converge uniformly on a compact subset* $\mathcal{K}$ *of* $\mathcal{D}$ *if its sequence of approximants* $\{f_n(\mathbf{z})\}_{n\in\mathbb{N}}$ *converges uniformly on* $\mathcal{K}$.

Let $\mathcal{O}$ be some region in $\mathbb{C}^2$. Here, the region refers to a domain (an open connected set) which may include all, part, or none of its boundary.

Definition 5.10. *If for each* $\mathbf{z} \in \mathcal{O}$ *the branched continued fraction (5.10) converges to the finite value* $f(\mathbf{z})$, *then, for* $n \geq, 1$

$$f(\mathbf{z}) - f_n(\mathbf{z})$$

is called the truncation error of the n*th approximant. For* $n \geq 1$ *the estimate of the form*

$$|f(\mathbf{z}) - f_n(\mathbf{z})| \leq C_n(\mathbf{z})$$

is called a priori bound (or truncation error bound), where $C_n(\mathbf{z}) \geq 0$, $n \geq, 1$ *and* $C_n(\mathbf{z}) \to 0$ *as* $n \to +\infty$ *for all* $\mathbf{z} \in \mathcal{O}$.

5.2 Extending analytic functions through branched continued fractions

An important application of branched continued fractions is, in particular, to represent functions expressed by hypergeometric series (or the ratio of hypergeometric series) as branched continued fractions. In this regard, the following two methods are used [9].

PC Method. This method is based on the so-called principle of correspondence [20, pp. 30–32]. It is used when the branched continued fraction expansion corresponds at $\mathbf{z} = \mathbf{0}$ to a hypergeometric series (or the ratio of hypergeometric series), and the sequence of its approximants converges uniformly on each compact subset of some neighborhood of the origin to a function that is holomorphic in this neighborhood. Then it remains to apply the Weierstrass' theorem [51, p. 23], prove the convergence of the above expansion in a wider domain than the domain of convergence of the hypergeometric series (or the ratio of hypergeometric series), and, finally, apply the principle of analytic continuation [56, p. 39].

Theorem 5.1 (Weierstrass' Theorem)**.** *Let a sequence $\{g_n(\mathbf{z})\}$ of holomorphic functions in a domain $\mathcal{D}$ of $\mathbb{C}^2$ converge to a function $g(\mathbf{z})$ uniformly on each compact subset in $\mathcal{D}$, then $f(\mathbf{z})$ is holomorphic in $\mathcal{D}$, and for any $r \geq 0$, $s \geq 0$,*

$$\frac{\partial^{r+s} g_n(\mathbf{z})}{\partial z_1^r \partial z_2^s} \to \frac{\partial^{r+s} g(\mathbf{z})}{\partial z_1^r \partial z_2^s} \quad \textit{as} \quad n \to +\infty$$

on each compact subset in $\mathcal{D}$.

Theorem 5.2 (Principle of Analytic Continuation)**.** *Let the functions $g_1(\mathbf{z})$ and $g_2(\mathbf{z})$ be holomorphic in the domains $\mathcal{D}_1$ and $\mathcal{D}_2$ of $\mathbb{C}^2$, respectively, and let $\mathcal{D}_1 \cap \mathcal{D}_2$ be the domain. Let, further, in a real neighborhood of the point $\mathbf{z}^0$ from $\mathcal{D}_1 \cap \mathcal{D}_2$ the functions $g_1(\mathbf{z})$ and $g_2(\mathbf{z})$ coincide. Then these functions are an analytic continuation of one another, i.e., there is a unique function $g(\mathbf{z})$ that is holomorphic in $\mathcal{D}_1 \cup \mathcal{D}_2$ and coincides with $g_1(\mathbf{z})$ in $\mathcal{D}_1$ and with $g_2(\mathbf{z})$ in $\mathcal{D}_2$.*

We will give the concept of correspondence at $\mathbf{z} = \mathbf{0}$ (see, [9] and also book [20, pp. 30–32)]).

Let

$$L(\mathbf{z}) = \sum_{r,s=0}^{+\infty} a_{r,s} z_1^r z_2^s$$

be a formal double power series at $\mathbf{z} = \mathbf{0}$, where $a_{r,s} \in \mathbb{C}$, $r \geq 0$, $s \geq 0$, $\mathbf{z} \in \mathbb{C}^2$. Let $f(\mathbf{z})$ be function holomorphic at a neighborhood of the origin $\mathbf{z} = \mathbf{0}$. Let the mapping $\Lambda : f(\mathbf{z}) \to \Lambda(f)$ associate with $f(\mathbf{z})$ its Taylor expansion at a neighborhood of the origin.

Definition 5.11. *A sequence $\{f_n(\mathbf{z})\}_{n\in\mathbb{N}_0}$ of functions holomorphic at the origin corresponds to a formal double power series $L(\mathbf{z})$ at $\mathbf{z}=\mathbf{0}$ if*

$$\lim_{n\to+\infty}\lambda(L-\Lambda(f_n))=+\infty,$$

where λ is the function defined as follows: $\lambda:\mathbb{L}\to\mathbb{Z}_{\geq 0}\cup\{+\infty\}$; if $L(\mathbf{z})\equiv 0$ then $\lambda(L)=+\infty$; if $L(\mathbf{z})\not\equiv 0$ then $\lambda(L)=n$, where n is the smallest degree of homogeneous terms for which $a_{r,s}\neq 0$, that is $n=r+s$.

If $\{f_n(\mathbf{z})\}_{n\in\mathbb{N}_0}$ corresponds at $\mathbf{z}=\mathbf{0}$ to a formal double power series $L(\mathbf{z})$, then the order of correspondence of $f_n(\mathbf{z})$ is defined to be

$$\nu_n=\lambda(L-\Lambda(f_n)).$$

By the definition of λ, the series $L(\mathbf{z})$ and $\Lambda(f_n)$ agree for all homogeneous terms up to and including degree (ν_n-1).

Definition 5.12. *A branched continued fraction (5.10) whose elements are polynomials in $\mathbb{C}^2$ corresponds to a formal double power series $L(\mathbf{z})$ at $\mathbf{z}=\mathbf{0}$ if its sequence of approximants corresponds to $L(\mathbf{z})$.*

PF Method. This method uses the property of fork (see, Proposition 5.1) and applies when the hypergeometric function (or the ratio of hypergeometric functions) and the elements of the branched continued fraction expansion are positive-valued functions in some domain $\mathcal{D}$ of $\mathbb{C}^2$. If it holds, then its approximants satisfy the property of fork: the sequence of even (odd) approximants increases (decreases) and is no greater (no less) than any odd (even) approximant. If, in addition, the above expansion converges, then it converges to the hypergeometric function (or the ratio of hypergeometric functions) in the domain $\mathcal{D}$. Finally, it remains to prove the convergence of the branched continued fraction expansion in a wider domain than the domain of convergence of the hypergeometric series (or the ratio of hypergeometric series) and apply the Theorem 5.2.

5.3 Branched continued fraction expansions of the Appell hypergeometric functions

While continued fraction expansions have only one structure with different elements, branched continued fraction expansions can have a wide range of both structures and elements. In both cases, the form of the expansion directly depends on the choice of recurrent relations of the hypergeometric functions. Some structures branched continued fractions were discussed in [12].

Appell hypergeometric function F_1. As already mentioned, the first expansions of the ratios of functions F_1 into branched continued fractions was considered in [32].

Let δ_j^i be the Kronecker symbol. By three-term recurrence relations

$$F_1(a,b_1,b_2;c;\mathbf{z}) = F_1(a,b_1+1,b_2;c;\mathbf{z}) - \frac{a}{c}z_1F_1(a+1,b_1+1,b_2;c+1;\mathbf{z}),$$
$$F_1(a,b_1,b_2;c;\mathbf{z}) = F_1(a,b_1,b_2+1;c;\mathbf{z}) - \frac{a}{c}z_1F_1(a+1,b,b_2+1;c+1;\mathbf{z}),$$
$$F_1(a,b_1,b_2;c;\mathbf{z}) = \frac{c-a}{c}F_1(a,b_1,b_2;c+1;\mathbf{z}) + \frac{a}{c}F_1(a+1,b_1,b_2;c+1;\mathbf{z}),$$

and four-term recurrence relation

$$F_1(a,b_1,b_2;c;\mathbf{z}) =$$
$$= F_1(a+1,b_1,b_2;c+1;\mathbf{z}) - \sum_{k=1}^{2}\frac{b_k}{c}z_kF_1(a+1,b_1+\delta_1^k,b_2+\delta_2^k;c+1;\mathbf{z}),$$

an algorithm was proposed for constructing the branched continued fraction expansions of the following ratios

$$\frac{F_1(a;b_1,b_2;c;\mathbf{z})}{F_1(a+1;b_1+\delta_1^i,b_2+\delta_2^i;c+1;\mathbf{z})}, \quad i=1,2. \tag{5.11}$$

However, no explicit formulas were found for the coefficients of the constructed expansions and no criteria of their convergence was established.

In [40], branched continued fractions was studied, which are expansions of the ratios of the Lauricella hypergeometric functions $F_D^{(N)}$. And, as is known, $F_D^{(N)}$ is equal to F_1 for $N=2$.

Let, for $i=1,2$,

$$p_{i(1)} = \delta_{i_1}^i, \quad i(1)\in\mathcal{I}_1, \quad p_{i(k)} = \sum_{r=1}^{k-1}\delta_{i_r}^{i_k} + \delta_{i_k}^i, \quad i(k)\in\mathcal{I}_k, \quad k\ge 2.$$

The following theorem gives explicit form of the expansions of the ratios (5.11) into branched continued fractions obtained in [32]. It also provides formulas for the coefficients of these expansions in terms of the parameters of the hypergeometric series F_1.

Theorem 5.3 ([46]). *For each $i=1,2$ the ratio (5.11) has a formal branched contin-*

ued fraction expansion

$$v_0^{(i)}(\mathbf{z}) + \cfrac{w_1}{1 + \sum_{i_1=1}^{2} \cfrac{u_{i(1)}^{(i)}(\mathbf{z})}{v_{i(1)}^{(i)}(\mathbf{z}) + \cfrac{w_2}{1 + \sum_{i_2=1}^{2} \cfrac{u_{i(2)}^{(i)}(\mathbf{z})}{v_{i(2)}^{(i)}(\mathbf{z}) + \ddots}}}}, \tag{5.12}$$

where

$$v_{i(k)}^{(i)}(\mathbf{z}) = \frac{a}{c+k}(1 - z_{i_k}), \quad i(k) \in \mathcal{I}_k, \quad k \geq 0,$$

$$w_k = 1 - \frac{a}{c+k-1}, \quad u_{i(k)}^{(i)}(\mathbf{z}) = \frac{b_{i_k} + p_{i(k)}}{c+k} z_{i_k}, \quad i(k) \in \mathcal{I}_k, \quad k \geq 1.$$

The three-term recurrence relations

$$F_1(a, b_1, b_2; c; \mathbf{z}) = \left(1 - \frac{a+1}{c} z_1 - \sum_{k=1}^{2} \frac{b_k}{c} z_k\right) F_1(a+1, b_1+1, b_2; c; \mathbf{z}) +$$

$$+ \sum_{k=1}^{2} \frac{(a+1)(b_k + \delta_k^1)}{c(c+1)} z_k (1 - z_k) F_1(a+2, b_1 + 1 + \delta_1^k, b_2 + \delta_2^k; c+2; \mathbf{z}),$$

$$F_1(a, b_1, b_2; c; \mathbf{z}) = \left(1 - \frac{a+1}{c} z_2 - \sum_{k=1}^{2} \frac{b_k}{c} z_k\right) F_1(a+1, b_1, b_2+1; c; \mathbf{z}) +$$

$$+ \sum_{k=1}^{2} \frac{(a+1)(b_k + \delta_k^2)}{c(c+1)} z_k (1 - z_k) F_1(a+2, b_1 + \delta_1^k, b_2 + 1 + \delta_2^k; c+2; \mathbf{z}),$$

give the following result.

Theorem 5.4 ([17]). *For each $i = 1, 2$ the ratio (5.11) has a formal expansion*

$$\widetilde{v}_0^{(i)}(\mathbf{z}) + \sum_{i_1=1}^{2} \cfrac{\widetilde{u}_{i(1)}^{(i)}(\mathbf{z})}{\widetilde{v}_{i(1)}^{(i)}(\mathbf{z}) + \sum_{i_2=1}^{2} \cfrac{\widetilde{u}_{i(2)}^{(i)}(\mathbf{z})}{\widetilde{v}_{i(2)}^{(i)}(\mathbf{z}) + \ddots}}, \tag{5.13}$$

where

$$\widetilde{v}_0^{(i)}(\mathbf{z}) = 1 - \frac{a+1}{c} z_i - \sum_{r=1}^{2} \frac{b_r}{c} z_r,$$

$$\widetilde{u}_{i(k)}^{(i)}(\mathbf{z}) = \frac{(a+k)(b_{i_k} + p_{i(k)})}{(c+k-1)(c+k)} z_{i_k} (1 - z_{i_k}), \quad i(k) \in \mathcal{I}_k, \quad k \geq 1,$$

$$\widetilde{v}_{i(k)}^{(i)}(\mathbf{z}) = 1 - \frac{a+k}{c+k} z_{i_k} - \sum_{r=1}^{2} \frac{b_r + p_{i(k),r}}{c+k} z_r, \quad i(k) \in \mathcal{I}_k, \quad k \geq 1.$$

Other formal expansions of the ratios (5.11) were obtained by compressing the combination of the fractional-linear mapping and the sum of fractional-linear mappings, which are the basis for constructing branched continued fractions (5.12). Namely, for $i = 1, 2$ it is obtained (see, [40, Prorosition 2.1.3])

$$\widehat{v}_{-1}^{(i)}(\mathbf{z}) + \cfrac{\widehat{u}_0}{\widehat{v}_0^{(i)}(\mathbf{z}) + \sum_{i_1=1}^{2} \cfrac{\widehat{u}_{i(1)}^{(i)}(\mathbf{z})}{\widehat{v}_{i(1)}^{(i)}(\mathbf{z}) + \sum_{i_2=1}^{2} \cfrac{\widehat{u}_{i(2)}^{(i)}(\mathbf{z})}{\widehat{v}_{i(2)}^{(i)}(\mathbf{z}) + \ddots}}}, \tag{5.14}$$

where

$$\widehat{v}_{-1}^{(i)}(\mathbf{z}) = \frac{a}{c}(1 - z_i), \quad \widehat{u}_0 = 1 - \frac{a}{c}, \quad \widehat{v}_0^{(i)}(\mathbf{z}) = 1 + \sum_{r=1}^{2} \frac{b_r + \delta_r^i}{a} \frac{z_r}{1 - z_r},$$

$$\widehat{u}_{i(1)}^{(i)}(\mathbf{z}) = -\frac{(c-a+1)(b_{i_1} + p_{i(1)})}{a(c+1)} \frac{z_{i_k}}{1 - z_{i_k}}, \quad i(1) \in \mathcal{I}_1,$$

$$\widehat{u}_{i(k)}^{(i)}(\mathbf{z}) = -\frac{(c-a+k)(b_{i_k} + p_{i(k)})}{(c+k-1)(c+k)} \frac{(1 - z_{i_{k-1}}) z_{i_k}}{1 - z_{i_k}}, \quad i(k) \in \mathcal{I}_k, \quad k \geq 2,$$

$$\widehat{v}_{i(k)}^{(i)}(\mathbf{z}) = 1 - \frac{a}{c+k} z_{i_k} + \sum_{r=1}^{2} \frac{b_r + p_{i(k),r}}{c+k} \frac{(1 - z_{i_k}) z_r}{1 - z_r}, \quad i(k) \in \mathcal{I}_k, \quad k \geq 1.$$

For each $i = 1, 2$, the branched continued fraction (5.14) with the sequence of approximants $\{\widehat{f}_k(\mathbf{z})\}_{k \in \mathbb{N}}$ is the even part of the expansion (5.12) with the sequence of approximants $\{f_k(\mathbf{z})\}_{k \in \mathbb{N}}$, i.e., $f_{2k}(\mathbf{z}) = \widehat{f}_k(\mathbf{z})$ for all $k \geq 1$ [41].

Finally, in [46], for another ratio

$$\frac{F_1(a-1; b_1, b_2; c; \mathbf{z})}{F_1(a; b_1, b_2; c; \mathbf{z})}$$

the formal branched continued fraction

$$1+\sum_{i=1}^{2}\cfrac{\overline{u}_0^{(i)}(\mathbf{z})}{1+\cfrac{\overline{w}_0^{(i)}(\mathbf{z})}{1+\sum\limits_{i_1=1}^{2}\cfrac{\overline{u}_{i(1)}^{(i)}(\mathbf{z})}{1+\cfrac{\overline{w}_{i(1)}^{(i)}(\mathbf{z})}{1+\sum\limits_{i_2=1}^{2}\cfrac{\overline{u}_{i(2)}^{(i)}(\mathbf{z})}{1+\cfrac{\overline{w}_{i(2)}^{(i)}(\mathbf{z})}{1+\ddots}}}}}} \tag{5.15}$$

was constructed using the additional relation

$$F_1(a;b_1,b_2;c;\mathbf{z})=(1-z_1)^{-b_1}(1-z_2)^{-b_2}F_1\left(c-a;b_1,b_2;c;\frac{z_1}{z_1-1},\frac{z_2}{z_2-1}\right),$$

where

$$\overline{u}_0^{(i)}(\mathbf{z})=\frac{b_i}{a-c}z_i,\quad \overline{w}_0=-\frac{a}{a-c}(1-z_i),$$
$$\overline{u}_{i(k)}^{(i)}(\mathbf{z})=-\frac{b_{i_k}+p_{i(k)}+\delta_{i_k}^{i}}{a-c}z_{i_k},\quad i(k)\in\mathcal{I}_k,\quad k\ge 1,$$
$$\overline{w}_{i(k)}^{(i)}(\mathbf{z})=-\frac{a+k}{a-c}(1-z_{i_k}),\quad i(k)\in\mathcal{I}_k,\quad k\ge 1.$$

Of all the constructed branched continued fractions, only (5.13) was proven to converge to the functions of which they are the expansions, and that they provide an analytic extension of these functions into the domain of their convergence.

Theorem 5.5 ([10]). *Suppose that a, b_1, b_2, and c are real constants such that*

$$a\ge 0,\quad b_1\ge 0,\quad b_2\ge 0,\quad 2c\ge a+b_1+b_2+1.$$

Then, for each $i=1,2$ the following statements hold:

(A) The branched continued fraction (5.13) converges uniformly on every compact subset of the domain

$$\Omega=\left\{\mathbf{z}\in\mathbb{C}^2:\ \mathrm{Re}(z_1)<\frac{1}{2},\ \mathrm{Re}(z_2)<\frac{1}{2}\right\} \tag{5.16}$$

to the function $\widetilde{f}^{(i)}(\mathbf{z})$ holomorphic in the domain Ω.

(B) The function $\widetilde{f}^{(i)}(\mathbf{z})$ is an analytic continuation of the function (5.11) in the domain (5.16).

Setting $a = 0$, replacing b_1 and c respectively by $b-1$ and $c-1$ in Theorem 5.5, it is obtained the following result.

Corollary 5.1. *Let b_1, b_2, and c be real constants satisfying the inequalities*

$$b_1 \geq 1, \quad b_2 > 0, \quad 2c > b_1 + b_2 + 2.$$

Then the branched continued fraction

$$\cfrac{1}{\widetilde{v}_0(\mathbf{z}) + \sum_{i_1=1}^{2} \cfrac{\widetilde{u}_{i(1)}(\mathbf{z})}{\widetilde{v}_{i(1)}(\mathbf{z}) + \sum_{i_2=1}^{2} \cfrac{\widetilde{u}_{i(2)}(\mathbf{z})}{\widetilde{v}_{i(2)}(\mathbf{z}) + \ddots}}} \tag{5.17}$$

converges uniformly on every compact subset of (5.16) to a function $\widehat{f}(\mathbf{z})$ holomorphic in Ω, and, in addition, $\widehat{f}(\mathbf{z})$ is an analytic continuation of the function $F_1(1; b_1, b_2; c; \mathbf{z})$ in the domain (5.16), where

$$\widetilde{v}_0(\mathbf{z}) = 1 - \sum_{r=1}^{2} \frac{b_r}{c-1} z_r,$$

$$\widetilde{u}_{i(k)}(\mathbf{z}) = \frac{k(b_{i_k} + p_{i(k)}) - \delta_{i_k}^1}{(c+k-2)(c+k-1)} z_{i_k}(1 - z_{i_k}), \quad i(k) \in \mathcal{I}_k, \quad k \geq 1,$$

$$\widetilde{v}_{i(k)}(\mathbf{z}) = 1 - \frac{k}{c+k-1} z_{i_k} - \sum_{r=1}^{2} \frac{b_r + p_{i(k),r} - \delta_{i_k}^1}{c+k-1} z_r, \quad i(k) \in \mathcal{I}_k, \quad k \geq 1.$$

The similar consequence is valid when $a = 0$, b_2 and c are replaced by $b_2 - 1$ and $c - 1$, respectively.

A number of convergence criteria were established for branched continued fractions (5.12) and (5.14), but the question of representing the functions by these branched continued fractions, as well as (5.15), the expansions of which they are, remains open.

Appell hypergeometric functions F_2. In [16], using three-term recurrence relations

$$F_2(a, b_1, b_2; c_1 c_2; \mathbf{z}) = F_2(a, b_1 + 1, b_2; c_1 + 1, c_2; \mathbf{z}) -$$
$$- \frac{a(c_1 - b_1)}{c_1(c_1 + 1)} z_1 F_2(a + 1, b_1 + 1, b_2; c_1 + 2, c_2; \mathbf{z}),$$
$$F_2(a, b_1, b_2; c_1 c_2; \mathbf{z}) = F_2(a, b_1, b_2 + 1; c_1, c_2 + 1; \mathbf{z}) -$$

$$-\frac{a(c_2-b_2)}{c_2(c_2+1)}z_2F_2(a+1,b_1,b_2+1;c_1,c_2+2;\mathbf{z}), \tag{5.18}$$

and four-term recurrence relation

$$\begin{gathered}
F_2(a,b_1,b_2;c_1c_2;\mathbf{z}) = F_2(a+1,b_1,b_2;c_1+1,c_2;\mathbf{z})- \\
-\frac{b_1(c_1-a)}{c_1(c_1+1)}z_1F_2(a+1,b_1+1,b_2;c_1+2,c_2;\mathbf{z})- \\
-\frac{b_2}{c_2}z_2F_2(a+1,b_1,b_2+1;c_1+1,c_2+1;\mathbf{z}), \\
F_2(a,b_1,b_2;c_1c_2;\mathbf{z}) = F_2(a+1,b_1,b_2;c_1,c_2+1;\mathbf{z})- \\
-\frac{b_1}{c_1}z_1F_2(a+1,b_1+1,b_2;c_1+1,c_2+1;\mathbf{z})- \\
-\frac{b_2(c_2-a)}{c_2(c_2+1)}z_2F_2(a+1,b_1,b_2+1;c_1,c_2+2;\mathbf{z}),
\end{gathered} \tag{5.19}$$

was obtained an expansion ratio

$$\frac{F_2(a,b_1,b_2;c_1,c_2;\mathbf{z})}{F_2(a+1,b_1,b_2;c_1+1,c_2;\mathbf{z})} \tag{5.20}$$

into formal branched continued fraction

$$1-\sum_{i_1=1}^{2}\cfrac{u_{i(1)}(\mathbf{z})}{1-\cfrac{w_{i(1)}(\mathbf{z})}{1-\sum\limits_{i_2=1}^{2}\cfrac{u_{i(2)}(\mathbf{z})}{1-\cfrac{w_{i(2)}(\mathbf{z})}{1-\ddots}}}},$$

where

$$u_{i(1)}(\mathbf{z}) = \frac{b_1(c_1-a)}{c_1(c_1+1)}\delta_{i_1}^1 z_1+\delta_{i_1}^2 z_2, \quad i(1)\in\mathcal{I}_1,$$

$$u_{i(k+1)}(\mathbf{z}) = \frac{(c_{i_{k+1}}-a+2p_{i(k)}-k-\delta_{i_{k+1}}^2)(b_{i_{k+1}}+p_{i(k)})}{(c_{i_{k+1}}+2p_{i(k)})(c_{i_{k+1}}+2p_{i(k)}+(-1)^{\delta_{i_{k+1}}^2})}\delta_{i_{k+1}}^{i_k}z_{i_{k+1}}$$

$$+\frac{b_{i_{k+1}}+p_{i(k)}}{c_{i_{k+1}}+2p_{i(k)}+\delta_{i_{k+1}}^1}\delta_{i_{k+1}+i_k}^3 z_{i_{k+1}}, \quad i(k+1)\in\mathcal{I}_{k+1}, \quad k\ge 1,$$

$$w_{i(k)}(\mathbf{z}) = \frac{(c_{i_k}-b_{i_k}+p_{i(k)}-\delta_{i_k}^2)(a+k)}{(c_{i_k}+2p_{i(k)}-1-\delta_{i_k}^2)(c_{i_k}+2p_{i(k)}-\delta_{i_k}^2)}z_{i_k}, \quad i(k)\in\mathcal{I}_k, \quad k\ge 1,$$

where

$$p_{i(k)} = \sum_{r=1}^{k}\delta_{i_r}^{i_k}, \quad i(k)\in\mathcal{I}_k, \quad k\ge 1.$$

Similarly, we can obtain the expansion of ratio

$$\frac{F_2(a, b_1, b_2; c_1, c_2; \mathbf{z})}{F_2(a+1, b_1, b_2; c_1, c_2+1; \mathbf{z})}, \tag{5.21}$$

symmetric to (5.20), into a branched continued fraction.

Expansions of other ratios of the Appell hypergeometric functions F_2 into branched continued fractions can be found in [45]. However, the question of their convergence remains open.

Setting $b_1 = c_1$ in (5.21), let

$$R^{(1)}_{F_2}(a, b_1, b_2; b_1, c_2; \mathbf{z}) = \frac{F_2(a, b_1, b_2; b_1, c_2; \mathbf{z})}{F_2(a+1, b_1, b_2; b_1, c_2+1; \mathbf{z})}, \tag{5.22}$$

$$R^{(2)}_{F_2}(a, b_1, b_2; b_1, c_2; \mathbf{z}) = \frac{F_2(a, b_1, b_2; b_1, c_2; \mathbf{z})}{F_2(a, b_1, b_2+1; b_1, c_2+1; \mathbf{z})}. \tag{5.23}$$

Then the following result holds [2].

Theorem 5.6. *The ratio (5.22) has a formal branched continued fraction expansion of the form*

$$1 - z_1 - \cfrac{u_1 z_2}{1 - \cfrac{u_2 z_2}{1 - z_1 - \cfrac{u_3 z_2}{1 - \cfrac{u_4 z_2}{1 - \ddots}}}}, \tag{5.24}$$

where

$$u_{2k-1} = \frac{(b_2+k-1)(c_2-a+k-1)}{(c_2+2k-2)(c_2+2k-1)}, \quad u_{2k} = \frac{(a+k)(c_2-b_2+k)}{(c_2+2k-1)(c_2+2k)}, \quad k \geq 1. \tag{5.25}$$

Note that (5.24) is a confluent branched continued fraction, not a continued fraction. Here, the fundamental difference between them lies in the different approach to understand the approximants. Namely, the sequence of approximants of the continued fraction for the branched continued fraction is a sequence of so-called figured approximants [15, pp. 18–20].

Proof Theorem 5.6. Since

$$F_2(a, b_1, b_2; b_1, c_2; \mathbf{z}) = F_2(a, b_1+1, b_2; b_1+1, c_2; \mathbf{z}),$$

then, dividing (5.19) and (5.18) by

$$F_2(a+1, b_1, b_2; b_1, c_2+1; \mathbf{z}) \quad \text{and} \quad F_2(a, b_1, b_2+1; b_1, c_2+1; \mathbf{z}),$$

respectively, we obtain

$$R^{(1)}_{F_2}(a,b_1,b_2;b_1,c_2;\mathbf{z}) = 1 - z_1 - \cfrac{\frac{b_2(c_2-a)}{c_2(c_2+1)}z_2}{R^{(2)}_{F_2}(a+1,b_1,b_2;b_1,c_2+1;\mathbf{z})}, \tag{5.26}$$

$$R^{(2)}_{F_2}(a,b_1,b_2;b_1,c_2;\mathbf{z}) = 1 - \cfrac{\frac{a(c_2-b_2)}{c_2(c_2+1)}z_2}{R^{(1)}_{F_2}(a,b_1,b_2+1;b_1,c_2+1;\mathbf{z})}. \tag{5.27}$$

In fact, in (5.26), we have Step 1.1 of constructing a branched continued fraction. At Step 1.2, replacing a, c_2 by $a+1$ and c_2+1, respectively, in (5.27), we get

$$R^{(1)}_{F_2}(a,b_1,b_2;b_1,c_2;\mathbf{z}) = \\ = 1 - z_1 - \cfrac{\frac{b_2(c_2-a)}{c_2(c_2+1)}z_2}{1 - \cfrac{\frac{(a+1)(c_2+1-b_2)}{(c_2+1)(c_2+2)}z_2}{R^{(1)}_{F_2}(a+1,b_1,b_2+1;b_1,c_2+2;\mathbf{z})}}. \tag{5.28}$$

Let us continue the next construction of the branched continued fraction in the same way as in Steps 1.1–1.2. It is clear that the following relation holds, for all $k \geq 1$,

$$R^{(1)}_{F_2}(a+k-1,b_1,b_2+k-1;b_1,c_2+2k-2;\mathbf{z}) = \\ = 1 - z_1 - \cfrac{\frac{(b_2+k-1)(c_2+k-1-a)}{(c_2+2k-2)(c_2+2k-1)}z_2}{1 - \cfrac{\frac{(a+k)(c_2+k-b_2)}{(c_2+2k-1)(c_2+2k)}z_2}{R^{(1)}_{K}(a_1+k,b_1,b_2+k;b_1,c_2+2k;\mathbf{z})}}. \tag{5.29}$$

At Steps 2.1–2.2, substituting (5.29) when $k=2$ in (5.28), we obtain

$$R^{(1)}_{F_2}(a,b_1,b_2;b_1,c_2;\mathbf{z}) =$$

$$= 1 - z_1 - \cfrac{\frac{b_2(c_2-a)}{c_2(c_2+1)}z_2}{1 - \cfrac{\frac{(a+1)(c_2+1-b_2)}{(c_2+1)(c_2+2)}z_2}{1 - z_1 - \cfrac{\frac{(b_2+1)(c_2+1-a)}{(c_2+2)(c_2+3)}z_2}{1 - \cfrac{\frac{(a+2)(c_2+2-b_2)}{(c_2+3)(c_2+4)}z_2}{R^{(1)}_{F_2}(a+2, b_1, b_2+2; b_1, c_2+4; \mathbf{z})}}}}.$$

Next, by (5.29) after the Steps k.1–k.2, we have

$$R^{(1)}_{F_2}(a, b_1, b_2; b_1, c_2; \mathbf{z}) =$$

$$= 1 - z_1 - \cfrac{\frac{b_2(c_2-a)}{c_2(c_2+1)}z_2}{1 - \cfrac{\frac{(a+1)(c_2+1-b_2)}{(c_2+1)(c_2+2)}z_2}{1 - \ddots - z_1 - \cfrac{\frac{(b_2+k-1)(c_2+k-1-a)}{(c_2+2k-2)(c_2+2k-1)}z_2}{1 - \cfrac{\frac{(a+k)(c_2+k-b_2)}{(c_2+2k-1)(c_2+2k)}z_2}{R^{(1)}_{K}(a_1+k, b_1, b_2+k; b_1, c_2+2k; \mathbf{z})}}}}.$$

Finally, as $k \to +\infty$, we obtain the formal expansion of (5.22) into branched continued fraction (5.24). □

The following theorem can be proved in much the same way as Theorem 5.6.

Theorem 5.7. *A ratio (5.23) has a formal branched continued fraction of the form*

$$1 - \cfrac{\widetilde{u}_1 z_2}{1 - z_1 - \cfrac{\widetilde{u}_2 z_2}{1 - \cfrac{\widetilde{u}_3 z_2}{1 - z_1 - \cfrac{\widetilde{u}_4 z_2}{1 - \ddots}}}},$$

where, for $k \geq 1$,

$$\widetilde{u}_{2k-1} = \frac{(a+k-1)(c_2-b_2+k-1)}{(c_2+2k-2)(c_2+2k-1)},$$
$$\widetilde{u}_{2k} = \frac{(b_2+k-1)(c_2-a+k-1)}{(c_2+2k-1)(c_2+2k)}.$$

The question of the convergence expansion (5.24) to the ratio (5.22) was investigated in works [2, 29, 31].

Theorem 5.8 ([29]). *Let a, b', and c' be real constants such that*

$$0 < u_k \le \tau, \quad k \ge 1, \tag{5.30}$$

where u_k, $k \ge 1$, are defined by (5.25), τ is a positive number. Then:

(A) *The branched continued fraction (5.24) converges uniformly on every compact subset of the domain*

$$\Omega_\tau = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ z_1 \notin [1, +\infty), \ z_2 \notin \left[\frac{1}{4\tau}, +\infty\right) \right\} \tag{5.31}$$

to a function $f(\mathbf{z})$ holomorphic in Ω_τ.

(B) *The function $f(\mathbf{z})$ is an analytic continuation of function (5.22) in the domain (5.31).*

Note that the assumption on the sequence $\{u_k\}_{k\in\mathbb{N}}$ in Theorem 5.7 involves (together with positivity) an upper bound τ, and that the domain of the analytic continuation also depends on this τ; and the smaller τ, the larger domain.

Setting $a = 0$ and replacing c_2 by $c_2 - 1$ in Theorem 5.8, we have the following corollary.

Corollary 5.2. *Let b_2 and c_2 be real constants such that*

$$0 < \frac{b_2}{c_2} \le \tau, \quad 0 < \frac{(b_2 + k)(c_2 + k - 1)}{(c_2 + 2k - 1)(c_2 + 2k)} \le \tau, \quad k \ge 1, \tag{5.32}$$

and

$$0 < \frac{k(c_2 - b_2 + k - 1)}{(c_2 + 2k - 2)(c_2 + 2k - 1)} \le \tau, \quad k \ge 1, \tag{5.33}$$

where τ is a positive number. Then the branched continued fraction

$$\cfrac{1}{1 - z_1 - \cfrac{\frac{b_2}{c_2} z_2}{1 - \cfrac{\frac{(c_2 - b_2)}{c_2(c_2+1)} z_2}{1 - z_1 - \cfrac{\frac{(b_2+1)c_2}{(c_2+1)(c_2+2)} z_2}{1 - \cfrac{\frac{2(c_2 - b_2 + 1)}{(c_2+2)(c_2+3)} z_2}{1 - \ddots}}}}} \tag{5.34}$$

converges uniformly on every compact subset of the domain (5.31) to a function $f(\mathbf{z})$ *holomorphic in* Ω_τ*, and, in addition, the function* $f(\mathbf{z})$ *is an analytic continuation of* $F_2(1, b_1, b_2; b_1, c_2; \mathbf{z})$ *in the domain (5.31).*

The following theorem provides a condition for the convergence of the branched continued fraction expansion in the case of complex coefficients.

Theorem 5.9 ([29]). *Suppose that* a $_2$*, and* c_2 *are complex constants such that*

$$|u_k| - \mathrm{Re}(u_k) \le \lambda\mu, \quad k \ge 1, \tag{5.35}$$

where u_k*,* $k \ge 1$*, are given in (5.25) herewith* $c_2 \notin \{0, -1, -2, \ldots\}$*,* $\lambda > 0$*, and* $0 < \mu < 1$*. Then, the following statements hold:*

(A) *The branched continued fraction*

$$1 - z_1 + \cfrac{u_1 z_2}{1 + \cfrac{u_2 z_2}{1 - z_1 + \cfrac{u_3 z_2}{1 + \cfrac{u_4 z_2}{1 - \ddots}}}}$$

converges uniformly on every compact subset of the domain

$$\Omega^{\eta,\tau}_{\lambda,\mu} = \Omega_{\lambda,\mu} \bigcup \Omega^{\eta,\tau}, \tag{5.36}$$

where

$$\Omega_{\lambda,\mu} = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ \mathrm{Re}(z_1 e^{-(i/2)\arg(z_2)}) < (1-\mu)\cos\left(\frac{\arg(z_2)}{2}\right), \right. \\ \left. |z_2| < \frac{1 + \cos(\arg(z_2))}{4\lambda} \right\}$$

and

$$\Omega^{\eta,\tau} = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ |z_1| < \frac{1-\eta}{2}, \ |z_2| < \frac{\eta(1-\eta)}{2\tau} \right\},$$

where

$$\tau = \sup_{k\in\mathbb{N}} |u_k| \quad \text{and} \quad 0 < \eta < 1,$$

to the function $f(\mathbf{z})$ *holomorphic in the domain* $\Omega^{\eta,\tau}_{\lambda,\mu}$*.*

(B) *The function* $f(\mathbf{z})$ *is an analytic continuation of the function*

$$\frac{F_2(a, b_1, b_2; b_1, c_2; z_1, -z_2)}{F_2(a+1, b_1, b_2; b_1, c_2+1; z_1, -z_2)} \tag{5.37}$$

in the domain (5.36).

Corollary 5.3. *Suppose that b_2 and c_2 are complex constants such that satisfy inequality (5.35), where*

$$u_1 = \frac{b_2}{c_2}, \quad u_{2k} = \frac{k(c_2 - b_2 + k - 1)}{(c_2 + 2k - 2)(c_2 + 2k - 1)},$$
$$u_{2k+1} = \frac{(b_2 + k)(c_2 + k - 1)}{(c_2 + 2k - 1)(c_2 + 2k)}, \quad k \geq 1,$$

herewith $c_2 \notin \{0, -1, -2, \ldots\}$, and where $\lambda > 0$ and $0 < \mu < 1$. Then the branched continued fraction

$$\cfrac{1}{1 - z_1 + \cfrac{\frac{b_2}{c_2} z_2}{1 + \cfrac{\frac{(c_2 - b_2)}{c_2(c_2+1)} z_2}{1 - z_1 + \cfrac{\frac{(b_2+1)c_2}{(c_2+1)(c_2+2)} z_2}{1 + \cfrac{\frac{2(c_2 - b_2 + 1)}{(c_2+2)(c_2+3)} z_2}{1 - \ddots}}}}}$$

converges uniformly on every compact subset of the domain (5.36) to the function $f(\mathbf{z})$ holomorphic in this domain, in addition, the function $f(\mathbf{z})$ is an analytic continuation of the function $F_2(1, b_1, b_2; b_1, c_2; z_1, -z_2)$ in the domain $\Omega^{\eta,\tau}_{\lambda,\mu}$.

Graphical illustrations of domains for variables z_1 and z_2 in (5.36) are shown in Figure 1a–c [29].

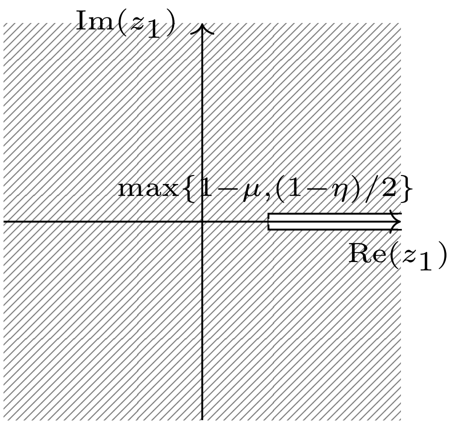
The following theorem establishes truncation errors bounds for branched continued fractions (5.24) in the case of non-negative elements.

Theorem 5.10 ([31])**.** *Suppose that a, b_2, and c_2 are real constants that satisfy the inequalities in (5.30). Then, the following apply:*

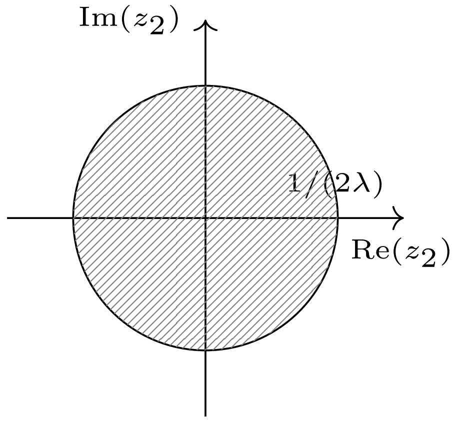
(A) The branched continued fraction (5.24) converges to a finite value $f(\mathbf{z})$ for each $\mathbf{z} \in \Omega_\kappa$, where

$$\Omega_\kappa = \{\mathbf{z} \in \mathbb{R}^2 : \ z_1 \leq \kappa, \ z_2 \leq 0\}, \quad 0 < \kappa < 1. \tag{5.38}$$

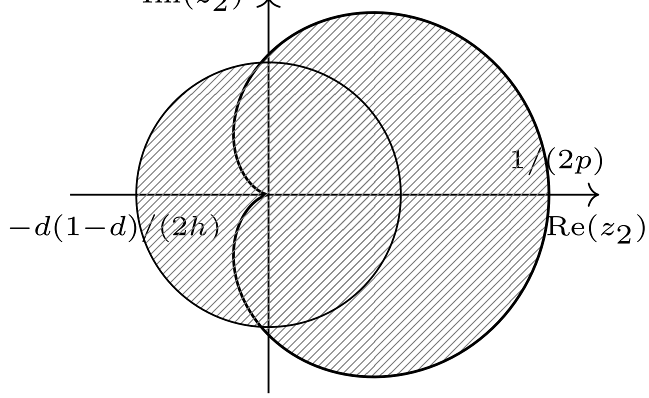
(B) The convergence is uniform on every compact subset of the domain $\mathrm{Int}(\Omega_\kappa)$, and $f(\mathbf{z})$ is analytic on $\mathrm{Int}(\Omega_\kappa)$.



(**a**) Domain for z_1.



(**b**) Domain for z_2 if $\tau = \lambda\eta(1-\eta)$.



(**c**) Domain for z_2 if $\tau > \lambda\eta(1-\eta)$.

Figure 1: Domains for variables z_1 and z_2 in (5.38).

(C) If $f_k(\mathbf{z})$ denotes the kth approximant of (5.24), then for each $\mathbf{z} \in \Omega_\kappa$

$$|f(\mathbf{z}) - f_k(\mathbf{z})| \le \frac{(\delta_1^{(-1)^k}|z_1|(1-\delta_1^{(-1)^{k+1}}z_1) + \tau|z_2|)(\tau|z_2|)^k}{(1-z_1)(1-z_1+\tau|z_2|)^{k-1}}, \quad k \ge 2.$$

(D) The function $f(\mathbf{z})$ is an analytic continuation of (5.22) in (5.38).

Setting $a = 0$ and replacing c_2 with $c_2 - 1$ in Theorem 5.10, we obtain a priori bound for the branched continued fraction (5.34).

Corollary 5.4. *Suppose that b_2 and c_2 are real constants that satisfy the inequality (5.32)–(5.33). Then, the following apply:*

(A) The branched continued fraction (5.34) converges to a finite value $f(\mathbf{z})$ for each $\mathbf{z} \in \Omega_\kappa$, where Ω_κ is defined by (5.38).

(B) The convergence is uniform on every compact subset of the domain $\mathrm{Int}(\Omega_\kappa)$, and $f(\mathbf{z})$ is analytic on $\mathrm{Int}(\Omega_\kappa)$.

(C) If $f_k(\mathbf{z})$ denotes the kth approximant of (5.34), then for each $\mathbf{z} \in \Omega H_\kappa$,

$$|f(\mathbf{z}) - f_k(\mathbf{z})| \le \frac{(\delta_1^{(-1)^k}|z_1|(1-\delta_1^{(-1)^{k+1}}z_1) + \kappa|z_2|)(\kappa|z_2|)^{k-1}}{(1-z_1)^3(1-z_1+\kappa|z_2|)^{k-3}}, \quad k \ge 3.$$

(D) The function $f(\mathbf{z})$ is an analytic continuation of $F_2(1, b, b_2; b, c_2; \mathbf{z})$ in (5.38).

Similar results to Theorems 5.8 and 5.10 and their consequences can be obtained for ratio (5.23). In addition, similar results to Theorem 5.9 and Corollary 5.3 can be also obtained for a ratio symmetric to (5.37).

Appell hypergeometric functions F_3. The following theorem on the expansion of the ratio

$$\frac{F_3(a_1, a_2, b_1, b_2; c; \mathbf{z})}{F_3(a_1+1, a_2, b_1+1, b_2; c+1; \mathbf{z})} \tag{5.39}$$

into a branched continued fraction was proved in [47] using four-term recurrence relations

$$\begin{aligned} F_3(a_1, a_2, b_1, b_2; c; \mathbf{z}) = &\left(1 - \frac{a_1+b_1+1}{c}z_1\right) F_3(a_1+1, a_2, b_1+1, b_2; c+1; \mathbf{z}) + \\ &+ \frac{(a_1+1)(b_1+1)}{c(c+1)} z_1(1-z_1) F_3(a_1+2, a_2, b_1+2, b_2; c+2; \mathbf{z}) + \\ &+ \frac{a_2 b_2}{c(c+1)} z_2 F_3(a_1+1, a_2+1, b_1+1, b_2+1; c+2; \mathbf{z}) \text{ and} \end{aligned}$$

$$\begin{aligned} F_3(a_1, a_2, b_1, b_2; c; \mathbf{z}) = &\left(1 - \frac{a_2+b_2+1}{c}z_2\right) F_3(a_1, a_2+1, b_1, b_2+1; c+1; \mathbf{z}) + \\ &+ \frac{a_1 b_1}{c(c+1)} z_1 F_3(a_1+1, a_2+1, b_1+1, b_2+1; c+2; \mathbf{z}) + \\ &+ \frac{(a_2+1)(b_2+1)}{c(c+1)} z_2(1-z_2) F_3(a_1, a_2+2, b_1, b_2+2; c+2; \mathbf{z}). \end{aligned}$$

Theorem 5.11. *The ratio (5.39) has a formal branched continued fraction expansion of the form*

$$v_0(\mathbf{z}) + \sum_{i_1=1}^{2} \frac{u_{i(1)}(\mathbf{z})}{v_{i(1)}(\mathbf{z}) + \sum_{i_1=1}^{2} \frac{u_{i(2)}(\mathbf{z})}{v_{i(2)}(\mathbf{z}) + \ddots}}, \tag{5.40}$$

where

$$v_0(\mathbf{z}) = 1 - \frac{a_1+b_1+1}{c} z_1,$$

$$u_{i(1)}(\mathbf{z}) = \frac{(a_{i_1}+\delta^1_{i_1})(b_{i_1}+\delta^1_{i_1})}{c(c+1)} z_{i_1}(1-\delta^1_{i_1} z_{i_1}), \quad i(1)\in\mathcal{I}_1,$$

$$u_{i(k)}(\mathbf{z}) = \frac{(a_{i_k}+p_{i(k)}-\delta^2_{i_k})(b_{i_k}+p_{i(k)}-\delta^2_{i_k})}{(c+n-1)(c+n)} z_{i_k}(1-\delta^{i_k}_{i_{k-1}} z_{i_k}),\ i(k)\in\mathcal{I}_k,\ k\ge 2;$$

and

$$v_{i(k)}(\mathbf{z}) = 1 - \frac{a_{i_k}+b_{i_k}+2p_{i(k)}+(-1)^{\delta^2_{i_k}}}{c+k}, \quad i(k)\in\mathcal{I}_k, \quad k\ge 1,$$

with $p_{i(k)} = \sum_{r=1}^{k} \delta^{i_k}_{i_r}$, $i(k)\in\mathcal{I}_k, \quad k\ge 1.$

The following result was also proved in [47].

Theorem 5.12. *Let* $a_1, a_2, b_1, b_2,$ *and* c *be real positive numbers such that*

$$c > \max\left\{ b_2 + \frac{a_1+b_1+1}{2},\ b_1 + \frac{a_2+b_2+1}{2} \right\}.$$

Then:

(A) The branched continued fraction (5.40) converges to a finite value $f(\mathbf{z})$ *for each* $\mathbf{z}\in\Omega$, *where*

$$\Omega = \left\{ \mathbf{z}\in\mathbb{C}^3 :\ \frac{1}{2} - \mathrm{Re}(z_k) \ge \sqrt{|z_k||1-z_k|},\ k=1,2, \right\}.$$

(B) The convergence is uniform on every compact subset of the domain $\mathrm{Int}(\Omega)$, *and* $f(\mathbf{z})$ *is analytic on* $\mathrm{Int}(\Omega)$.

(C) The function $f(\mathbf{z})$ *is an analytic continuation of the function (5.39) in the domain* $\mathrm{Int}(\Omega)$.

Similar results to Theorems 5.11 and 5.12 can be also obtained for a ratio symmetric to (5.39). A formal expansion of another ratio of the Appell hypergeometric functions F_3 into a branched continued fraction can be found in [45].

Appell hypergeometric functions F_4. The expansions of the ratios of the Appell hypergeometric functions F_4 is the least studied. In work [18], based on the four-term recurrence relations

$$F_4(a,b;c_1,c_2;\mathbf{z}) = F_4(a+1,b;c_1+1,c_2;\mathbf{z}) -$$

$$-\frac{b(c_1-a)}{c_1(c_1+1)}z_1F_4(a+1,b+1;c_1+2,c_2;\mathbf{z})-\frac{b}{c_2}z_2F_4(a+1,b+1;c_1+1,c_2+1;\mathbf{z}),$$

$$F_4(a,b;c_1,c_2;\mathbf{z})=F_4(a,b+1;c_1+1,c_2;\mathbf{z})-$$

$$-\frac{a(c_1-b)}{c_1(c_1+1)}z_1F_4(a+1,b+1;c_1+2,c_2;\mathbf{z})-\frac{a}{c_2}z_2F_4(a+1,b+1;c_1+1,c_2+1;\mathbf{z}),$$

$$F_4(a,b;c_1,c_2;\mathbf{z})=F_4(a+1,b;c_1,c_2+1;\mathbf{z})-$$

$$-\frac{b}{c_1}z_1F_4(a+1,b+1;c_1+1,c_2+1;\mathbf{z})-\frac{b(c_2-a)}{c_2(c_2+1)}z_2F_4(a+1,b+1;c_1,c_2+2;\mathbf{z})$$

$$F_4(a,b;c_1,c_2;\mathbf{z})=F_4(a,b+1;c_1,c_2+1;\mathbf{z})-$$

$$-\frac{a}{c_1}z_1F_4(a+1,b+1;c_1+1,c_2+1;\mathbf{z})-\frac{a(c_2-b)}{c_2(c_2+1)}z_2F_4(a+1,b+1;c_1,c_2+2;\mathbf{z}),$$

a formal expansion of the ratio

$$\frac{F_4(a,b;c_1,c_2;\mathbf{z})}{F_4(a+1,b;c_1+1,c_2;\mathbf{z})} \tag{5.41}$$

into a branched continued fraction

$$1-\sum_{i_1=1}^{2}\frac{u_{i(1)}(\mathbf{z})}{1-\sum_{i_2=1}^{2}\frac{u_{i(2)}(\mathbf{z})}{1-\sum_{i_3=1}^{2}\frac{u_{i(3)}(\mathbf{z})}{1-\ddots}}}$$

was constructed, where

$$u_{i(1)}(\mathbf{z})=\frac{b(c_1-a)}{c_1(c_1+1)}\delta_{i_1}^1z_1+\frac{b}{c_2}\delta_{i_1}^2z_2,\quad i(1)\in\mathcal{I}_1,$$

$$u_{i(2k)}(\mathbf{z})=\frac{(a+k)(c_{i_{2k}}-b+p_{i(2k-1)}-k+\delta_{i_{2k}}^1)}{(c_{i_{2k}}+p_{2k-1}-\delta_{i_{2k}}^2)(c_{i_{2k}}+p_{i(k-1)}+\delta_{i_{2k}}^1)}\delta_{i_{2k}}^{i_{2k-1}}z_{i_{2k}}+$$

$$+\frac{a+k}{c_{i_{2k}}+p_{i(k-1)}+\delta_{i_{2k}}^1}\delta_{i_{2k-1}+i_{2k}}^3z_{i_{2k}},\quad i(2k)\in\mathcal{I}_{2k},\quad k\ge1,$$

$$u_{i(2k+1)}(\mathbf{z})=\frac{(b+k)(c_{i_{2k+1}}-a+p_{i(2k)}-k)}{(c_{i_{2k+1}}+p_{2k}-\delta_{i_{2k+1}}^2)(c_{i_{2k+1}}+p_{i(k)}+\delta_{i_{2k+1}}^1)}\delta_{i_{2k+1}}^{i_{2k}}z_{i_{2k+1}}+$$

$$+\frac{b+k}{c_{i_{2k+1}}+p_{i(k)}+\delta_{i_{2k+1}}^1}\delta_{i_{2k}+i_{2k+1}}^3z_{i_{2k+1}},\quad i(2k+1)\in\mathcal{I}_{2k+1},\quad k\ge1.$$

The question of the convergence of this expansion remains open. Similar results can also be established for ratios symmetric to (5.41).

5.4 Approximations of some special functions by branched continued fractions.

Analytical solutions to a wide range of equations in applied mathematics often appear in the form of hypergeometric functions, which, in turn, can be represented as branched continued fractions. We will give some examples of approximations of special functions by branched continued fractions.

Example 1 [2]. By Corollary 5.2 we have

$$\ln\left(1+\frac{z_2}{1+z_1}\right) = z_2 F_2(1,b,1;b,2;-z_1,-z_2) = = \cfrac{z_2}{1+z_1+\cfrac{\frac{1}{2}z_2}{1+\cfrac{\frac{1}{6}z_2}{1+z_1+\cfrac{\frac{1}{3}z_2}{1+\cfrac{\frac{1}{5}z_2}{1+\ddots}}}}} \tag{5.42}$$

at some neighborhood of the origin. In addition, the branched continued fraction in (5.42) converges and represents a single-valued branch of the analytic function

$$\ln\left(1+\frac{z_2}{1+z_1}\right) \tag{5.43}$$

in the domain

$$\Omega = \left\{\mathbf{z}\in\mathbb{C}^2 : \ |\arg(z_1+1)|<\pi, \ \left|\arg\left(z_2+\frac{1}{2}\right)\right|<\pi\right\}.$$

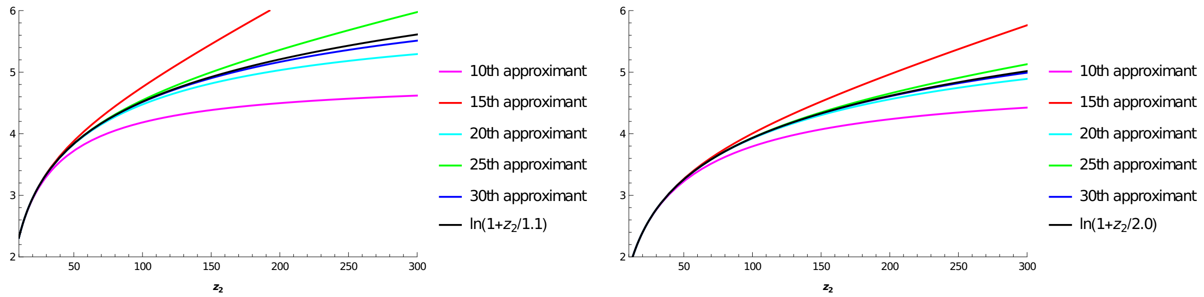
Table 1 shows the results of the evaluations (5.42) and

$$\ln\left(1+\frac{z_2}{1+z_1}\right) = z_2F_2(1,b,1;b,2;-z_1,-z_2) = -\sum_{r,s=0}^{\infty}\frac{(1)_{r+s}(1)_s}{(2)_s}\frac{(-z_1)^r}{r!}\frac{(-z_2)^{s+1}}{s!}.$$

Plots of the values of the nth approximants of (5.42) are shown in Figure 2a–b. Here we can see the fork property for a branched continued fraction. That is, the plots of the values of even (odd) approximations of(5.42) approaches from below (above) to the plot of the function $\ln(1+z_2/(1+z_1))$ at fixed values of z_1. The plots at fixed values of z_2 are similar. Figure 3a–d shows the plots where the 10th approximant of (5.42) guarantees certain truncation error bounds for function $\ln(1+z_2/(1+z_1))$.

Table 1: Relative error of 10th partial sum and 10th approximants for $\ln(1 + z_2/(1 + z_1))$.

z	**(5.43)**	**(5.42)**	**(5.4)**
$(-0.01, 0.1)$	0.096228	2.8844×10^{-16}	2.2613×10^{-13}
$(0.1 + 0.01i, 0.1 + 0.01i)$	$0.0871089 + 0.00757447i$	9.5235×10^{-16}	3.4973×10^{-9}
$(1.0, 1.0)$	0.40546511	1.7002×10^{-12}	$2.9649 \times 10^{+3}$
$(1 + i, 1 - i)$	$0.2938933 - 0.4636476i$	6.9884×10^{-11}	$2.2319 \times 10^{+2}$
$(3.0, 5.0)$	0.81093022	6.0879×10^{-09}	$8.4668 \times 10^{+8}$
$(6.0, 30.0)$	1.66500777	1.9580×10^{-05}	$6.9735 \times 10^{+15}$
$(10 + 10i, -10 - 10i)$	$-2.6990814 - 0.7378151i$	3.6187×10^{-03}	$1.3918 \times 10^{+11}$
$(10.0, 100.0)$	2.311634929	5.3044×10^{-04}	$1.1109 \times 10^{+21}$
$(1 + 100i, 1 + 100i)$	$0.6930597 + 0.0049985i$	9.7041×10^{-10}	$2.6863 \times 10^{+24}$
$(1000, 1000)$	0.6926476	9.6335×10^{-10}	$2.6852 \times 10^{+35}$
$(1 - 1000i, 1 - 1000i)$	$0.6931463 - 0.0004999i$	9.7152×10^{-10}	$2.6847 \times 10^{+35}$
$(-1000 + 1000i, 1000 - 1000i)$	$-7.2538289 - 2.3556942i$	4.2875×10^{-01}	$5.3972 \times 10^{+32}$
$(10000, 10000)$	0.6930972	9.7071×10^{-10}	$2.6848 \times 10^{+46}$



(a) At $z_1 = 0.1$. **(b)** At $z_1 = 1.0$.

Figure 2: The plots of values of the nth approximants of (5.42) for function (5.43).

Example 2 [2]. By Corollary 5.2 we obtain

$$\arctan\sqrt{\frac{z_2}{1+z_1}} = \sqrt{z_2(1+z_1)}F_2\left(1, b, \frac{1}{2}; b, \frac{3}{2}; -z_1, -z_2\right) = \\ = \cfrac{\sqrt{z_2(1+z_1)}}{1 + z_1 + \cfrac{\frac{1}{3}z_2}{1 + \cfrac{\frac{4}{15}z_2}{1 + z_1 + \cfrac{\frac{9}{35}z_2}{1 + \cfrac{\frac{16}{63}z_2}{1 + \ddots}}}}}, \tag{5.44}$$

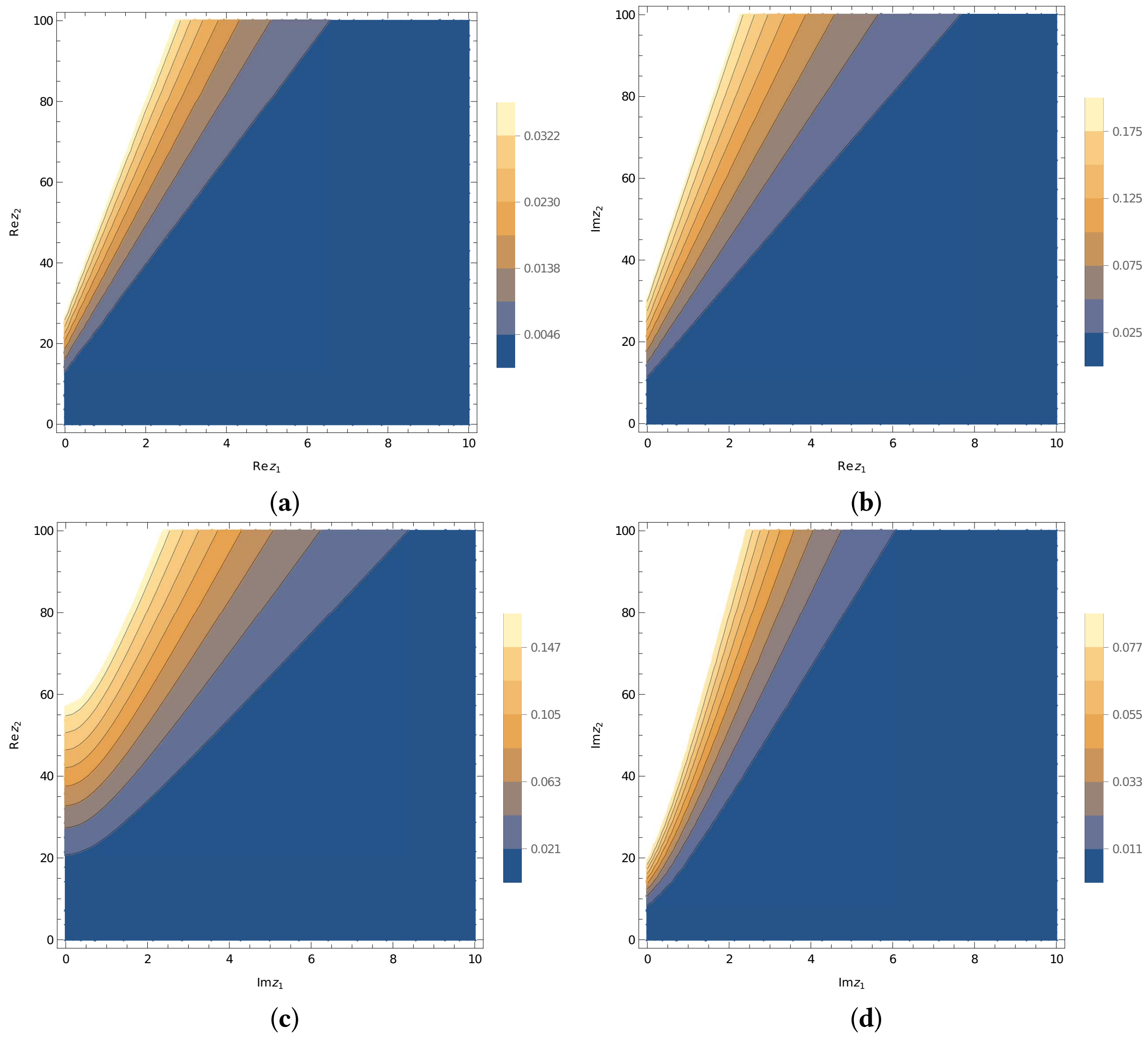


Figure 3: The plots where the 10th approximant of (5.42) guarantees certain truncation error bounds for function (5.43).

where the branched continued fraction converges and represents a single-valued branch of the analytic function of two variables

$$\arctan\sqrt{\frac{z_2}{1+z_1}} \tag{5.45}$$

in the domain

$$\Omega=\left\{\mathbf{z}\in\mathbb{C}^2:\ |\arg(z_1+1)|<\pi,\ \left|\arg\left(z_2+\frac{3}{4}\right)\right|<\pi\right\}.$$

The numerical illustration of (5.44) and

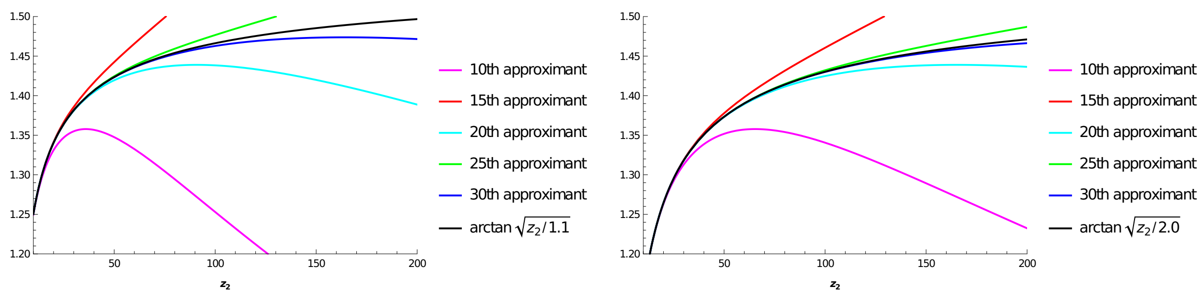
$$\arctan\sqrt{\frac{z_2}{1+z_1}}=\sqrt{z_2(1+z_1)}F_2\left(1,b,\frac{1}{2};b,\frac{3}{2};-z_1,-z_2\right)=$$
$$=\sqrt{z_2(1+z_1)}\sum_{r,s=0}^{\infty}\frac{(1)_{r+s}(1/2)_s}{(3/2)_s}\frac{(-z_1)^r}{r!}\frac{(-z_2)^s}{s!} \tag{5.46}$$

is given in the Table 2.

Table 2: Relative error of 10th partial sum and 10th approximants for $\arctan\sqrt{z_2/(1+z_1)}$.

z	(5.45)	(5.44)	(5.46)
$(-0.01, 0.01)$	0.100167	0	1.3855×10^{-16}
$(-0.01, 0.1)$	0.307725	0	1.1545×10^{-13}
$(0.1+0.01i, 0.1+0.01i)$	$0.293289 + 0.0125413i$	3.7861×10^{-16}	1.9078×10^{-9}
$(0.2, 0.3)$	0.463648	1.4367×10^{-15}	3.2554×10^{-5}
$(0.9, 0.1)$	0.225513	1.2308×10^{-16}	4.9359×10^{-1}
$(-0.001, 0.9)$	0.75932	4.3178×10^{-10}	9.2141×10^{-3}
$(1.0, 1.0)$	0.61547979	1.8029×10^{-12}	$1.5376 \times 10^{+2}$
$(1+i, 1-i)$	$0.6466485 - 0.3235375i$	7.3031×10^{-11}	$1.2140 \times 10^{+2}$
$(4.0, 10.0)$	0.95531662	2.3049×10^{-07}	$1.3529 \times 10^{+11}$
$(10+10i, -10-10i)$	$0.3806548 - 2.0302143i$	3.1341×10^{-03}	$5.7608 \times 10^{+11}$
$(10.0, 100.0)$	1.25055029	7.2305×10^{-04}	$3.5865 \times 10^{+20}$
$(1+100i, 1+100i)$	$0.7853607 + 0.0024994i$	1.0618×10^{-09}	$1.3261 \times 10^{+24}$
$(1000, 1000)$	0.7851483	1.0539×10^{-09}	$1.3257 \times 10^{+35}$
$(1-1000i, 1-1000i)$	$0.7853978 - 0.0002499i$	1.0629×10^{-09}	$1.3254 \times 10^{+35}$
$(-1000+1000i, 1000-1000i)$	$1.1779722 - 4.3201866i$	3.7200×10^{-01}	$2.7321 \times 10^{+33}$
$(10000, 10000)$	0.7853732	1.0621×10^{-09}	$1.3254 \times 10^{+46}$

The graphical illustrations of (5.44) and (5.45) are given in Figures 4a–b and 5a–d. Here we have results like the results in the previous example.



(a) At $z_1 = 0.1$. (b) At $z_1 = 1.0$.

Figure 4: The plots of values of the nth approximants of (5.44) for function (5.45).

Example 3 [31]. From [1,53], Corollaries 5.2 and 5.4, it follows that the function

$$\ln\left(\sqrt{\frac{z_2}{1+z_1}} + \sqrt{1+\frac{z_2}{1+z_1}}\right) \tag{5.47}$$

has representations in the series

$$\begin{aligned} &\sqrt{z_2}\sqrt{1+z_1+z_2}F_2(1, b_1, 1; b_1, 3/2; -z_1, -z_2) = \\ &= \sqrt{z_2}\sqrt{1+z_1+z_2}\sum_{p,q=0}^{\infty}\frac{(1)_{p+q}(1)_q}{(3/2)_q}\frac{(-z_1)^p}{p!}\frac{(-z_2)^q}{q!} \end{aligned} \tag{5.48}$$

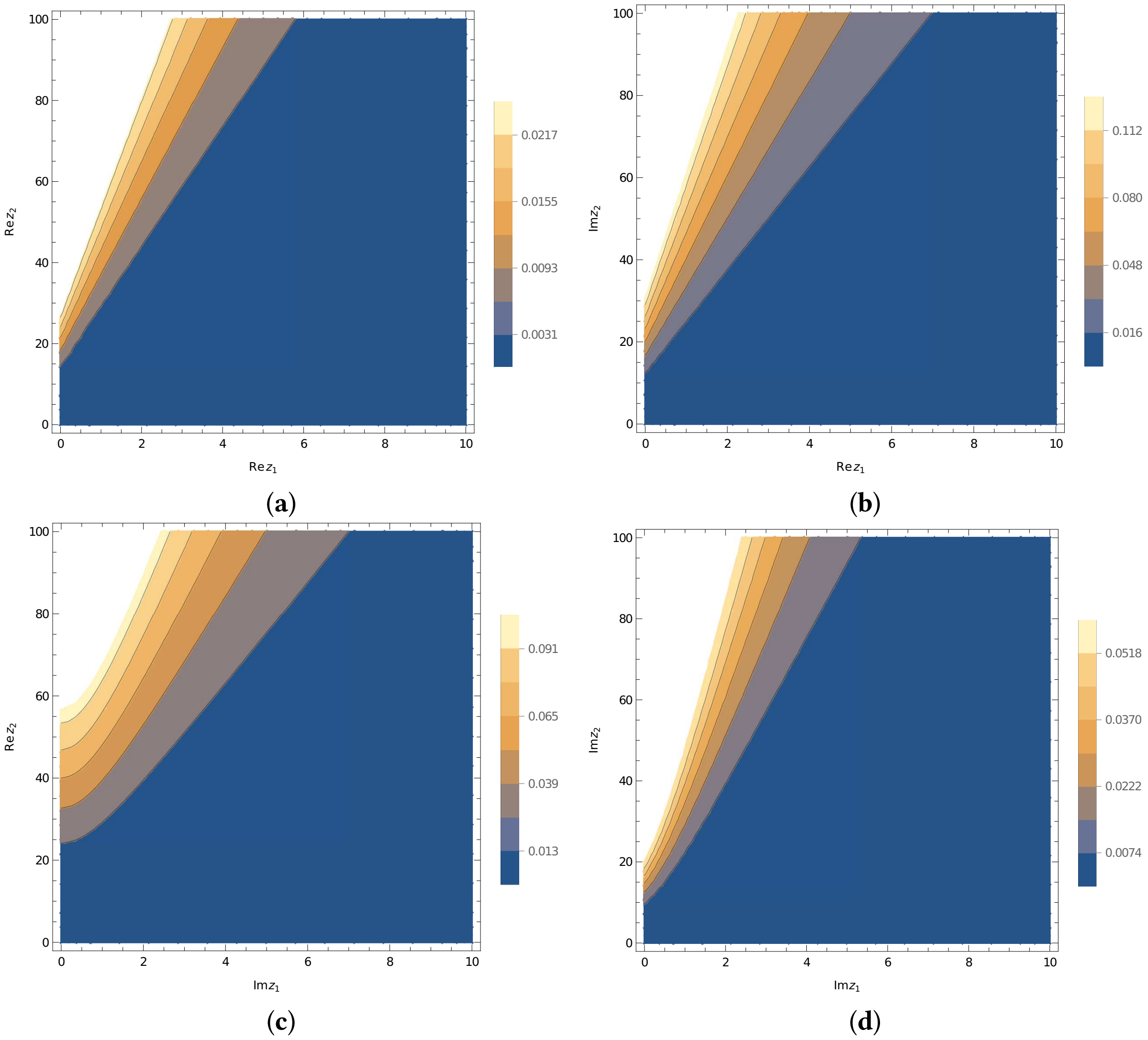
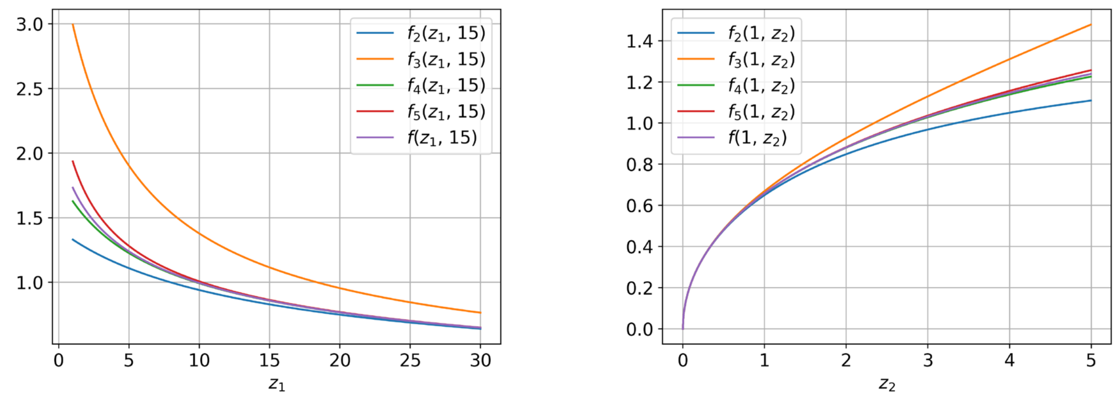


Figure 5: The plots where the 10th approximant of (5.44) guarantees certain truncation error bounds for function (5.45).

and a branched continued fraction

$$\frac{\sqrt{z_2}\sqrt{1+z_1+z_2}}{1+z_1+\cfrac{\frac{2}{3}z_2}{1+\cfrac{\frac{2}{15}z_2}{1+z_1+\cfrac{\frac{12}{35}z_2}{1+\cfrac{\frac{12}{63}z_2}{1+\ddots}}}}}, \tag{5.49}$$



(**a**) At $z_2 = 15$. (**b**) At $z_1 = 1$.

Figure 6: The plots of values of the nth approximants of (5.49) for function (5.47).

which converges and represents a single-valued branch of the function (5.47) in the domain

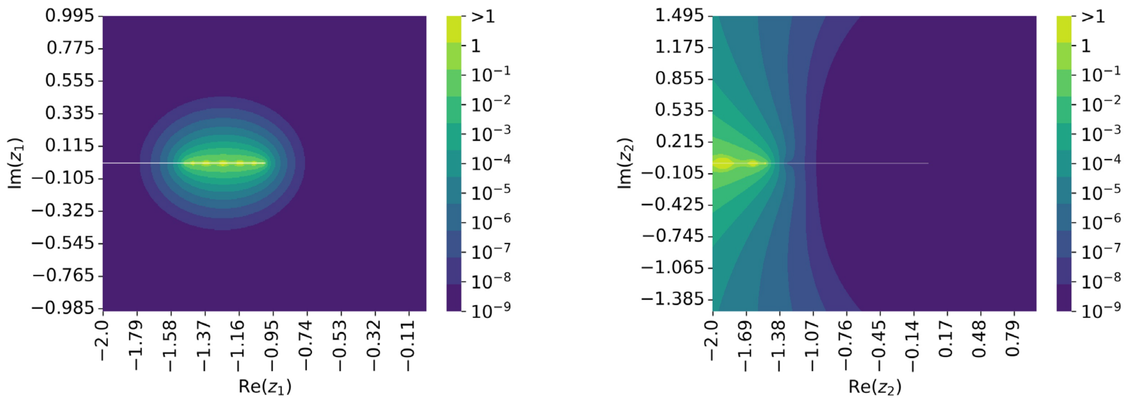
$$\Omega = \left\{ \mathbf{z} \in \mathbb{C}^2 : \ z_1 \notin (-\infty, -1], \ z_2 \notin (-\infty, 0] \right\}.$$

Plots of the values of the nth approximants of branched continued fraction expansion (5.49) are shown in Figure 6a–b. On the sets given in color there, all the elements of (5.49) are positive, so we see the fork property for a branched continued fraction. That is, the plots of the values of even (odd) approximations of expansion (5.49) approaches from below (above) to the plot of function (5.47) at a fixed value of z_2 (Figure 6a). The plots at fixed values of z_1 are similar (see Figure 6b).

Figure 7a–b show 2D plots where the 15th approximant of the branched continued fraction expansion (5.49) guarantees certain truncation error bounds for (5.47). Here we see the symmetrical regions, with a cut along the real axis from $-\infty$ to -1 (Figure 7a) and a cut along the real axis from $-\infty$ to 0 (Figure 7b). Figure 8a–d show 2D plots in different planes in $\mathbb{C}^2$, where the 15th approximant of (5.49) guarantees certain truncation error bounds for function (5.47). Here we observe symmetrical regions in all cases except Figure 8a.

The results of evaluations (5.48) and (5.49) are displayed in Table 3. The analysis of these computation results shows that the approximation of function (5.47) is better using the 20th approximant of (5.49) and then the 20th partial sum of (5.47). We also see numerical stability of the 20th approximant of the branched continued fraction (5.49) at extreme inputs (e.g., $\mathbf{z} = (100, 100)$, $\mathbf{z} = (100 + 45i, 100 + 45i)$).

Plots of the values of the relative errors of the nth approximants of expansion (5.49) for function (5.47) are shown in Figure 9a–b. Here we observe that the relative error values tend to 0 as $k \to +\infty$.



(**a**) At $z_2 = 0.5$. (**b**) At $z_1 = 0.5$.

Figure 7: Two-dimensional plots in which the approximant $f_{15}(\mathbf{z})$ of (5.49) guarantees certain truncation error bounds for (5.47).

Table 3: Relative error of 20th partial sum and 20th approximants for (5.47).

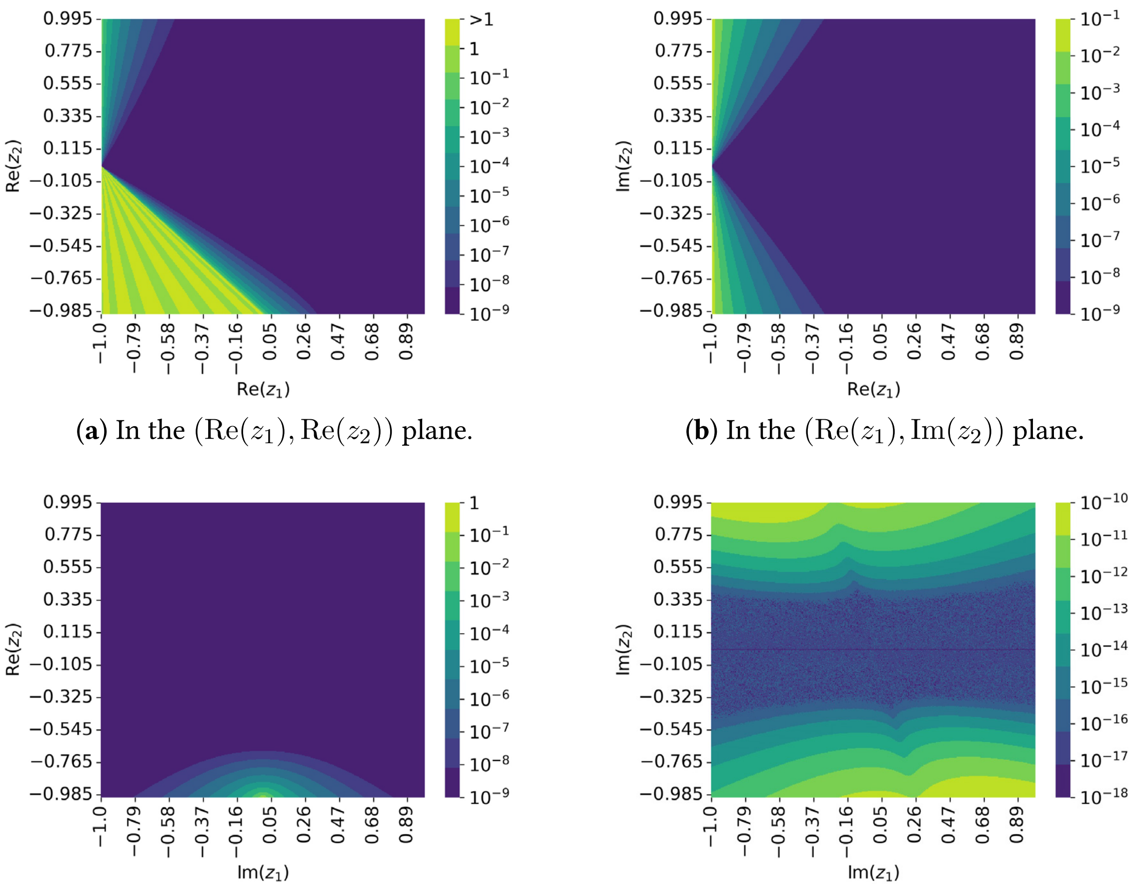
$\mathbf{z}$	**(5.47)**	**(5.49)**	**(5.48)**
$(0.1, 0.1)$	0.2971	3.74×10^{-16}	7.47×10^{-16}
$(-0.3, 0.3)$	0.4636	3.59×10^{-16}	7.77×10^{-12}
$(1.5, 0.3)$	0.3398	4.90×10^{-16}	2.82×10^{3}
$(3, 3)$	0.2612	2.13×10^{-16}	3.89×10^{10}
$(10, 15)$	0.2116	5.25×10^{-16}	1.36×10^{25}
$(100, 100)$	0.0728	2.10×10^{-15}	3.60×10^{51}
$(0.5 + 0.5i, 0.5 + 0.5i)$	$0.6165 + 0.12881i$	1.97×10^{-16}	2.02×10^{4}
$(1 - 3i, 5 - 2i)$	$0.9556 - 0.5421i$	6.52×10^{-12}	5.49×10^{35}
$(10 - 5i, 4 + 15i)$	$0.8402 + 0.6810i$	1.12×10^{-11}	2.43×10^{56}
$(100 + 45i, 100 + 45i)$	$0.8784 + 0.0013i$	7.59×10^{-16}	1.88×10^{94}

Example 4 [31]. From [1, 53], Corollaries 5.2 and 5.4, it follows that the function

$$\ln \frac{1 + \sqrt{\dfrac{z_2}{1 - z_1}}}{1 - \sqrt{\dfrac{z_2}{1 - z_1}}} \tag{5.50}$$

has representations in the series

$$2\sqrt{z_2(1 - z_1)}F_2(1, b_1, 1/2; b_1, 3/2; z_1, z_2) = 2\sqrt{z_2(1 - z_1)} \sum_{p,q=0}^{\infty} \frac{(1)_{p+q}(1/2)_q}{(3/2)_q} \frac{z_1^p}{p!} \frac{z_2^q}{q!} \tag{5.51}$$



(**a**) In the $(\mathrm{Re}(z_1), \mathrm{Re}(z_2))$ plane. (**b**) In the $(\mathrm{Re}(z_1), \mathrm{Im}(z_2))$ plane.

(**c**) In the $(\mathrm{Im}(z_1), \mathrm{Re}(z_2))$ plane. (**d**) In the $(\mathrm{Im}(z_1), \mathrm{Im}(z_2))$ plane.

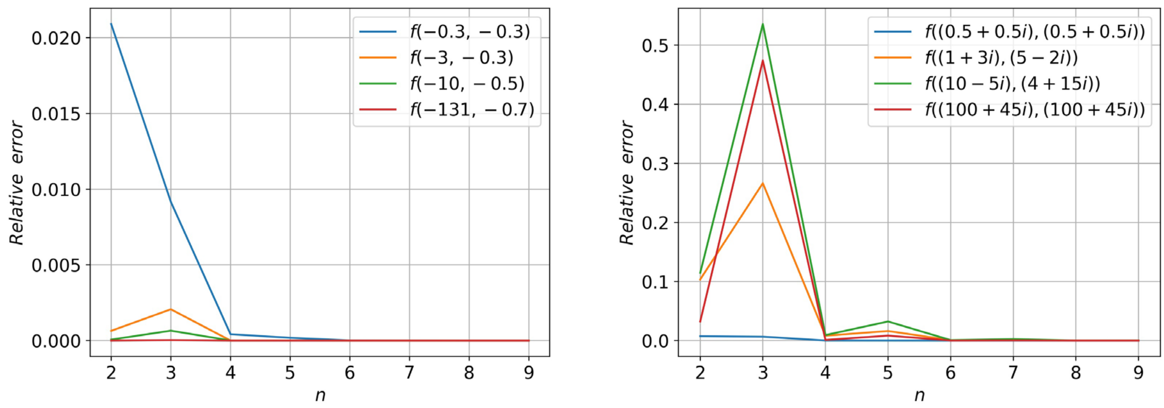
Figure 8: The plots where the approximant $f_{15}(\mathbf{z})$ of (5.49) guarantees certain truncation error bounds for (5.47).

and a branched continued fraction

$$\cfrac{2\sqrt{z_2(1-z_1)}}{1-z_1-\cfrac{\frac{1}{3}z_2}{1-\cfrac{\frac{4}{15}z_2}{1-z_1-\cfrac{\frac{9}{35}z_2}{1-\cfrac{\frac{16}{63}z_2}{1-\ddots}}}}}, \tag{5.52}$$

which converges and represents a single-valued branch of function (5.17) in the domain

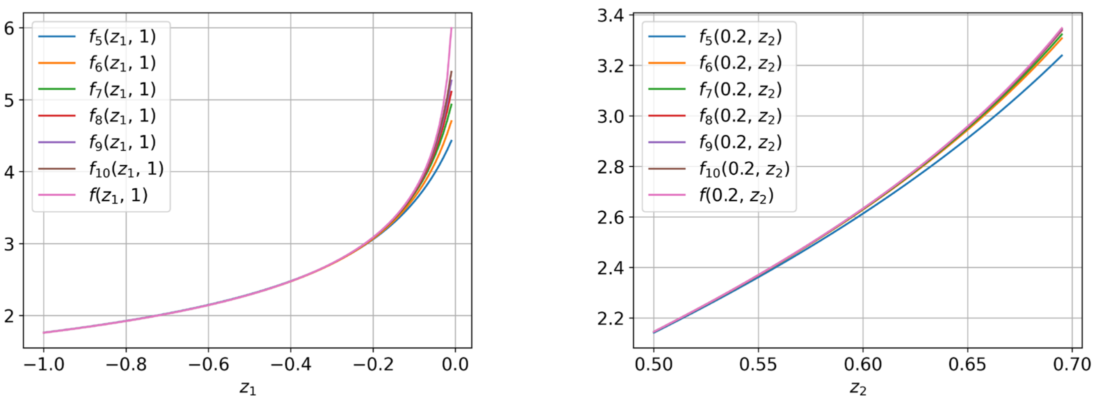
$$\Omega = \left\{ \mathbf{z} \in \mathbb{C}^2 : \; z_1 \notin [1, +\infty), \; z_2 \notin (-\infty, 0] \cup \left[\frac{3}{4}, +\infty\right) \right\}.$$



(**a**) At real points. (**b**) At complex points.

Figure 9: Plots of the values of the relative errors of the nth approximants of expansion (5.49) for (5.47).

In Figure 10a–b, the sets where the elements of (5.50) are not positive are shown, so we cannot see the fork property as in the previous example.



(**a**) At $z_2 = 1$. (**b**) At $z_1 = 0.2$.

Figure 10: The plots of values of the nth approximants of (5.52) for function (5.50).

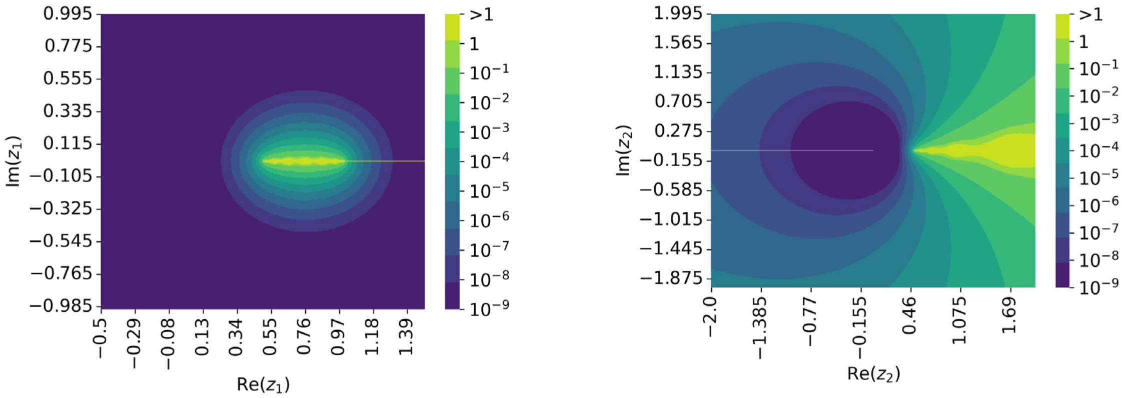
The graphical illustrations of (5.50) and (5.52) are given in Figures 11a–b and 12a–d.

The numerical illustrations of (5.51) and (5.52) are given in Table 4.

Finally, plots of the values of the relative errors of the nth approximants of (5.52) for function (5.50) are shown in Figure 13a–b.

Here we have similar results to those in the previous example.

Thus, the numerical experiments confirmed the expediency and effectiveness of using branched continued fractions as an approximation tool of analytic functions.



(**a**) At $z_2 = 0.5$. (**b**) At $z_1 = 0.5$.

Figure 11: Two-dimensional plots where the approximant $f_{15}(\mathbf{z})$ of (5.52) guarantees certain truncation error bounds for (5.50).

Table 4: Relative errors of 20th partial sum and 20th approximants for (5.50).

$\mathbf{z}$	**(5.50)**	**(5.52)**	**(5.51)**
$(-0.1, 0.1)$	0.6224	$3,57 \times 10^{-16}$	8.92×10^{-16}
$(-0.5, 0.3)$	0.9624	2.31×10^{-16}	6.05×10^{-6}
$(-0.8, 0.3)$	0.8670	1.28×10^{-16}	1.1448
$(-3, 0.3)$	0.5621	1.98×10^{-16}	1.20×10^{12}
$(-11, 0.5)$	0.4141	1.34×10^{-16}	2.55×10^{26}
$(-131, 0.7)$	0.1459	5.71×10^{-16}	2.70×10^{51}
$(0.5 + 0.5i, 0.5 + 0.5i)$	$0.8814 + 1.5708i$	8.52×10^{-14}	5.16×10^{3}
$(1 + 3i, 5 - 2i)$	$1.0818 + 2.0576i$	6.74×10^{-9}	5.59×10^{34}
$(10 - 5i, 4 + 15i)$	$1.02611 - 1.8952i$	2.16×10^{-10}	2.55×10^{55}
$(100 + 45i, 100 + 45i)$	$0.0019 + 1.5750i$	1.57×10^{-15}	1.88×10^{93}

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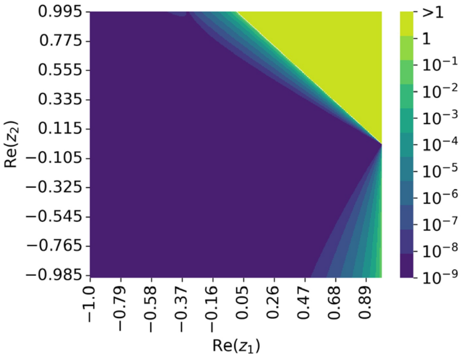
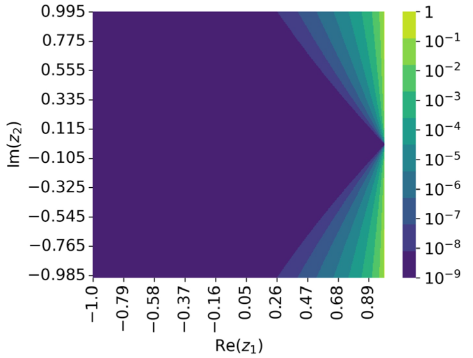
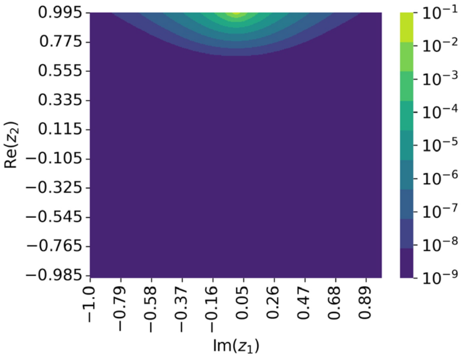


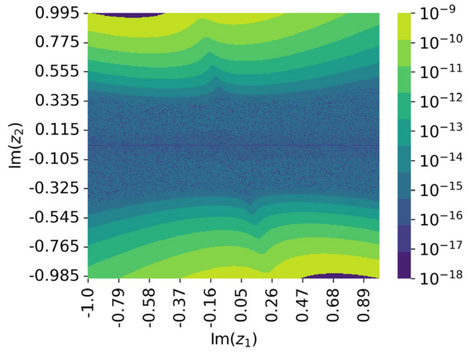
(**a**) In the $(\mathrm{Re}(z_1), \mathrm{Re}(z_2))$ plane.



(**b**) In the $(\mathrm{Re}(z_1), \mathrm{Im}(z_2))$ plane.

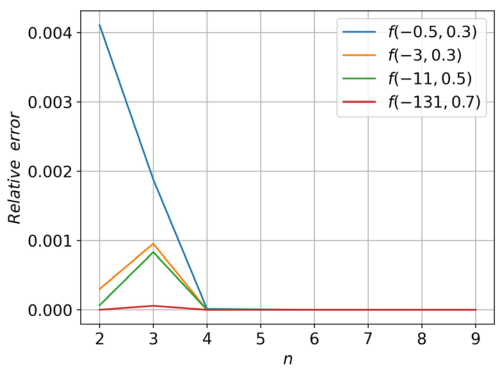


(**c**) In the $(\mathrm{Im}(z_1), \mathrm{Re}(z_2))$ plane.

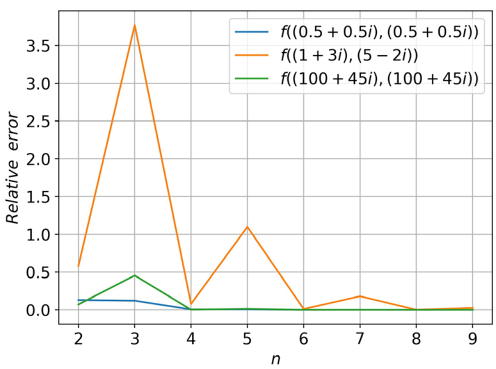


(**d**) In the $(\mathrm{Im}(z_1), \mathrm{Im}(z_2))$ plane.

Figure 12: The plots where the approximant $f_{15}(\mathbf{z})$ of (5.52) guarantees certain truncation error bounds for (5.50).



(**a**) At real points.



(**b**) At complex points.

Figure 13: Plots of the values of the relative errors of the nth approximants of expansion (5.52) for (5.50).

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Volodymyr HLADUN

6 Stability to perturbations of continued fractions with applications

Methods of rational approximation, particularly continued fractions and their multidimensional generalizations (branched continued fractions), serve as efficient instruments in modern computational mathematics [4, 5, 24, 27, 28]. Their application spectrum extends across theoretical and applied physics, engineering, applied mathematics, and computer science, encompassing areas such as signal processing, cryptography, and machine learning [25, 26, 30–33]. A critical advantage of continued and branched continued fractions is their capability to approximate special functions of one and several variables effectively [2, 6, 10–13, 15, 20, 21].

In high-precision computing, ensuring the reliability of numerical results is paramount. This necessitates a rigorous analysis of the computational methods used [16]. Two primary aspects of stability are crucial for continued fractions: stability to perturbations of the fraction's elements [8, 17–19] and the numerical stability of the algorithms used to compute their approximants [9, 14, 22]. Stability implies that small variations in the elements – whether due to measurement errors or floating-point representation – result in bounded and predictable deviations in the final value.

While various algorithms exist for evaluating continued fractions, such as the forward recurrence algorithm and the backward recurrence algorithm, they exhibit different sensitivities to error propagation. Previous studies indicate that the backward recurrence algorithm often demonstrates superior stability properties compared to forward algorithm, which are more prone to error accumulation [3, 7, 23, 29].

This chapter is devoted to a study of stability for specific classes of continued fractions with complex elements. We derive formulas for relative errors, establish sufficient conditions for stability to perturbations, and construct the sets of stability to perturbations. The theoretical findings are applied to the approximation of special functions, including Bessel functions and ratios of Horn's confluent hypergeometric functions. The results presented in this chapter summarize and develop the authors' research published in [8, 22].

Volodymyr HLADUN, Lviv Polytechnic National University, 12 Bandera str., 79013, Lviv, Ukraine, e-mail: volodymyr.r.hladun@lpnu.ua

6.1 Stability to perturbations of continued fractions with complex partial denominators and partial numerators equal to one

Let us consider the continued fraction

$$b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots}}, \tag{6.1}$$

where $b_k \in \mathbb{C}$, $k \geq 0$.

The finite continued fractions

$$f_n = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots + \cfrac{1}{b_n}}}, \quad n \geq 1,$$

are called the nth approximants of the continued fraction (6.1). The quantities

$$G_k^{(n)} = b_k + \cfrac{1}{b_{k+1} + \cfrac{1}{b_{k+2} + \ddots + \cfrac{1}{b_n}}}, \quad 0 \leq k \leq n, \quad n \geq 1,$$

are called the tails of the nth approximant. From the definition of the tail of the nth approximant, it follows that $f_n = G_0^{(n)}$, $n \geq 1$. For the tails $G_k^{(n)}$, $0 \leq k \leq n$, $n \geq 1$, the following recurrence relations hold

$$G_k^{(n)} = b_k + \frac{1}{G_{k+1}^{(n)}}, \quad 0 \leq k \leq n, \quad n \geq 1,$$

with the initial condition $G_n^{(n)} = b_n$, $n \geq 1$.

Let $\widehat{b}_k, k \geq 0$, be the perturbed values of the elements b_k of the continued fraction (6.1). The continued fraction

$$\widehat{b}_0 + \cfrac{1}{\widehat{b}_1 + \cfrac{1}{\widehat{b}_2 + \ddots}}, \tag{6.2}$$

is called the perturbed continued fraction corresponding to the continued fraction (6.1).

Let $\{E_k\}$, $\emptyset \neq E_k \subset \mathbb{C}$, $k \geq 0$, be a sequence of element sets of the continued fraction (6.1) and its perturbed counterpart (6.2):

$$b_k \in E_k, \ \widehat{b}_k \in E_k, \ k \geq 0.$$

If for all $k \geq 0$, $E_k = E$, the set E is called a simple element set. If for all $k \geq 0$, $E_{2k+1} = E_1$, $E_{2k} = E_2$, the sets E_1, E_2 are called twin element sets.

Definition 6.1. *A sequence $\{V_k\}$, $\emptyset \neq V_k \subset \mathbb{C}$, $k \geq 0$, is called a sequence of value sets for the tails of the approximants of the continued fraction* (6.1)*, corresponding to the sequence of element sets $\{E_k\}$, if for every $b_k \in E_k$, $k \geq 0$,*

$$b_k \in V_k, \tag{6.3}$$

and for every $v \in V_{k+1}$ and every $b_k \in E_k$, $k \geq 0$,

$$b_k + \frac{1}{v} \in V_k. \tag{6.4}$$

From this definition, it follows that $G_k^{(n)} \in V_k$, $0 \leq k \leq n$, $n \geq 1$.

Definition 6.2. *A sequence of element sets $\{E_k\}$, $k \geq 0$, is called a sequence sets of stability to perturbations for the continued fraction* (6.1) *if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $b_k, \widehat{b}_k \in E_k$, $b_k \neq 0$, $k \geq 0$, satisfying $|\widehat{b}_k - b_k|/|b_k| < \delta$, the following inequality holds*

$$\left| \frac{\widehat{f}_n - f_n}{f_n} \right| < \varepsilon,$$

where f_n and $\widehat{f}_n$ are the approximants of the continued fractions (6.1) *and* (6.2)*, respectively, herewith $f_n \neq 0$, $n \geq 1$.*

Let us denote by $\epsilon_k^{(b)}$, $k \geq 0$, $\epsilon_n^{(f)}$, $n \geq 1$, and $\epsilon_{k,n}^{(G)}$, $0 \leq k \leq n$, $n \geq 1$, the relative errors of the elements b_k, the approximant f_n, and the tails $G_k^{(n)}$ of the continued fraction (6.1), respectively, namely,

$$\begin{aligned} \widehat{b}_k &= b_k(1 + \epsilon_k^{(b)}), \quad k \geq 0, \\ \widehat{f}_n &= f_n(1 + \epsilon_n^{(f)}), \quad n \geq 1, \\ \widehat{G}_k^{(n)} &= G_k^{(n)}(1 + \epsilon_{k,n}^{(G)}), \quad 0 \leq k \leq n, \quad n \geq 1. \end{aligned}$$

Since $f_n = G_0^{(n)}$ and $\widehat{f}_n = \widehat{G}_0^{(n)}$, $n \geq 1$, then $\epsilon_n^{(f)} = \epsilon_{0,n}^{(G)}$, $n \geq 1$.

Let us consider the relative errors $\widehat{\epsilon}_k^{(b)}$, $k \geq 0$, $\widehat{\epsilon}_{k,n}^{(G)}$, $0 \leq k \leq n$, $n \geq 1$, defined by the relations

$$\begin{aligned} b_k &= \widehat{b}_k(1 + \widehat{\epsilon}_k^{(b)}), \quad k \geq 0, \\ G_k^{(n)} &= \widehat{G}_k^{(n)}(1 + \widehat{\epsilon}_{k,n}^{(G)}), \quad 0 \leq k \leq n, \quad n \geq 1. \end{aligned}$$

Assuming that $b_k \neq 0, \widehat{b}_k \neq 0, k \geq 0$, and $G_k^{(n)} \neq 0, \widehat{G}_k^{(n)} \neq 0, 0 \leq k \leq n$, $n \geq 1$, we will prove that the following recurrence formulas hold

$$\epsilon_{k,n}^{(G)} = \epsilon_k^{(b)} + g_{k+1}^{(n)}(\widehat{\epsilon}_{k+1,n}^{(G)} - \epsilon_k^{(b)}), \quad 0 \leq k \leq n-1, \quad n \geq 1, \tag{6.5}$$

and

$$\widehat{\epsilon}_{k,n}^{(G)} = -\frac{\epsilon_k^{(b)}}{1+\epsilon_k^{(b)}} + \widehat{g}_{k+1,n}^{(G)}\left(\epsilon_{k+1}^{(n)} + \frac{\epsilon_k^{(b)}}{1+\epsilon_k^{(b)}}\right), \quad 0 \le k \le n-1, \quad n \ge 1,$$

with the initial conditions

$$\epsilon_{n,n}^{(G)} = \epsilon_n^{(b)}, \quad \widehat{\epsilon}_{n,n}^{(G)} = -\frac{\epsilon_n^{(b)}}{1+\epsilon_n^{(b)}}, \quad n \ge 1, \tag{6.6}$$

where

$$g_k^{(n)} = \frac{1}{G_{k-1}^{(n)} G_k^{(n)}}, \; \widehat{g}_k^{(n)} = \frac{1}{\widehat{G}_{k-1}^{(n)} \widehat{G}_k^{(n)}}, \quad 1 \le k \le n, \quad n \ge 1. \tag{6.7}$$

Let n be a fixed natural number. Since $G_n^{(n)} = b_n$, $\widehat{G}_n^{(n)} = \widehat{b}_n$, then

$$\epsilon_{n,n}^{(G)} = \epsilon_n^{(b)}, \quad \widehat{\epsilon}_{n,n}^{(G)} = \widehat{\epsilon}_n^{(b)} = -\frac{\epsilon_n^{(b)}}{1+\epsilon_n^{(b)}}.$$

For arbitrary k, $0 \le k \le n-1$, we have

$$\epsilon_{k,n}^{(G)} = \frac{\widehat{G}_k^{(n)} - G_k^{(n)}}{G_k^{(n)}} = \frac{1}{G_k^{(n)}}\left(b_k(1+\epsilon_k^{(b)}) + \frac{1}{G_{k+1}^{(n)}(1+\epsilon_{k+1,n}^{(G)})}\right) - 1 =$$

$$= \frac{b_k}{G_k^{(n)}}(1+\epsilon_k^{(b)}) + \frac{1+\widehat{\epsilon}_{k+1,n}^{(G)}}{G_k^{(n)} G_{k+1}^{(n)}} - 1 = (1-g_{k+1}^{(n)})(1+\epsilon_k^{(b)}) + g_{k+1}^{(n)}(1+\widehat{\epsilon}_{k+1,n}^{(G)}) - 1 =$$

$$= \epsilon_k^{(b)} + g_{k+1}^{(n)}(\widehat{\epsilon}_{k+1,n}^{(G)} - \epsilon_k^{(b)})$$

and

$$\widehat{\epsilon}_{k,n}^{(G)} = \frac{\widehat{G}_k^{(n)} - G_k^{(n)}}{\widehat{G}_k^{(n)}} = \frac{1}{\widehat{G}_k^{(n)}}\left(\widehat{b}_k(1+\widehat{\epsilon}_k^{(b)}) + \frac{1}{\widehat{G}_{k+1}^{(n)}(1+\widehat{\epsilon}_{k+1,n}^{(G)})}\right) - 1 =$$

$$= \frac{\widehat{b}_k}{\widehat{G}_k^{(n)}}(1+\widehat{\epsilon}_k^{(b)}) + \frac{1+\epsilon_{k+1,n}^{(G)}}{\widehat{G}_k^{(n)} \widehat{G}_{k+1}^{(n)}} - 1 = (1-\widehat{g}_{k+1}^{(n)})(1+\widehat{\epsilon}_k^{(b)}) + \widehat{g}_{k+1}^{(n)}(1+\epsilon_{k+1,n}^{(G)}) - 1 =$$

$$= \widehat{\epsilon}_k^{(b)} + \widehat{g}_{k+1}^{(n)}(\epsilon_{k+1,n}^{(G)} - \widehat{\epsilon}_k^{(b)}) = -\frac{\epsilon_k^{(b)}}{1+\epsilon_k^{(b)}} + \widehat{g}_{k+1}^{(n)}\left(\epsilon_{k+1,n}^{(G)} + \frac{\epsilon_k^{(b)}}{1+\epsilon_k^{(b)}}\right).$$

Thus, from (6.5)–(6.6) it follows

$$\varepsilon_n^{(f)} = \tilde{\epsilon}_0^{(b)} + \sum_{k=1}^{n}(\tilde{\epsilon}_k^{(b)} - \tilde{\epsilon}_{k-1}^{(b)}) \prod_{m=1}^{k} \tilde{g}_m^{(n)}, \quad n \ge 1, \tag{6.8}$$

where

$$\tilde{g}_k^{(n)} = \begin{cases} g_k^{(n)}, & k = 2m+1, \\ \widehat{g}_k^{(n)}, & k = 2m, \end{cases} \quad \text{and} \quad \tilde{\epsilon}_k^{(b)} = \begin{cases} \epsilon_k^{(b)}, & k = 2m, \\ -\dfrac{\epsilon_k^{(b)}}{1+\epsilon_k^{(b)}}, & k = 2m+1. \end{cases} \tag{6.9}$$

The following result is true.

Theorem 6.1. *The simple element set*

$$E = (0, +\infty) \tag{6.10}$$

is a set of stability to perturbations for the continued fraction (6.1)*, if there exists a constant* $\epsilon^{(b)}$*,* $0 < \epsilon^{(b)} < 1$*, such that the relative errors of its elements satisfy*

$$|\epsilon_k^{(b)}| \le \epsilon^{(b)}, \quad k \ge 0. \tag{6.11}$$

Moreover, the following estimate holds

$$|\epsilon_n^{(f)}| \le \frac{\epsilon^{(b)}}{1-\epsilon^{(b)}}, \quad n \ge 1. \tag{6.12}$$

Proof. Since $b_k \in E$, $\widehat{b}_k \in E$, $k \ge 0$, where E is defined by (6.10), then $G_k^{(n)} > 0$ and $\widehat{G}_k^{(n)} > 0$, $0 \le k \le n$, $n \ge 1$, and for $1 \le k \le n$, $n \ge 1$, we have

$$0 < g_k^{(n)} = \frac{1}{G_{k-1}^{(n)} G_k^{(n)}} = \frac{1/G_k^{(n)}}{b_{k-1} + 1/G_k^{(n)}} < 1 \tag{6.13}$$

and

$$0 < \widehat{g}_k^{(n)} = \frac{1}{\widehat{G}_{k-1}^{(n)} \widehat{G}_k^{(n)}} = \frac{1/\widehat{G}_k^{(n)}}{\widehat{b}_{k-1} + 1/\widehat{G}_k^{(n)}} < 1. \tag{6.14}$$

Let us rewrite the formula (6.8) as follows

$$\varepsilon_n^{(f)} = \tilde{\epsilon}_0^{(b)} + \sum_{k=1}^{n} (\tilde{\epsilon}_k^{(b)} - \tilde{\epsilon}_{k-1}^{(b)}) \prod_{m=1}^{k} \tilde{g}_m^{(n)} =$$

$$= \tilde{\epsilon}_0^{(b)} + \sum_{k=1}^{n} \tilde{\epsilon}_k^{(b)} \prod_{m=1}^{k} \tilde{g}_m^{(n)} - \sum_{k=0}^{n-1} \tilde{\epsilon}_k^{(b)} \prod_{m=1}^{k+1} \tilde{g}_m^{(n)} =$$

$$= \tilde{\epsilon}_0^{(b)} - \tilde{\epsilon}_0^{(b)} \tilde{g}_1^{(n)} + \sum_{k=1}^{n-1} \tilde{\epsilon}_k^{(b)} \left(\prod_{m=1}^{k} \tilde{g}_m^{(n)} - \prod_{m=1}^{k+1} \tilde{g}_m^{(n)} \right) + \tilde{\epsilon}_n^{(b)} \prod_{m=1}^{n} \tilde{g}_m^{(n)} =$$

$$= \tilde{\epsilon}_0^{(b)}(1-\tilde{g}_1^{(n)}) + \sum_{k=1}^{n-1} \tilde{\epsilon}_k^{(b)}(1-\tilde{g}_{k+1}^{(n)}) \prod_{m=1}^{k} \tilde{g}_m^{(n)} + \tilde{\epsilon}_n^{(b)} \prod_{m=1}^{n} \tilde{g}_m^{(n)} =$$

$$= \sum_{k=0}^{n-1} \tilde{\epsilon}_k^{(b)}(1-\tilde{g}_{k+1}^{(n)}) \prod_{m=1}^{k} \tilde{g}_m^{(n)} + \tilde{\epsilon}_n^{(b)} \prod_{m=1}^{n} \tilde{g}_m^{(n)}.$$

Then, taking into account the inequalities (6.11), (6.13), and (6.14), we have

$$|\varepsilon_n^{(f)}| = \left| \sum_{k=0}^{n-1} \tilde{\epsilon}_k^{(b)}(1-\tilde{g}_{k+1}^{(n)}) \prod_{m=1}^{k} \tilde{g}_m^{(n)} + \tilde{\epsilon}_n^{(b)} \prod_{m=1}^{n} \tilde{g}_m^{(n)} \right| \le$$

$$\le \sum_{k=0}^{n-1} |\tilde{\epsilon}_k^{(b)}|(1-\tilde{g}_{k+1}^{(n)}) \prod_{m=1}^{k} \tilde{g}_m^{(n)} + |\tilde{\epsilon}_n^{(b)}| \prod_{m=1}^{n} \tilde{g}_m^{(n)} \le$$

$$\le \frac{\epsilon^{(b)}}{1-\epsilon^{(b)}} \left(\sum_{k=0}^{n-1} (1-\tilde{g}_{k+1}^{(n)}) \prod_{m=1}^{k} \tilde{g}_m^{(n)} + \prod_{m=1}^{n} \tilde{g}_m^{(n)} \right) = \frac{\epsilon^{(b)}}{1-\epsilon^{(b)}}.$$

This proves the estimate (6.12).

Let us consider the function

$$\psi(\epsilon^{(b)}) = \frac{\epsilon^{(b)}}{1-\epsilon^{(b)}}.$$

Since

$$\lim_{\epsilon^{(b)} \to 0^+} \psi(\epsilon^{(b)}) = 0,$$

for any $\varepsilon > 0$ there exists $\delta_\varepsilon = \varepsilon/(1+\varepsilon)$ such that for any $0 < \epsilon^{(b)} < \delta_\varepsilon$ the inequality $\psi(\epsilon^{(b)}) < \varepsilon$ holds.

Thus, if $|\epsilon_k^{(b)}| \le \epsilon^{(b)} < \delta_\varepsilon$, $k \ge 0$, then $|\varepsilon_n^{(f)}| \le \psi(\epsilon^{(b)}) < \varepsilon$, $n \ge 1$, which proves the stability to perturbations of the continued fraction (6.1) in the set E. □

The following result concerns symmetric sets of stability to perturbations of the continued fraction (6.1).

Theorem 6.2. *The sequence of element sets*

$$E_k = \{z \in \mathbb{C} : \ \arg z = (-1)^k \varphi, \ |z| \ge \rho_k\}, \quad k \ge 0, \tag{6.15}$$

is a sequence of sets of stability to perturbations for the continued fraction (6.1) *if the relative errors of its elements satisfy condition* (6.11) *and the series*

$$\sum_{k=1}^{\infty} \prod_{m=1}^{k} \frac{1}{1+\rho_{m-1}\rho_m} \tag{6.16}$$

converges, where $0 \leq \varphi < 2\pi$ *and* $\rho_k > 0$, $k \geq 0$, *are real constants. Moreover, the following estimate holds:*

$$|\epsilon_n^{(f)}| \leq \frac{\epsilon^{(b)}}{1-\epsilon^{(b)}} \left(1 + (2-\epsilon^{(b)}) \sum_{k=1}^{n} \prod_{m=1}^{k} \frac{1}{1+\rho_{m-1}\rho_m} \right), \quad n \geq 1.$$

Proof. Let us prove that the sequence $\{E_k\}$, where E_k, $k \geq 0$, are defined by (6.15), is a sequence of value sets for the tails of the approximants of the continued fraction (6.1).

Let n be an arbitrary natural number and k be an arbitrary integer such that $k \geq 0$. Then the function $t(z) = 1/z$ maps the set

$$E_{k+1} = \{z \in \mathbb{C} : \ \arg z = (-1)^{k+1}\varphi, \ |z| \geq \rho_{k+1}\}$$

into the set

$$E'_{k+1} = \{z \in \mathbb{C} : \ \arg z = (-1)^{k}\varphi, \ |z| \leq 1/\rho_{k+1}\}.$$

For all $z \in E'_{k+1}$ we have

$$b_k + z = |b_k| \exp((-1)^k i\varphi) + |z| \exp((-1)^k i\varphi) = (|b_k| + |z|) \exp((-1)^k i\varphi).$$

It follows that

$$\arg(b_k + z) = (-1)^k \varphi, \quad |b_k + z| = |b_k| + |z| \geq \rho_k.$$

This proves that condition (6.4) is satisfied. Therefore, the element sets E_k are the value sets for the tails $G_k^{(n)}$.

Let us estimate the quantities $g_k^{(n)}$, $1 \leq k \leq n$. For $1 \leq k \leq n$, we write the quantities $g_k^{(n)}$ as

$$g_k^{(n)} = \frac{1}{G_{k-1}^{(n)} G_k^{(n)}} = \frac{1}{(b_{k-1} + 1/G_k^{(n)}) G_k^{(n)}} = \frac{1}{1 + b_{k-1} G_k^{(n)}}.$$

Let us denote

$$b_{k-1} = |b_{k-1}| \exp((-1)^{k-1} i\varphi), \quad G_k^{(n)} = |G_k^{(n)}| \exp((-1)^k i\varphi).$$

Then

$$|1 + b_{k-1} G_k^{(n)}| = \sqrt{1 + |b_{k-1}|^2 |G_k^{(n)}|^2 + 2|b_{k-1}||G_k^{(n)}|} = 1 + |b_{k-1}||G_k^{(n)}| \geq$$
$$\geq 1 + \rho_{k-1}\rho_k.$$

Therefore,

$$|g_k^{(n)}| \leq \frac{1}{1 + \rho_{k-1}\rho_k}, \quad 1 \leq k \leq n. \tag{6.17}$$

Since $\widehat{b}_k \in E_k$, $k \geq 0$, then

$$|\widehat{g}_k^{(n)}| \leq \frac{1}{1+\rho_{k-1}\rho_k}, \quad 1 \leq k \leq n. \tag{6.18}$$

If the inequalities (6.14), (6.17), and (6.18) are satisfied, then from the formula (6.8) for the relative error of the nth approximant of the continued fraction (6.1), we obtain

$$|\varepsilon_n^{(f)}| \leq |\tilde{\epsilon}_0^{(b)}| + \sum_{k=1}^{n} \left(|\tilde{\epsilon}_k^{(b)}| + |\tilde{\epsilon}_{k-1}^{(b)}|\right) \prod_{m=1}^{k} |\tilde{g}_m^{(n)}| \leq$$
$$\leq \frac{\epsilon^{(b)}}{1-\epsilon^{(b)}} \left(1 + (2-\epsilon^{(b)}) \sum_{k=1}^{n} \prod_{m=1}^{k} \frac{1}{1+\rho_{m-1}\rho_m}\right).$$

If the series (6.16) converges, then there exists a positive constant C such that

$$\sum_{k=1}^{n} \prod_{m=1}^{k} \frac{1}{1+\rho_{m-1}\rho_m} \leq C, \quad n \geq 1.$$

Then

$$|\varepsilon^{(n)}| \leq \frac{\epsilon^{(b)}}{1-\epsilon^{(b)}}(1 + (2-\epsilon^{(b)})C), \quad n \geq 1.$$

Let us consider the function

$$\psi(\beta) = \frac{\epsilon^{(b)}}{1-\epsilon^{(b)}}(1 + (2-\epsilon^{(b)})C).$$

Since $\lim_{\epsilon^{(b)} \to 0^+} \psi(\epsilon^{(b)}) = 0$, then for any $\varepsilon > 0$ there exists

$$\delta_\varepsilon = \frac{1+\varepsilon+2C - \sqrt{(1+\varepsilon+2C)^2 - 4\varepsilon C}}{2C}$$

such that for any $0 < \epsilon^{(b)} < \delta_\varepsilon$, the inequality $\psi(\epsilon^{(b)}) < \varepsilon$ holds. Therefore, if $|\epsilon_k^{(b)}| \leq \epsilon^{(b)} < \delta_\varepsilon$, $k \geq 0$, then $|\varepsilon_n^{(f)}| \leq \psi(\epsilon^{(b)}) < \varepsilon$, which proves that the conditions for determining the sets of stability to perturbations of the continued fraction (6.1) are satisfied. □

Setting $\rho_{2k+1} = \rho_1$, $\rho_{2k} = \rho_2$, $k \geq 0$, $\rho_1 > 0$, and $\rho_2 > 0$ in Theorem 6.2, we obtain the following corollary.

Corollary 6.1. *The twin element sets*

$$E_1 = \{z \in \mathbb{C} : \ \arg z = -\varphi, \ |z| \geq \rho_1\}, \ E_2 = \{z \in \mathbb{C} : \ \arg z = \varphi, \quad |z| \geq \rho_2\},$$

where $0 \leq \varphi < 2\pi$, $\rho_1 > 0$, *and* $\rho_2 > 0$ *are real constants, are sets of stability to perturbations of the continued fraction* (6.1), *if the relative errors of its elements satisfy condition* (6.11). *Moreover, the following estimate holds:*

$$|\varepsilon_n^{(f)}| \leq \frac{\epsilon^{(b)}}{1-\epsilon^{(b)}}\left(1+\frac{2-\epsilon^{(b)}}{\rho_1\rho_2}\left(1-\frac{1}{(1+\rho_1\rho_2)^n}\right)\right), \quad n \geq 1.$$

Theorem 6.3. *The sequence of element sets*

$$E_k = \{z \in \mathbb{C}: \ |\arg z| \leq \pi/4, \ |z| \geq \rho_k\}, \quad k \geq 0, \tag{6.19}$$

where ρ_k, $k \geq 0$, *are positive real constants, is a sequence of sets of stability to perturbations of the continued fraction* (6.1), *if the relative errors of its elements satisfy condition* (6.11) *and the series*

$$\sum_{k=1}^{\infty}\prod_{m=1}^{k}\frac{1}{\sqrt{1+\rho_{m-1}^2\rho_m^2}} \tag{6.20}$$

converges. Moreover, the following estimate holds:

$$|\varepsilon_n^{(f)}| \leq \frac{\epsilon^{(b)}}{1-\epsilon^{(b)}}\left(1+(2-\epsilon^{(b)})\sum_{k=1}^{n}\prod_{m=1}^{k}\frac{1}{\sqrt{1+\rho_{m-1}^2\rho_m^2}}\right), \quad n \geq 1. \tag{6.21}$$

Proof. Let us prove that the sequence $\{E_k\}$, where the sets E_k, $k \geq 0$, are defined by (6.19), is a sequence of value sets for the tails of the approximants of the continued fraction (6.1).

Let n be an arbitrary natural number and k be an arbitrary integer such that $k \geq 0$. The function $t(z) = 1/z$ maps the set

$$E_{k+1} = \{z \in \mathbb{C}: \ |\arg z| \leq \pi/4, \ |z| \geq \rho_{k+1}\}$$

into the set

$$E'_{k+1} = \{z \in \mathbb{C}: \ |\arg z| \leq \pi/4, \ |z| \geq 1/\rho_{k+1}\}.$$

Then condition (6.4) is satisfied if for all $z \in E'_{k+1}$

$$|\arg(b_k + z)| \leq \frac{\pi}{4}, \tag{6.22}$$

and

$$|b_k + z| \geq \rho_k. \tag{6.23}$$

Let us denote $b_k = \operatorname{Re} b_k + i \operatorname{Im} b_k$, $z = \operatorname{Re} z + i \operatorname{Im} z$. Since $|\arg b_k| \le \pi/4$, $|\arg z| \le \pi/4$, then

$$|\operatorname{Im} b_k| \le \operatorname{Re} b_k, \quad |\operatorname{Im} z| \le \operatorname{Re} z.$$

Taking into account that $\operatorname{Re} b_k > 0$, $\operatorname{Re} z \ge 0$, we have

$$\left|\frac{\operatorname{Im}(b_k + z)}{\operatorname{Re}(b_k + z)}\right| = \frac{|\operatorname{Im} b_k + \operatorname{Im} z|}{\operatorname{Re} b_k + \operatorname{Re} z} \le \frac{|\operatorname{Im} b_k| + |\operatorname{Im} z|}{\operatorname{Re} b_k + \operatorname{Re} z} \le 1.$$

This proves that the inequality (6.22) holds for all $z \in E'_{k+1}$.

To prove inequality (6.23), we denote $b_k = |b_k| \exp(i\varphi_k)$, $z = |z| \exp(i\varphi)$. Then

$$|b_k + z| = \sqrt{|b_k|^2 + |z|^2 + 2|b_k||z| \cos(\varphi_k - \varphi)}.$$

Since $|\varphi_k - \varphi| \le \pi/2$, one has $|b_k + z| \ge |b_k| \ge \rho_k$. Therefore, the element sets G_k are the value sets for the tails $G_k^{(n)}$.

Let us estimate the quantities $|1 + b_{k-1} G_k^{(n)}|$ from below, for $1 \le k \le n$. Let us denote

$$b_{k-1} = |b_{k-1}| \exp(i\varphi_{k-1}), \quad G_k^{(n)} = |G_k^{(n)}| \exp(i\varphi_k^{(n)}).$$

Then

$$|1 + b_{k-1} G_k^{(n)}| = \sqrt{1 + |b_{k-1}|^2 |G_k^{(n)}|^2 + 2|b_{k-1}||G_k^{(n)}| \cos(\varphi_{k-1} + \varphi_k^{(n)})}.$$

Since $|\varphi_{k-1} + \varphi_k^{(n)}| \le \pi/2$, then

$$|1 + b_{k-1} G_k^{(n)}| \ge \sqrt{1 + |b_{k-1}|^2 |G_k^{(n)}|^2} \ge \sqrt{1 + \rho_{k-1}^2 \rho_k^2}.$$

Therefore, for $1 \le k \le n$

$$|g_k^{(n)}| = \frac{1}{|1 + b_{k-1} G_k^{(n)}|} \le \frac{1}{\sqrt{1 + \rho_{k-1}^2 \rho_k^2}}. \tag{6.24}$$

Since $\widehat{b}_k \in G_k$, $k \ge 0$, then

$$|\widehat{g}_k^{(n)}| \le \frac{1}{\sqrt{1 + \rho_{k-1}^2 \rho_k^2}}, \quad 1 \le k \le n. \tag{6.25}$$

If inequalities (6.11), (6.24), (6.25) hold, then from formula (6.8) for the relative error of the nth approximant of the continued fraction (6.1), we have

$$|\varepsilon_n^{(f)}| \le |\tilde{\epsilon}_0^{(b)}| + \sum_{k=1}^{n} (|\tilde{\epsilon}_k^{(b)}| + |\tilde{\epsilon}_{k-1}^{(b)}|) \prod_{m=1}^{k} |\tilde{g}_m^{(n)}| \le$$

$$\leq \frac{\epsilon^{(b)}}{1-\epsilon^{(b)}}\left(1+(2-\epsilon^{(b)})\sum_{k=1}^{n}\prod_{m=1}^{k}\frac{1}{\sqrt{1+\rho_{m-1}^2\rho_m^2}}\right).$$

If the series (6.20) converges, then there exists a positive constant C such that

$$\sum_{k=1}^{n}\prod_{m=1}^{k}\frac{1}{\sqrt{1+\rho_{m-1}^2\rho_m^2}} \leq C, \quad n \geq 1.$$

Then, for the errors of the nth approximant of the continued fraction (6.1), the estimate

$$|\varepsilon_n^{(f)}| \leq \frac{\epsilon^{(b)}}{1-\epsilon^{(b)}}(1+(2-\epsilon^{(b)})C), \quad n \geq 1,$$

is valid, from which, as in the Theorem 6.2, it follows that the conditions for determining the sets of stability to perturbations are satisfied. □

Setting $\rho_k = \rho$, $k \geq 0$, $\rho > 0$ in Theorem 6.1, we obtain the following corollary.

Corollary 6.2. *The simple element set*

$$E = \{z \in \mathbb{C}: \ |\arg z| \leq \pi/4, \ |z| \geq \rho\},$$

where ρ is a positive real constant, is a set of stability to perturbations of the continued fraction (6.1), *if the relative errors of its elements satisfy condition* (6.11). *Moreover, the following estimate holds:*

$$|\varepsilon_n^{(f)}| \leq \frac{\epsilon^{(b)}}{1-\epsilon^{(b)}}\left(1+(2-\epsilon^{(b)})\frac{1-\sqrt{(1+\rho^4)}^{-n}}{\sqrt{1+\rho^4}-1}\right), \quad n \geq 1. \tag{6.26}$$

Theorem 6.4. *The sequence of element sets*

$$E_k(\varphi) = \{z \in \mathbb{C}: \ |\arg z + (-1)^{k+1}\varphi| \leq \pi/4, \ |z| \geq \rho_k\}, \quad k \geq 0, \tag{6.27}$$

where $0 \leq \varphi < 2\pi$ and $\rho_k > 0$, $k \geq 0$, are real constants, is a sequence of stability sets to perturbations of the continued fraction (6.1), *if the relative errors of its elements satisfy condition* (6.11) *and the series* (6.20) *converges. Moreover, the estimate* (6.21) *holds for the relative errors of the approximants.*

Proof. Let us consider the continued fractions

$$r_0b_0 + \cfrac{r_0r_1}{r_1b_1 + \cfrac{r_1r_2}{r_2b_2 + \ddots}}, \quad r_0\widehat{b}_0 + \cfrac{r_0r_1}{r_1\widehat{b}_1 + \cfrac{r_1r_2}{r_2\widehat{b}_2 + \ddots}}, \tag{6.28}$$

where $r_k = \exp((-1)^k i\varphi)$, $k \geq 0$, $0 \leq \varphi < 2\pi$, $b_k \in E_k, \widehat{b}_k \in E_k$, $k \geq 0$, and the sets E_k, $k \geq 0$, are defined by (6.19). Then $b_k \exp((-1)^k i\varphi) \in E_k(\varphi)$, $\widehat{b}_k \exp((-1)^k i\varphi) \in E_k(\varphi)$, where the sets $E_k(\varphi)$, $k \geq 0$, are defined by (6.27).

Let us denote by

$$f_n^* = r_0 b_0 + \cfrac{r_0 r_1}{r_1 b_1 + \cfrac{r_1 r_2}{r_2 b_2 + \ddots + \cfrac{r_{n-1} r_n}{r_n b_n}}}, \quad n \geq 1,$$

$$\widehat{f}_n^* = r_0 \widehat{b}_0 + \cfrac{r_0 r_1}{r_1 \widehat{b}_1 + \cfrac{r_1 r_2}{r_2 \widehat{b}_2 + \ddots + \cfrac{r_{n-1} r_n}{r_n \widehat{b}_n}}}, \quad n \geq 1,$$

the approximants of the continued fractions (6.28), respectively, and

$$G_k^{*(n)} = r_k b_k + \cfrac{r_k r_{k+1}}{r_{k+1} b_{k+1} + \cfrac{r_{k+1} r_{k+2}}{r_{k+2} b_{k+2} + \ddots + \cfrac{r_{n-1} r_n}{r_n b_n}}}, \quad 0 \leq k \leq n,$$

$$\widehat{G}_k^{*(n)} = r_k \widehat{b}_k + \cfrac{r_k r_{k+1}}{r_{k+1} \widehat{b}_{k+1} + \cfrac{r_{k+1} r_{k+2}}{r_{k+2} \widehat{b}_{k+2} + \ddots + \cfrac{r_{n-1} r_n}{r_n \widehat{b}_n}}}, \quad 0 \leq k \leq n,$$

the tails of the approximants f_n^*, $\widehat{f}_n^*$, respectively.

It is known that $g_k^{*(n)} = g_k^{(n)}$, $\widehat{g}_k^{*(n)} = \widehat{g}_k^{(n)}$, $1 \leq k \leq n$, [23], where

$$g_k^{*(n)} = \frac{1}{G_{k-1}^{*(n)} G_k^{*(n)}}, \quad \widehat{g}_k^{*(n)} = \frac{1}{\widehat{G}_{k-1}^{*(n)} \widehat{G}_k^{*(n)}}, \quad 1 \leq k \leq n,$$

and the quantities $g_k^{(n)}$, $\widehat{g}_k^{(n)}$, $1 \leq k \leq n$, are defined by (6.7). Then, if $b_k \in E_k, \widehat{b}_k \in E_k$, $k \geq 0$, then for the quantities $g_k^{*(n)}, \widehat{g}_k^{*(n)}$, $1 \leq k \leq n$, the following estimates hold

$$|g_k^{*(n)}| \leq \frac{1}{\sqrt{1 + \rho_{k-1}^2 \rho_k^2}}, \quad |\widehat{g}_k^{*(n)}| \leq \frac{1}{\sqrt{1 + \rho_{k-1}^2 \rho_k^2}}, \quad 1 \leq k \leq n. \tag{6.29}$$

Therefore, if condition (6.11) and inequalities (6.29) are satisfied, then for the relative error of the nth approximant, the estimate (6.21) holds. Then, according to Theorem 6.1, if the series (6.20) converges, the sequence of sets (6.27) is a sequence sets of stability to perturbations of the continued fraction (6.1). □

Setting $\rho_{2k+1} = \rho_1$, $\rho_{2k} = \rho_2$, $k \geq 0$, $\rho_1 > 0$, $\rho_2 > 0$ in Theorem 6.4, we obtain the following corollary.

Corollary 6.3. *The twin element sets*

$$E_1(\varphi) = \left\{z \in \mathbb{C} : \ |\arg z + \varphi| \le \pi/4, \ |z| \ge \rho_1 \right\},$$
$$E_2(\varphi) = \left\{z \in \mathbb{C} : \ |\arg z - \varphi| \le \pi/4, \ |z| \ge \rho_2 \right\},$$

where $0 \le \varphi < 2\pi$, $\rho_1 > 0$ and $\rho_2 > 0$ are real constants, are sets of stability to perturbations of the continued fraction (6.1)*, if the relative errors of its elements satisfy condition* (6.11)*. Moreover, the estimate* (6.26) *holds for the relative errors of the approximants.*

Theorem 6.5. *The sequence of element sets*

$$E_k = \{z \in \mathbb{C} : \ |z| \ge \rho_k + 1/\rho_{k+1}\}, \quad k \ge 0, \tag{6.30}$$

where ρ_k, $k \ge 0$, are positive real constants, is a sequence sets of stability to perturbations of the continued fraction (6.1)*, if the relative errors of its elements satisfy condition* (6.11) *and the series*

$$\sum_{k=1}^{\infty} \prod_{m=1}^{k} \frac{1}{\rho_{m-1}\rho_m} \tag{6.31}$$

converges. Moreover, the following estimate holds:

$$|\varepsilon^{(n)}| \le \frac{\epsilon^{(b)}}{1-\epsilon^{(b)}} \left(1 + (2-\epsilon^{(b)}) \sum_{k=1}^{n} \prod_{m=1}^{k} \frac{1}{\rho_{m-1}\rho_m} \right), \quad n \ge 1.$$

Proof. Let us consider the sequence of sets $\{V_k\}$, where

$$V_k = \{z \in \mathbb{C} : \ |z| \ge \rho_k\}, \quad k \ge 0.$$

Then $E_k \subset V_k$, $k \ge 0$, where the sets G_k, $k \ge 0$, are defined by (6.30), which ensures that condition (6.3) is satisfied.

Let k be an arbitrary non-negative integer number. The function $t(z) = 1/z$ maps V_{k+1} to the set $V'_{k+1} = \{z \in \mathbb{C} : \ |z| \le 1/\rho_{k+1}\}$. Then condition (6.4) is satisfied if inequality (6.23) holds for all $z \in V'_{k+1}$.

Let us denote $b_k = |b_k| \exp(i\varphi_k)$, $z = |z| \exp(i\varphi)$. Then

$$|b_k + z| = \sqrt{|b_k|^2 + |z|^2 + 2|b_k||z|\cos(\varphi_k - \varphi)} \ge \sqrt{|b_k|^2 + |z|^2 - 2|b_k||z|} =$$
$$= ||b_k| - |z|| \ge \rho_k + 1/\rho_{k+1} - 1/\rho_{k+1} = \rho_k.$$

This proves that condition (6.4) is satisfied. Therefore, the sequence of sets $\{V_k\}$ is a sequence of value sets for the tails $G_k^{(n)}$, $k \ge 0$.

Let us denote $b_{k-1} = |b_{k-1}| \exp(i\varphi_{k-1})$, $G_k^{(n)} = |G_k^{(n)}| \exp(i\varphi_k^{(n)})$. Then

$$|1 + b_{k-1}G_k^{(n)}| = \sqrt{1 + |b_{k-1}|^2|G_k^{(n)}|^2 + 2|b_{k-1}||G_k^{(n)}|\cos(\varphi_{k-1} + \varphi_k^{(n)})} \geq$$
$$\geq \sqrt{1 + |b_{k-1}|^2|G_k^{(n)}|^2 - 2|b_{k-1}||G_k^{(n)}|} = ||b_{k-1}||G_k^{(n)}| - 1| \geq$$
$$\geq \left(\rho_{k-1} + \frac{1}{\rho_k}\right)\rho_k - 1 = \rho_{k-1}\rho_k.$$

Thus,

$$|g_k^{(n)}| = \frac{1}{|1 + b_{k-1}G_k^{(n)}|} \leq \frac{1}{\rho_{k-1}\rho_k}, \quad 1 \leq k \leq n. \tag{6.32}$$

Since $\widehat{b}_k \in G_k$, $k \geq 0$, then

$$|\widehat{g}_k^{(n)}| \leq \frac{1}{\rho_{k-1}\rho_k}, \quad 1 \leq k \leq n. \tag{6.33}$$

Taking into account the fulfillment of inequalities (6.11), (6.32), (6.33), from formula (6.8) for the relative error of the nth approximant of the continued fraction (6.1), we have

$$|\varepsilon_n^{(f)}| \leq |\tilde{\epsilon}_0^{(b)}| + \sum_{k=1}^{n}(|\tilde{\epsilon}_k^{(b)}| + |\tilde{\epsilon}_{k-1}^{(b)}|)\prod_{m=1}^{k}|\tilde{g}_m^{(n)}| \leq$$
$$\leq \frac{\epsilon^{(b)}}{1 - \epsilon^{(b)}}\left(1 + (2 - \epsilon^{(b)})\sum_{k=1}^{n}\prod_{m=1}^{k}\frac{1}{\rho_{m-1}\rho_m}\right).$$

As in the proof of Theorem 6.2, we conclude that the convergence of the series (6.31) ensures the fulfillment of the conditions of the definition sets of stability to perturbations of the continued fraction (6.1). □

Setting $\rho_k = \rho$, $k \geq 0$ herewith $\rho > 1$ in Theorem 6.5, we obtain the following result.

Corollary 6.4. *The simple element set*

$$E = \{z \in \mathbb{C} : \ |z| \geq \rho + 1/\rho\},$$

where $\rho > 1$ is a real constant, is a set of stability to perturbations of the continued fraction (6.1)*, if the relative errors of its elements satisfy condition* (6.11)*. Moreover, the following estimate holds:*

$$|\varepsilon_n^{(f)}| \leq \frac{\epsilon^{(b)}}{1 - \epsilon^{(b)}}\left(1 + (2 - \epsilon^{(b)})\frac{1 - \rho^{-2n}}{\rho^2 - 1}\right), \quad n \geq 1.$$

Theorem 6.6. *The sequence of sets*

$$E_k = \{z \in \mathbb{C}: \ \operatorname{Re} z \geq \rho_k\}, \quad k \geq 0, \tag{6.34}$$

where ρ_k, $k \geq 0$, are positive real constants, is a sequence sets of stability to perturbations of the continued fraction (6.1), *if the relative errors of its elements satisfy condition* (6.11) *and the series*

$$\sum_{k=1}^{\infty} \prod_{m=1}^{k} \eta_m, \tag{6.35}$$

converges, where

$$\eta_k = \begin{cases} \dfrac{1}{1+\rho_{k-1}\rho_k}, & \rho_{k-1}\rho_k \geq 1, \\ \dfrac{1}{2\sqrt{\rho_{k-1}\rho_k}}, & 0 < \rho_{k-1}\rho_k < 1, \end{cases} \quad k \geq 1. \tag{6.36}$$

Moreover, the following estimate holds:

$$|\varepsilon_n^{(f)}| \leq \frac{\epsilon^{(b)}}{1-\epsilon^{(b)}} \left(1 + (2-\epsilon^{(b)}) \sum_{k=1}^{n} \prod_{m=1}^{k} \eta_m\right), \quad n \geq 1. \tag{6.37}$$

Proof. Let us prove that the sequence $\{E_k\}$, where the sets E_k, $k \geq 0$, are defined by (6.34), is a sequence of value sets for the tails of the approximants of the continued fraction (6.1).

Let k be an arbitrary non-negative integer number. Since $0 \notin G_k$, the function $t(z) = 1/z$ maps the set E_{k+1} to a circle with center at the point $1/(2\rho_{k+1})$ and radius $1/(2\rho_{k+1})$. Then

$$b_k + \frac{1}{E_{k+1}} = \{z: \ |z - q_k| \leq r_k\},$$

where $r_k = 1/(2\rho_{k+1})$, $q_k = b_k + 1/(2\rho_{k+1}) \in E_k$. Condition (6.4) is satisfied if $q_k \in V_k$ and $\operatorname{Re}(q_k) - \rho_k \geq r_k$. The last inequality is equivalent to the inequality by which the sets (6.34) are defined. Thus, the sets (6.34) are the value sets for the tails of the approximants of the continued fraction (6.1).

Let n be an arbitrary natural number and k be an arbitrary positive integer number. Let us denote $b_{k-1} = x_1 + iy_1$, $G_k^{(n)} = x_2 + iy_2$. Then

$$|g_k^{(n)}| = \frac{1}{|1 + b_{k-1}G_k^{(n)}|} = \frac{1}{\sqrt{(1 + x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2}}.$$

Since

$$\min_{\substack{x_1 \geq \rho_{k-1}, \\ x_2 \geq \rho_k, \\ y_1, y_2 \in \mathbb{R}}} \sqrt{(1 + x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2} = \begin{cases} 1 + \rho_{k-1}\rho_k, & \rho_{k-1}\rho_k \in [1, \infty), \\ 2\sqrt{\rho_{k-1}\rho_k}, & \rho_{k-1}\rho_k \in (0; 1), \end{cases}$$

then
$$|g_k^{(n)}| \le \eta_k, \tag{6.38}$$
where η_k is defined by (6.36). In addition, since $\widehat{b}_k \in E_k$, then
$$|\widehat{g}_k^{(n)}| \le \eta_k. \tag{6.39}$$

Therefore, if inequalities (6.11), (6.38), (6.39) hold, then for the relative error of the nth approximant of the continued fraction (6.1), the estimate (6.37) holds. Then, if the series (38) converges, the sets (6.34) are sets of stability to perturbations of the continued fraction (6.1). □

Setting $\rho_k = \rho$, $k \ge 0$, herewith $\rho > 1/2$ in Theorem 6.6, we obtain the following corollary.

Corollary 6.5. *The simple element set* $E_k = \{z \in \mathbb{C} : \operatorname{Re} z \ge \rho\}$, *with* $\rho > 1/2$ *is a set of stability to perturbations of the continued fraction* (6.1), *if the relative errors of its elements satisfy condition* (6.11). *Moreover, the following estimate holds:*
$$|\varepsilon_n^{(f)}| \le \frac{\epsilon^{(b)}}{1-\epsilon^{(b)}}\left(1+(2-\epsilon^{(b)})\frac{\eta(1-\eta^n)}{1-\eta}\right), \quad n \ge 1, \tag{6.40}$$
where
$$\eta = \begin{cases} \dfrac{1}{1+\rho^2}, & \rho \ge 1, \\ \dfrac{1}{2\rho}, & 1/2 < \rho < 1. \end{cases}$$

By the same way of proving Theorem 6.4, we obtain the following result.

Theorem 6.7. *The sequence of sets*
$$E_k = \{z \in \mathbb{C} : \operatorname{Re}(z \exp((-1)^{k+1} i\varphi)) \ge \rho_k\}, \quad k \ge 0,$$
where $\rho_k > 0$, $k \ge 0$, *and* $0 \le \varphi < 2\pi$ *are real constants, is a sequence sets of stability to perturbations of the continued fraction* (6.1), *if the relative errors of its elements satisfy condition* (6.11) *and the series* (6.35) *converges. Moreover, the estimate* (6.37) *holds for the relative errors of the approximants.*

Setting $\rho_{2k+1} = \rho_1$, $\rho_{2k} = \rho_2$, $k \ge 0$, herewith $\rho_1 > 0$, $\rho_2 > 0$, and $\rho_1\rho_2 > 1/2$ in Theorem 6.7, we obtain the following corollary.

Corollary 6.6. *The twin element sets*
$$E_1 = \{z \in \mathbb{C} : \operatorname{Re}(z \exp(i\varphi)) \ge \rho_1\}, \quad E_2 = \{z \in \mathbb{C} : \operatorname{Re}(z \exp(-i\varphi)) \ge \rho_2\},$$

where $0 \le \varphi < 2\pi$, $\rho_1 > 0$, $\rho_2 > 0$, $\rho_1\rho_2 > 1/2$, *are real constants, are sets of stability to perturbations of the continued fraction* (6.1), *if the relative errors of its elements satisfy condition* (6.11). *Moreover, the estimate* (6.40) *holds, where*

$$\eta = \begin{cases} \dfrac{1}{1+\rho_1\rho_2}, & \rho_1\rho_2 \ge 1, \\ \dfrac{1}{2\sqrt{\rho_1\rho_2}}, & 1/2 < \rho_1\rho_2 < 1. \end{cases}$$

The theoretical results presented in Theorems 6.1–6.7 describe the conditions under which continued fractions with partial numerators equal to one are stable to perturbations of their elements. As a practical illustration of these properties, we examine Bessel functions of the first kind, which play a fundamental role in various branches of mathematics and applied sciences.

The Bessel function of the first kind is defined as

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{2k}, \quad |\arg z| < \pi,$$

where $\nu \in \mathbb{C}$ is the order of the Bessel function, and $\Gamma(z)$ is the gamma function.

For our investigation, we consider the continued fraction expansion for the ratio of Bessel functions [6]

$$\frac{J_{\nu+1}(z)}{J_\nu(z)} = \cfrac{z/(2(\nu+1))}{1 - \cfrac{z^2/(4(\nu+1)(\nu+2))}{1 - \cfrac{z^2/(4(\nu+2)(\nu+3))}{1 - \ddots}}}.$$

By an equivalence transformation, this continued fraction can be converted to a continued fraction with partial denominators $b_k(z) = (-1)^{k-1}2(\nu+k)/z$ and unit partial numerators:

$$\cfrac{1}{2(\nu+1)/z + \cfrac{1}{-2(\nu+2)/z + \cfrac{1}{2(\nu+3)/z - \ddots}}}. \tag{6.41}$$

For (6.41), we consider the perturbed continued fraction

$$\cfrac{1}{2(\widehat{\nu}+1)/\widehat{z} + \cfrac{1}{-2(\widehat{\nu}+2)/\widehat{z} + \cfrac{1}{2(\widehat{\nu}+3)/\widehat{z} - \ddots}}}, \tag{6.42}$$

Let $z, \widehat{z} \in E_M$, where

$$E_M = \{z \in \mathbb{C} : \ |z| \leq M\}. \tag{6.43}$$

where M is a positive real constant.

Theorem 6.8. *The set* (6.43) *is a set of stability to perturbations for the continued fraction* (6.41), *if there exists a constant* $\epsilon^{(b)}$, $0 < \epsilon^{(b)} < 1$, *such that the relative errors of its elements* $b_k(z) = (-1)^{k-1}2(\nu + k)/z$ *satisfy condition* (6.11) *and* $M < (1 + \vartheta)$, *where* $\vartheta = \min\{\nu, \widehat{\nu}\}$. *Moreover, the following estimate holds:*

$$|\epsilon_n^{(f)}| \leq \frac{\epsilon^{(b)}}{1 - \epsilon^{(b)}} \left(1 + (2 - \epsilon^{(b)})\frac{1 - \rho^{-2(n-1)}}{\rho^2 - 1}\right), \quad n \geq 1,$$

where $\rho = (a + \sqrt{a^2 - 4})/2 > 1$, $a = 2(\vartheta + 1)/M$.

Proof. Since $b_0(z) = \widehat{b}_0(z) = 0$, then $\epsilon_0^{(b)} = 0$ and

$$g_1^{(n)} = \frac{1}{1 + b_0 G_1^{(n)}} = 1.$$

Then the formula for the relative error of the nth approximant takes the form

$$\varepsilon_n^{(f)} = \tilde{\epsilon}_1^{(b)} + \sum_{k=2}^{n} (\tilde{\epsilon}_k^{(b)} - \tilde{\epsilon}_{k-1}^{(b)}) \prod_{m=2}^{k} \tilde{g}_m^{(n)}, \tag{6.44}$$

where the quantities $\tilde{\epsilon}_k^{(b)}$, $k \geq 1$, $\tilde{g}_k^{(n)}$, $k \geq 2$, are defined by (6.9).

Since $z, \widehat{z} \in E_M$ with E_M given by (6.1), for the elements of the continued fraction (6.41) and its perturbed continued fraction (6.42) for $k \geq 1$, the following estimates hold

$$|b_k(z)| = |(-1)^{k-1}2(\nu + k)/z| = 2(\nu + k)/|z| \geq 2(\vartheta + 1)/M,$$
$$|\widehat{b}_k(\widehat{z})| = |(-1)^{k-1}2(\widehat{\nu} + k)/\widehat{z}| = 2(\widehat{\nu} + k)/|\widehat{z}| \geq 2(\vartheta + 1)/M.$$

If $M < (1 + \vartheta)$, then the quantity $a = 2(\vartheta + 1)/M$ can be written as $a = \rho + 1/\rho$, where $\rho = (a + \sqrt{a^2 - 4})/2 > 1$. Then $|b_k(z)| \geq \rho + 1/\rho$, $|\widehat{b}_k(\widehat{z})| \geq \rho + 1/\rho$, $k \geq 1$, and according to corollary 6.4, the set (6.43) is a set of stability to perturbations of the continued fraction (6.41). In addition,

$$|g_k^{(n)}(z)| \leq \rho^{-2}, \quad |\widehat{g}_k^{(n)}(\widehat{z})| \leq \rho^{-2}, \quad k \geq 2.$$

and from formula (6.44) we have for $n \geq 1$

$$|\varepsilon_n^{(f)}| \leq |\tilde{\epsilon}_1^{(b)}| + \sum_{k=2}^{n} (|\tilde{\epsilon}_k^{(b)}| - |\tilde{\epsilon}_{k-1}^{(b)}|) \prod_{m=2}^{k} |\tilde{g}_m^{(n)}| \leq$$

$$\leq \frac{\epsilon^{(b)}}{1 - \epsilon^{(b)}} \left(1 + (2 - \epsilon^{(b)}) \sum_{k=2}^{n} \rho^{-2(k-1)}\right) = \frac{\epsilon^{(b)}}{1 - \epsilon^{(b)}} \left(1 + (2 - \epsilon^{(b)})\frac{1 - \rho^{-2(n-1)}}{\rho^2 - 1}\right).$$

□

6.2 Stability to perturbations of continued fraction with complex partial numerators and partial denominators equal to one

In this section, we establish sufficient conditions for the stability to perturbations of the continued fraction with complex partial numerators

$$1+\cfrac{a_1}{1+\cfrac{a_2}{1+\ddots}}. \tag{6.45}$$

Let n be a fixed natural number. The nth approximant

$$f_n = 1+\cfrac{a_1}{1+\cfrac{a_2}{1+\ddots+\cfrac{a_n}{1}}}.$$

of the continued fraction (6.45) is equal to $P_0^{(n)}$, where

$$P_k^{(n)} = 1+\cfrac{a_{k+1}}{1+\cfrac{a_{k+2}}{1+\ddots+\cfrac{a_n}{1}}}, \qquad k=n-1,n-2,\ldots,0. \tag{6.46}$$

For the tails $P_k^{(n)}$, $0 \le k \le n$, $n \ge 1$, the recurrence relations hold:

$$P_k^{(n)} = 1+\frac{a_{k+1}}{P_{k+1}^{(n)}}, \qquad 0 \le k \le n,$$

under the initial condition $P_n^{(n)} = 1$. We set

$$p_k^{(n)} = \frac{a_k}{P_{k-1}^{(n)}P_k^{(n)}}, \qquad 1 \le k \le n. \tag{6.47}$$

When the elements a_k, $k \ge 1$, of the continued fraction (6.45) are perturbed, the relative error arises

$$\varepsilon_k^{(a)} = \frac{\widehat{a}_k - a_k}{a_k}. \tag{6.48}$$

If $\widehat{a}_k = a_k = 0$, we assume that $\varepsilon_k^{(a)} = 0$. In addition, the number

$$\widehat{f}_n = \widehat{P}_0^{(n)} = 1+\cfrac{\widehat{a}_1}{1+\cfrac{\widehat{a}_2}{1+\ddots+\cfrac{\widehat{a}_n}{1}}}, \tag{6.49}$$

where $\widehat{P}_k^{(n)} = 1+\dfrac{\widehat{a}_{k+1}}{\widehat{P}_{k+1}^{(n)}}$, $k = n-1, n-2, \dots, 0$, under the initial condition $\widehat{P}_n^{(n)} = 1$, is an perturbed value of the approximant f_n.

Definition 6.3. *The continued fraction* (6.45) *is called stable to perturbations if for arbitrary* $\varepsilon > 0$ *there exists* $\delta_\varepsilon > 0$ *such that for every* $\widehat{a}_k \in \mathbb{C}$*,* $k \geq 1$*, satisfying*

$$\left|\frac{\widehat{a}_k - a_k}{a_k}\right| < \delta_\varepsilon,$$

with $a_k \neq 0$*,* $k \geq 1$*, the following inequality holds*

$$\left|\frac{\hat{f}_n - f_n}{f_n}\right| < \varepsilon, \quad n \geq 1,$$

where $f_n \neq 0$*,* $n \geq 1$*, and* $\widehat{f}_n$ *denotes the* n*th approximant (6.49) of the continued fraction with perturbed elements* $\widehat{a}_k$*.*

Let

$$\varepsilon_{k,n}^{(P)} = \frac{\widehat{P}_k^{(n)} - P_k^{(n)}}{P_k^{(n)}}$$

be a relative errors of $P_k^{(n)}$, $1 \leq k \leq n$. Since $P_n^{(n)} = \widehat{P}_n^{(n)} = 1$, then $\varepsilon_{n,n}^{(P)} = 0$. We prove that

$$\varepsilon_{k,n}^{(P)} = -p_{k+1}^{(n)} + \frac{p_{k+1}^{(n)}(1+\varepsilon_{k+1}^{(a)})}{1+\varepsilon_{k+1,n}^{(P)}}, \quad 0 \leq k \leq n-1, \tag{6.50}$$

where $\varepsilon_k^{(a)}$, $1 \leq k \leq n$, are defined by (6.48), and $g_k^{(n)}$, $1 \leq k \leq n$, are defined by (6.47).

For any $0 \leq k \leq n-1$ we have

$$\varepsilon_{k,n}^{(P)} = \frac{\widehat{P}_k^{(n)} - P_k^{(n)}}{P_k^{(n)}} = \frac{1}{P_k^{(n)}} + \frac{a_{k+1}(1+\varepsilon_{k+1}^{(a)})}{P_k^{(n)} P_{k+1}^{(n)}(1+\varepsilon_{k+1,n}^{(P)})} - 1.$$

From here we get (6.50) since

$$\frac{1}{P_k^{(n)}} = \frac{1}{P_k^{(n)}}\left(P_k^{(n)} - \frac{a_{k+1}}{P_{k+1}^{(n)}}\right) = 1 - p_{k+1}^{(n)}.$$

Using the recurrent relation (6.50), for any $0 \leq k \leq n-1$ we obtain

$$\varepsilon_{k,n}^{(P)} = -p_{k+1}^{(n)} + \cfrac{p_{k+1}^{(n)}(1+\varepsilon_{k+1}^{(a)})}{1 - p_{k+2}^{(n)} + \cfrac{p_{k+2}^{(n)}(1+\varepsilon_{k+2}^{(a)})}{1 - p_{k+3}^{(n)} + \ddots + \cfrac{p_n^{(n)}(1+\varepsilon_n^{(a)})}{1}}}, \tag{6.51}$$

where $0 \le k \le n-1$. It follows that for the relative error of the nth approximant of the continued fraction (6.45) the following formula holds

$$\varepsilon_n^{(f)} = \frac{\widehat{f}_n - f_n}{f_n} = -p_1^{(n)} + \cfrac{p_1^{(n)}(1+\varepsilon_1^{(a)})}{1 - p_2^{(n)} + \cfrac{p_2^{(n)}(1+\varepsilon_2^{(a)})}{1 - p_3^{(n)} + \ddots + \cfrac{p_n^{(n)}(1+\varepsilon_n^{(a)})}{1}}}, \quad n \ge 1.$$

Theorem 6.9. *The continued fraction (6.45) is stable to perturbations if there exist positive constant $\varepsilon^{(a)}$ such that*

$$|\varepsilon_k^{(a)}| \le \varepsilon^{(a)}, \quad k \ge 1, \tag{6.52}$$

where $\varepsilon_k^{(a)}$ are defined by (6.48), and there exists a constant η, $0 < \eta < 1$, such that

$$|p_k^{(n)}| \le \eta, \quad 1 \le k \le n, \quad n \ge 1, \text{ and}$$

$$\frac{\eta(1+\varepsilon^{(a)})}{(1+\eta)^2} \le \frac{1}{4}. \tag{6.53}$$

Moreover, for the relative error of the nth approximant of the continued fraction (6.45), the following

$$|\varepsilon_n^{(f)}| \le \frac{1-\eta}{2} - \frac{\sqrt{(1+\eta)^2 - 4\eta(1+\varepsilon^{(a)})}}{2}, \quad n \ge 1, \tag{6.54}$$

is valid, if $(1+\eta)^2 > 4\eta(1+\varepsilon^{(a)})$, and

$$|\varepsilon_n^{(f)}| \le \sqrt{(\varepsilon^{(a)})^2 + \varepsilon^{(a)}} - \varepsilon^{(a)}, \quad n \ge 1, \tag{6.55}$$

if $(1+\eta)^2 = 4\eta(1+\varepsilon^{(a)})$.

Proof. Using (6.51), we estimate the absolute value of the relative error of the quantities $P_k^{(n)}$ defined in (6.46).

Let $\{t_n(\omega)\}_{n\ge0}$ and $\{T_n(\omega)\}_{n\ge0}$ be the sequence of linear fractional transformations

$$t_0(\omega) = 1 + \eta + \omega, \quad t_n(\omega) = -\frac{\eta(1+\varepsilon^{(a)})}{1+\eta+\omega},$$
$$T_0(\omega) = t_0(\omega), \quad T_n(\omega) = T_{n-1}(t_n(\omega)), \quad n \ge 1.$$

Let us establish the conditions under which $T_n(\omega) > 0$, $n \ge 1$. To do this, let us set

$$\omega' = \frac{\omega}{1+\eta}$$

and consider sequences $\{t'_n(\omega')\}_{n\geq 0}$ and $\{T'_n(\omega')\}_{n\geq 0}$ of linear fractional transformations

$$t'_0(\omega) = (1+\eta)(1+\omega'), t'_n(\omega) = -\frac{\eta(1+\varepsilon^{(a)})/(1+\eta)^2}{1+\omega'},$$
$$T'_0(\omega') = t'_0, \quad T'_n(\omega') = T_{n-1}(t'_n(\omega')), \quad n \geq 1.$$

With (6.53), we write

$$\frac{\eta(1+\varepsilon^{(a)})}{(1+\eta)^2} = r(1-r),$$

where

$$r = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4\eta(1+\varepsilon^{(a)})}{(1+\eta)^2}}\right), \quad 0 < r < \frac{1}{2}.$$

Now, we consider a periodic continued fraction

$$1 - \cfrac{r(1-r)}{1 - \cfrac{r(1-r)}{1 - \ddots}}. \tag{6.56}$$

Let $h_n = A_n/B_n$ be the nth approximant of the continued fraction (6.56), $n \geq 1$. According to [24, Theorem 3.2] the continued fraction (6.56) converges to the value $(1 - \sqrt{1-4r(1-r)})/2$. In addition, it is easy to show that the approximants h_n, $n \geq 1$, form a monotonically decreasing sequence such that

$$1 > h_n > \frac{1 - \sqrt{1-4r(1-r)}}{2}, \quad n \geq 1, \tag{6.57}$$

and

$$B_n = \sum_{i=0}^{n} r^i(1-r)^{n-i}, \quad A_n = B_{n+1}, \quad n \geq 1.$$

Let n be an arbitrary natural number. Then

$$T'_n(\omega') = (1+\eta)\frac{A_n + \omega' A_{n-1}}{B_n + \omega' B_{n-1}} = (1+\eta)\frac{B_{n+1} + \omega' B_n}{B_n + \omega' B_{n-1}}.$$

Hence $T'_n(\omega') > 0$, if

$$-\omega' < \frac{B_{n+1}}{B_n} \quad \text{or} \quad -\omega' > \frac{B_{n+1}}{B_n}, \quad n \geq 1.$$

It follows from (6.57) that $T_n'(\omega') > 0$, if

$$-\omega' < \frac{1-\sqrt{1-4r(1-r)}}{2} \quad \text{or} \quad -\omega' > 1.$$

We choose $\omega = -\eta$. Then $\omega' = -\frac{\eta}{1+\eta}$ and $T_n'(\omega') > 0$, if

$$\omega' < \frac{1}{2}\left(1-\sqrt{1-\frac{4\eta(1+\varepsilon^{(a)})}{(1+\eta)^2}}\right).$$

Note that the last inequality holds for all $0 < \eta < 1$.

Using (6.50), by induction on k, we show that

$$|\varepsilon_{k,n}^{(P)}| \le -\eta - \cfrac{c_1}{d_1 + \cfrac{c_2}{d_2 + \ddots + \cfrac{c_{n-k}}{d_{n-k}}}}, \qquad k = n-1, n-2, \ldots, 0, \tag{6.58}$$

where

$$c_m = -\eta(1+\varepsilon^{(a)}), \quad 1 \le m \le n-k,$$
$$d_m = 1+\eta, \quad 1 \le m \le n-k-1, \quad d_{n-k} = 1.$$

Note that

$$1+\eta+\cfrac{c_1}{d_1 + \cfrac{c_2}{d_2 + \ddots + \cfrac{c_{n-k}}{d_{n-k}}}} = T_{n-k-1}'\left(-\frac{\eta}{1+\eta}\right) > 0$$

if $0 < \eta < 1$ and hence the condition (6.53) holds.

For $k = n-1$ we have

$$|\varepsilon_{n-1,n}^{(P)}| = |-p_n^{(n)} + p_n^{(n)}(1+\varepsilon_n^{(a)})| \le |p_n^{(n)}||\varepsilon_n^{(a)}| \le \eta\varepsilon^{(a)} = -\eta + \eta(1+\varepsilon^{(a)}).$$

Let (6.58) holds for $k = m+1$, $0 \le m \le n-2$. Then, for $k = m$,

$$|\varepsilon_{m,n}^{(P)}| = \left|-p_{m+1}^{(n)} + \frac{p_{m+1}^{(n)}(1+\varepsilon_{m+1}^{(a)})}{1+\varepsilon_{m+1,n}^{(P)}}\right| = \left|p_{m+1}^{(n)}\left(\frac{1+\varepsilon_{m+1}^{(a)}}{1+\varepsilon_{m+1,n}^{(P)}} - 1\right)\right| \le$$

$$\le |p_{m+1}^{(n)}|\frac{|\varepsilon_{m+1}^{(a)}| + |\varepsilon_{m+1,n}^{(P)}|}{1-|\varepsilon_{m+1,n}^{(P)}|} \le \eta\left(\frac{1+\varepsilon^{(a)}}{1-|\varepsilon_{m+1,n}^{(P)}|} - 1\right) =$$

$$= -\eta + \frac{\eta(1+\varepsilon^{(a)})}{1-|\varepsilon_{m+1,n}^{(P)}|} \le -\eta + \cfrac{\eta(1+\varepsilon^{(a)})}{1+\eta+\cfrac{c_1}{d_1+\cfrac{c_2}{d_2+\ddots+\cfrac{c_{n-m-1}}{d_{n-m-1}}}}} =$$

$$= -\eta - \cfrac{c_1}{d_1 + \cfrac{c_2}{d_2 + \ddots + \cfrac{c_{n-m}}{d_{n-m}}}}.$$

Setting $k = 0$ in (6.58), for the relative error of the nth approximant of the continued fraction (6.45), we obtain the following

$$|\varepsilon_n^{(f)}| \le -\eta - \cfrac{c_1}{d_1 + \cfrac{c_2}{d_2 + \ddots + \cfrac{c_n}{d_n}}}. \tag{6.59}$$

Now, we find the value of the right-hand side of (6.59). It is sufficient to find

$$\lim_{n\to+\infty} \cfrac{c_1}{d_1 + \cfrac{c_2}{d_2 + \ddots + \cfrac{c_n}{d_n}}}.$$

The transformation

$$t(\omega) = -\frac{\eta(1+\varepsilon^{(a)})}{1+\eta+\omega}$$

is a linear fractional transformation with fixed points

$$\omega_1 = \frac{(1-\eta)(1+\varepsilon^{(a)})}{2} + \frac{\sqrt{(1+\eta)^2 - 4\eta(1+\varepsilon^{(a)})}}{2} - 1,$$
$$\omega_2 = \frac{(1-\eta)(1+\varepsilon^{(a)})}{2} - \frac{\sqrt{(1+\eta)^2 - 4\eta(1+\varepsilon^{(a)})}}{2} - 1,$$

herewith ω_1 is an attractive point.

Let $(1+\eta)^2 - 4\eta(1+\varepsilon^{(a)}) > 0$. Then $\omega_1 \neq \omega_2$. Since $\eta \neq t^{-1}(\omega_2)$, then [24]

$$\lim_{n\to+\infty} \cfrac{c_1}{d_1 + \cfrac{c_2}{d_2 + \ddots + \cfrac{c_n}{d_n}}} = \omega_1$$

and, therefore,

$$-\eta - \omega_1 = \frac{1-\eta}{2} - \frac{\sqrt{(1+\eta)^2 - 4\eta(1+\varepsilon^{(a)})}}{2}.$$

Using the formula for the difference between compositions of linear fractional transformations $T_{n+1}(-\eta) - T_n(-\eta)$ [4], we have

$$\frac{\eta(1+\varepsilon^{(a)})}{T_{n+1}(-\eta)} - \frac{\eta(1+\varepsilon^{(a)})}{T_n(-\eta)} \ge 0,$$

that is, the sequence

$$-\eta-\cfrac{c_1}{d_1+\cfrac{c_2}{d_2+\ddots+\cfrac{c_n}{d_n}}},\quad n\geq 1,$$

is monotonically non-decreasing and the estimate (6.54) holds for the relative error of the nth approximant of the continued fraction (6.45).

Next, we show that from (6.54) it follows that the conditions of definition 6.3 are fulfilled. Consider the function

$$\varphi(\varepsilon^{(a)})=\frac{1-\eta}{2}-\frac{\sqrt{(1+\eta)^2-4\eta(1+\varepsilon^{(a)})}}{2}.$$

Since $\lim_{\varepsilon^{(a)}\to+0}\varphi(\varepsilon^{(a)})=0$, then for any $\varepsilon>0$ there exists $\delta_\epsilon>0$ such that for all $\varepsilon^{(a)}$ such that $0<\varepsilon^{(a)}<\delta_\epsilon$, the inequality $\varphi(\varepsilon^{(a)})<\varepsilon$ holds. If $\varepsilon^{(a)}<\delta_\epsilon$, then provide $|\varepsilon_k^{(a)}|\leq\varepsilon^{(a)}<\delta_\epsilon$, $k\geq 1$, the inequality $|\varepsilon_n^{(f)}|<\varepsilon$ holds, which proves the stability to perturbations of the continued fraction (6.45).

Finally, let

$$(1+\eta)^2-4\eta(1+\varepsilon^{(a)})=0.$$

Then $\omega_1=\omega_2$ and

$$-\eta-\lim_{n\to+\infty}\cfrac{c_1}{d_1+\cfrac{c_2}{d_2+\ddots+\cfrac{c_n}{d_n}}}=\psi(\varepsilon^{(a)}),$$

where $\psi(\varepsilon^{(a)})=\sqrt{(\varepsilon^{(a)})^2+\varepsilon^{(a)}}-\varepsilon^{(a)}$. Thus, the estimate (6.55) is valid and, as in the previous case, we obtain the stability to perturbations of the continued fraction (6.45). □

We consider the stability to perturbations of the continued fraction expansion of ratio of the Horn‘s confluent functions H_7.

Recall than the Horn‘s confluent function H_7 is defined as follows

$$\mathrm{H}_7(\alpha;\gamma_1,\gamma_2;\mathbf{z})=\sum_{r,s=0}^{\infty}\frac{(\alpha)_{2r+s}}{(\gamma_1)_r(\gamma_2)_s}\frac{z_1^r z_2^s}{r!\,s!},\quad |z_1|<1/4,\quad |z_2|<+\infty,$$

where $\alpha,\gamma_1,\gamma_2\in\mathbb{C}$, $\gamma_1,\gamma_2\notin\{0,-1,-2,\ldots\}$, $(\alpha)_k$ is the Pochhammer‘s symbol, $\mathbf{z}=(z_1,z_2)\in\mathbb{C}^2$.

In [1] it was established, in particular, that

$$\frac{\mathrm{H}_7(\alpha;(\alpha+1)/2,\alpha;\mathbf{z})}{\mathrm{H}_7(\alpha+1;(\alpha+3)/2,\alpha;\mathbf{z})}=1+\cfrac{d_1z_1}{1+\cfrac{d_2z_2}{1+\cfrac{d_3z_2}{1+\cfrac{d_4z_1}{1+\cfrac{d_5z_2}{1+\cfrac{d_6z_2}{1+\ddots}}}}}}. \tag{6.60}$$

where

$$d_{3k+1}=-4,\quad d_{3k+2}=-\frac{1}{\alpha+2k+1},\quad d_{3k+3}=\frac{1}{\alpha+2k+1},\quad k\ge 0. \tag{6.61}$$

Theorem 6.10. *The continued fraction (6.60) is stable to perturbations if the relative errors of their elements satisfy the conditions (6.52), and* $\mathbf{z}\in\Omega_{\alpha,\rho}$, *where*

$$\Omega_{\alpha,\rho}=\left\{\mathbf{z}\in\mathbb{C}^2:\ |z_1|\le\frac{\rho(1-\rho)}{4},\ |z_2|\le|1+\alpha|\rho(1-\rho)\right\},$$

$0<\rho<\frac{1}{2}$, $\mathrm{Re}(\alpha)>-1$, *and for* $\eta=\rho/(1-\rho)$, *the inequality (6.53) holds.*

Proof. For any $\mathbf{z}\in\Omega_{\alpha,\rho}$ and $k\ge 0$, we have

$$|d_{3k+1}z_1|\le\rho(1-\rho),\ \ |d_{3k+2}z_2|\le\frac{|\alpha+1|\rho(1-\rho)}{|\alpha+2k+1|}\le\rho(1-\rho),$$
$$|d_{3k+3}z_2|\le\rho(1-\rho).$$

Then it is easy to see that $V=\{\omega\in\mathbb{C}:\ |\omega|\le\rho\}$ is the value set corresponding to the element set $E=\{\omega\in\mathbb{C}:\ |\omega|\le\rho(1-\rho)\}$ of the continued fraction (6.60) (see, [6, Section 3.2]).

Let n be an arbitrary natural number. By setting

$$a_{3k+1}=d_{3k+1}z_1,\quad a_{3k+2}=d_{3k+2}z_2,\quad a_{3k+3}=d_{3k+3}z_2,\quad k\ge 0, \tag{6.62}$$

we use the settings (6.46) and (6.47).

It follows from $0\in V$ and $\frac{a_k}{1+V}\subseteq V$ that [23]

$$\frac{a_k}{P_k^{(n)}}\in V,\quad 1\le k\le n.$$

For any $1\le k\le n$ we rewrite $p_k^{(n)}$ as

$$p_k^{(n)}=\frac{a_k}{P_{k-1}^{(n)}P_k^{(n)}}=\frac{a_k}{\left(1+\dfrac{a_k}{P_k^{(n)}}\right)P_k^{(n)}}=1-\frac{1}{1+\dfrac{a_k}{P_k^{(n)}}}.$$

Then,

$$p_k^{(n)} \in 1 - \frac{1}{1+V}, \quad 1 \le k \le n.$$

Now, since $0 < \rho < 1/2$, then $0 \notin 1 + V$ and the function

$$t(\omega) = 1 - \frac{1}{1+\omega}$$

maps the set V into the closed disk

$$1 - \frac{1}{1+V} = \left\{ w \in \mathbb{C} : \ |w + \frac{\rho^2}{1-\rho^2}| \le \frac{\rho}{1-\rho^2} \right\}.$$

Thus,

$$|p_k^{(n)}| \le \frac{\rho}{1-\rho}, \quad 1 \le k \le n.$$

Finally, we set $\eta = \rho/(1-\rho)$. It follows from $0 < \rho < 1/2$ that $0 < \eta < 1$. Thus, according to (6.53), we have the stability to perturbations of the continued fraction (6.60). □

Theorem 6.11. *The continued fraction (6.60) is stable to perturbations if the relative errors of their elements satisfy the conditions (6.52), the parameter* α *is such that* $\operatorname{Re}\alpha > -1$ *and for* $k \ge 0$

$$|d_{3k+2}| - \operatorname{Re} d_{3k+2} \le r q_{3k+1}(1 - q_{3k+2}), |d_{3k+3}| - \operatorname{Re} d_{3k+3} \le r q_{3k+2}(1 - q_{3k+3}),$$

where d_{3k+2} *and* d_{3k+3} *are defined by (6.61),* r *is a positive constant,* $\{q_k\}$ *is a sequence of real numbers such that* $\delta \le q_k \le 1 - \delta$, $0 < \delta \le 1/2$, *and*

$$8 \le s q_{3k}(1 - q_{3k+1}), \quad k \ge 0, \tag{6.63}$$

where s *is a positive constant, in addition,* $\mathbf{z} \in \Omega_{\alpha,\rho}^{r,s}$, *where*

$$\Omega_{\alpha,\rho}^{r,s} = \left\{ \mathbf{z} \in \mathbb{C}^2 : |z_1| \le \frac{1+\cos(\arg(z_1))}{s}, |z_2| \le \frac{1+\cos(\arg(z_2))}{r}, \arg(z_1) = \arg(z_1) \right\}, \tag{6.64}$$

and for $\eta = \max\left\{ \frac{8}{s\delta^2}, \frac{2}{r|1+\alpha|\delta^2} \right\}$ *inequality (6.53) holds.*

Proof. It follows from (6.64) and (6.63) that for $k \ge 0$

$$|d_{3k+1}| - \operatorname{Re} d_{3k+1} \le s q_{3k}(1 - q_{3k+1}),$$

where d_{3k+1}, $k \ge 0$, are defined by (6.61).

We set $z_1 = |z_1|e^{i\varphi}$, $z_2 = |z_2|e^{i\varphi}$, and choose $q_k = \frac{\xi_k}{\cos(\varphi/2)}$, $k \geq 0$. Then for $k \geq 0$

$$|d_{3k+1}z_1| - \mathrm{Re}(d_{3k+1}z_1e^{-i\varphi}) \leq \frac{s|z_1|\xi_{3k}(\cos\frac{\varphi}{2} - \xi_{3k+1})}{\cos^2\frac{\varphi}{2}} \leq 2\xi_{3k}(\cos\frac{\varphi}{2} - \xi_{3k+1}).$$

Similarly,

$$\begin{aligned} |d_{3k+2}z_2| - \mathrm{Re}(d_{3k+2}z_2e^{-i\varphi}) &\leq \xi_{3k+1}(\cos\psi - \xi_{3k+2}), \quad k \geq 0, \\ |d_{3k+3}z_2| - \mathrm{Re}(d_{3k+3}z_2e^{-i\varphi}) &\leq \xi_{3k+2}(\cos\psi - \xi_{3k+3}), \quad k \geq 0. \end{aligned}$$

Thus,

$$V_k = \{\omega \in \mathbb{C} : \ \mathrm{Re}(\omega e^{-i\varphi/2}) \geq -\xi_k\}, \quad k \geq 0,$$

is the sequence of value sets corresponding the the sequence of element sets

$$E_k = \{\omega \in \mathbb{C} : \ |\omega| - \mathrm{Re}(\omega e^{-i\varphi}) \leq 2\xi_{k-1}(\cos(\varphi/2) - \xi_k)\}, \quad k \geq 1,$$

of the continued fraction (6.60) (see, [6, Section 3.2]).

Setting as (6.46), (6.47), and (6.62), it follows from

$$0 \in V_k \quad \text{and} \quad \frac{d_{k+1}}{1 + V_{k+1}} \subseteq V_k, \quad k \geq 0,$$

that

$$\frac{a_{k+1}}{P_{k+1}^{(n)}} \in V_k \quad \text{and} \quad P_k^{(n)} \in 1 + V_k, \quad k \geq 0.$$

Next, since

$$0 \notin 1 + V_k = \{\omega \in \mathbb{C} : \ \mathrm{Re}((\omega - 1)e^{-i\varphi/2}) \geq -\xi_{k-1}\},$$

then

$$\min_{\omega \in 1+V_k} |\omega| = \cos(\varphi/2) - \xi_k = \cos(\varphi/2)(1 - q_k).$$

Thus,

$$|p_{3k+1}^{(n)}| = \frac{|d_{3k+1}z_1|}{|P_{3k}^{(n)}||P_{3k+1}^{(n)}|} \leq \frac{4(1+\cos\varphi)}{s\cos^2(\varphi/2)(1-q_{3k})(1-q_{3k+1})} = \frac{8}{s\delta^2},$$

$$|p_{3k+2}^{(n)}| = \frac{|d_{3k+2}z_2|}{|P_{3k+1}^{(n)}||P_{3k+2}^{(n)}|} \leq \frac{1+\cos\varphi}{r|\alpha+2k+1|\cos^2(\varphi/2)(1-q_{3k})(1-q_{3k+1})} \leq \frac{2/r}{|\alpha+1|\delta^2},$$

and similarly

$$|p_{3k+3}^{(n)}| \leq \frac{2}{r|\alpha+1|\delta^2}.$$

Let

$$\eta = \max\left\{\frac{8}{s\delta^2}, \frac{2}{r|1+\alpha|\delta^2}\right\}.$$

Then, according to the Theorem 6.9, the continued fraction (6.60) is stable to perturbations. □

Note that results similar to Theorems 6.10 and 6.11 can be obtained in the same way for expansions

$$\frac{\mathrm{H}_7(\alpha;\alpha/2,\alpha;\mathbf{z})}{\mathrm{H}_7(\alpha+1;\alpha/2,\alpha+1;\mathbf{z})} = 1 + \cfrac{c_1 z_2}{1+\cfrac{c_2 z_2}{1+\cfrac{c_3 z_1}{1+\cfrac{c_4 z_2}{1+\cfrac{c_5 z_2}{1+\cfrac{c_6 z_1}{1+\ddots}}}}}},$$

where $c_{3k+1} = -1/(\alpha+2k)$, $c_{3k+2} = 1/(\alpha+2k)$, $c_{3k+3} = -4$, $k \geq 0$, and

$$\frac{\mathrm{H}_7(\alpha;(\alpha+1)/2,\alpha-1;\mathbf{z})}{\mathrm{H}_7(\alpha;(\alpha+1)/2,\alpha;\mathbf{z})} = 1 + \cfrac{b_1 z_2}{1+\cfrac{b_2 z_1}{1+\cfrac{b_3 z_2}{1+\cfrac{b_4 z_2}{1+\cfrac{b_5 z_1}{1+\cfrac{b_6 z_2}{1+\ddots}}}}}},$$

where $b_{3k+1} = 1/(\alpha+2k-1)$, $b_{3k+2} = -4$, $b_{3k+3} = -1/(\alpha+2k+1)$, $k \geq 0$.

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Victoria KRAVTSIV, Taras VASYLYSHYN, and Andriy ZAGORODNYUK

7 Various generalisations of symmetric functions on Banach spaces

One of the most natural generalisations of symmetric functions to the case of infinitely many variables is the notion of a symmetric function on a sequence Banach space with a symmetric Schauder basis, such as spaces of absolutely summable in a power $p \in [1, +\infty)$ sequences. A function on such a space is called symmetric if it is invariant under the action on its argument of every linear continuous operator that permutes coordinates of the argument. Such functions were first considered in [25]. The paper [25] also introduced a generalisation of the notion of symmetry to functions on Lebesgue-integrable function spaces over an interval. Namely, a function on such a space is called symmetric if it is invariant under the action on its argument of every linear operator that composes the argument with a measurable automorphism of the interval. Symmetric functions in the two aforementioned senses were used in [25], in particular, to study the possibility of approximating continuous functions on a Hilbert space by smooth ones, especially by polynomials. Results of [25] were generalised in [9], where symmetric continuous polynomials on real separable rearrangement-invariant function spaces were used to analyse the existence of separating polynomials. The study of the spectrum of the Fréchet algebra of entire symmetric functions of bounded type on the complex Banach space ℓ_1 of all absolutely summable sequences was initiated in [1], further developed in [4–7], and ultimately completed with a full description in [8]. The non-separable case was first considered in [10], where the spectrum of the Fréchet algebra of entire symmetric functions of bounded type on the complex Banach space of Lebesgue measurable essentially bounded functions on $[0, 1]$ was described. In [12] this algebra was represented as the algebra of analytic functions on its spectrum. Symmetric functions on the non-separable sequence space ℓ_∞ were studied in [11]. The algebras of symmetric polynomials and symmetric analytic functions of bounded type on Cartesian products of Banach spaces $\ell_p, p \in [1, +\infty)$ were studied in [20–23,40]. The algebras of symmetric continuous polynomials and symmetric analytic func-

Victoria KRAVTSIV, Vasyl Stefanyk Carpathian National University, 57 Shevchenka Str., 76018 Ivano-Frankivsk, Ukraine, e-mail: viktoriia.kravtsiv@pnu.edu.ua

Taras VASYLYSHYN, Vasyl Stefanyk Carpathian National University, 57 Shevchenka Str., 76018 Ivano-Frankivsk, Ukraine, e-mail: taras.vasylyshyn@pnu.edu.ua

Andriy ZAGORODNYUK, Vasyl Stefanyk Carpathian National University, 57 Shevchenka Str., 76018 Ivano-Frankivsk, Ukraine, e-mail: andriy.zagorodnyuk@pnu.edu.ua

tions on Cartesian products of Banach spaces of Lebesgue measurable functions were studied in [3, 26, 27, 29–38].

Let X be a Banach space, and let $\mathcal{P}(X)$ be the algebra of all continuous polynomials defined on X. Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$. A sequence $(Q_i)_i$ of polynomials is called an algebraic basis of $\mathcal{P}_0(X)$ if for every $P \in \mathcal{P}_0(X)$ there is a unique polynomial $q \in \mathcal{P}(\mathbb{C}^n)$ for some n such that $P(x) = q(Q_1(x), \ldots, Q_n(x))$. In other words, if Q is mapping $x \in X \rightsquigarrow (Q_1(x), \ldots, Q_n(x)) \in \mathbb{C}^n$, then $P = q \circ Q$ and this representation is unique. Typical examples of such kind of algebras are algebras of polynomials which are invariant with respect to a (semi)group $\mathcal{S}$ of operators on X. If X has an unconditional basis (e_n), we can consider the group $\mathcal{S} = S_{\mathbb{N}}$ of all permutations of natural numbers $\mathbb{N}$ acting on X by

$$\sigma\colon x = \sum_{n=1}^{\infty} x_n e_n \rightsquigarrow \sum_{n=1}^{\infty} x_{\sigma(n)} e_n.$$

$S_{\mathbb{N}}$-invariant polynomials on X are called *symmetric*.

Let $\mathcal{P}_s(\ell_p)$ be the algebra of all symmetric polynomials on ℓ_p. In [25] Nemirovskii and Semenov described algebraic bases of algebra of continuous symmetric polynomials on real spaces ℓ_p, where $1 \leq p < \infty$. Their results were generalized by Gonzalez et al. [9] for real separable rearrangement-invariant sequence spaces. Also in [9] was proved that polynomials

$$F_k(x) = \sum_{i=1}^{\infty} x_i^k,$$

$k \geq \lceil p \rceil$ form an algebraic basis in $\mathcal{P}_s(\ell_p)$, where $x = (x_1, x_2, \ldots) \in \ell_p$, $\lceil p \rceil$ is the smallest integer, greater than p. Polynomials $F_k(x)$ are called *power* symmetric polynomials. In the case $p = 1$ we can consider another natural algebraic bases in $\mathcal{P}_s(\ell_1)$: the *elementary* symmetric polynomials basis, given by

$$G_k(x) = \sum_{i_1 < i_2 < \ldots < i_k}^{\infty} x_{i_1} x_{i_2} \ldots x_{i_k}, \tag{7.1}$$

and the *complete* symmetric polynomial basis, defined as

$$H_k(x) = \sum_{i_1 \leq \ldots \leq i_k} x_{i_1} \cdots x_{i_k}.$$

These bases are connected through the well-known Newton identities [24]:

$$nG_n = \sum_{k=1}^{n} (-1)^{k-1} G_{n-k} F_k, \quad n \in \mathbb{N}, \tag{7.2}$$

$$nH_n = \sum_{k=1}^{n} H_{n-k}F_k, \quad n \in \mathbb{N}. \tag{7.3}$$

Let $\mathbb{Z}_+$ denote the set of all nonnegative integers, $\mathbb{Z}_+^n$ — the set of all vectors $\lambda = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_i \in \mathbb{Z}_+$. We will use standard notations $|\lambda|_1 = \lambda_1 + 2\lambda_2 + \ldots + n\lambda_n$, $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_n$, $\lambda! = \lambda_1! \cdot \ldots \cdot \lambda_n!$, $F^\lambda = F_1^{\lambda_1} \cdot \ldots \cdot F_n^{\lambda_n}$. Let we denote by $1^{\lambda_1} \cdot 2^{\lambda_2} \cdot \ldots \cdot n^{\lambda_n} = z_n^\lambda$ for $z_n = (1, 2, \ldots, n)$. It is well known that elementary and complete symmetric polynomials can be expressed in terms of power symmetric polynomials using the Waring-Girard formulas [24] for $\lambda \in \mathbb{Z}_+^n$:

$$G_n = \sum_{|\lambda|_1 = n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda \lambda!} F^\lambda, \tag{7.4}$$

$$H_n = \sum_{|\lambda|_1 = n} \frac{1}{z_n^\lambda \lambda!} F^\lambda.$$

Let we denote by $z_{p,n} = (p, \ldots, n)$, $\lambda_{p,n} = (\lambda_p, \ldots, \lambda_n)$, $|\lambda_{p,n}|_1 = p\lambda_p + \ldots + n\lambda_n$, $|\lambda_{p,n}| = \lambda_p + \ldots + \lambda_n$, $z_{p,n}^{\lambda_{p,n}} = p^{\lambda_p} \cdot \ldots \cdot n^{\lambda_n}$, $\lambda_{p,n}! = \lambda_p! \cdot \ldots \cdot \lambda_n!$, $F_{p,n} = (F_p, \ldots, F_n)$ and $F_{p,n}^{\lambda_{p,n}} = (F_p)^{\lambda_p} \ldots (F_n)^{\lambda_n}$.

In the case $p > 1$ we have no elementary symmetric polynomials, because the series (7.1) does not converge for any k. But putting in the Newton formulas $F_k = 0$ for $k < p$, we can define elementary symmetric polynomials on ℓ_p by

$$G_n^{(p)} = \sum_{k=\lceil p \rceil}^{n-\lceil p \rceil} (-1)^{k-1} F_k G_{n-k}.$$

It is easy to check that the sequence $\{G_n^{(p)}\}_{n>p}$ forms an algebraic basis in $\mathcal{P}_s(\ell_p)$. In [13], the Waring-Girard formulas were extended to the Banach space $\ell_p, p > 1$:

$$G_n^{(p)} = \sum_{|\lambda_{p,n}|_1 = n} \frac{(-1)^{n+|\lambda_{p,n}|}}{z_{p,n}^{\lambda_{p,n}} \lambda_{p,n}!} F_{p,n}^{\lambda_{p,n}},$$

$$H_n^{(p)} = \sum_{|\lambda_{p,n}|_1 = n} \frac{-1}{z_{p,n}^{\lambda_{p,n}} \lambda_{p,n}!} F_{p,n}^{\lambda_{p,n}},$$

where $G_n^{(p)}$ and $H_n^{(p)}$ are elementary and complete symmetric polynomials on ℓ_p.

7.1 Algebraic basis of algebra of block-symmetric polynomials on $\ell_p(\mathbb{C}^n)$.

Let $n \in \mathbb{N}$ and $p \in [1, +\infty)$. Let us denote $\ell_p(\mathbb{C}^n)$ the vector space of all sequences

$$x = (x_1, x_2, \ldots),$$

where $x_j = (x_j^{(1)}, \ldots, x_j^{(n)}) \in \mathbb{C}^n$ for $j \in \mathbb{C}$, such that the series $\sum_{j=1}^{\infty} \sum_{s=1}^{n} |x_j^{(s)}|^p$ is convergent. The space $\ell_p(\mathbb{C}^n)$ with norm

$$\|x\|_p = \left(\sum_{j=1}^{\infty} \sum_{s=1}^{n} |x_j^{(s)}|^p \right)^{1/p}$$

is a Banach space.

A polynomial P on the space $\ell_p(\mathbb{C}^n)$ is called *block-symmetric (or vector-symmetric)* if:

$$P(x_1, x_2, \ldots, x_m, \ldots) = P(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(m)}, \ldots)$$

for every permutation $\sigma \in S_{\mathbb{N}}$ and $x_m \in \mathbb{C}^n$. We denote by $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^n))$ the algebra of all block-symmetric polynomials on $\ell_p(\mathbb{C}^n)$.

The block-symmetric or the McMahon polynomials are natural generalizations of symmetric polynomials and can be considered symmetric polynomials on linear spaces of vector sequences. Combinatorial properties of such polynomials are described in [24, 28, 41]. In this section we construct an algebraic basis of the algebra of all symmetric continuous complex-valued polynomials on the Cartesian power of the complex Banach space ℓ_p for some fixed $1 \leq p < \infty$ (see [18]). In the finite case, we have algebraically dependent polynomials, and in [19] it is shown how to find these algebraic dependencies. For more details on the real case, see [39]. Algebras of continuous symmetric polynomials on Cartesian products $\ell_{p_1} \times \ldots \ell_{p_n}$ for different $p_1, \ldots p_n$ were considered in [2]. Block-symmetric polynomials on $\ell_1 \otimes \ell_\infty$ were considered in [15].

Power block-symmetric polynomials on $\ell_p(\mathbb{C}^n)$. For a multi-index $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ let $|\mathbf{k}| = k_1 + \ldots + k_n$. For every $\mathbf{k} \in \mathbb{Z}_+^n$ such that $|\mathbf{k}| \geq \lceil p \rceil$, where $\lceil p \rceil$ is a ceiling of p, let us define a mapping $H^{\mathbf{k}} : \ell_p(\mathbb{C}^n) \to \mathbb{C}$ by

$$H^{\mathbf{k}}(x) = \sum_{j=1}^{\infty} \prod_{\substack{s=1 \\ k_s > 0}}^{n} (x_j^{(s)})^{k_s}. \tag{7.5}$$

Also we set $H^{(0,\ldots,0)}(x) \equiv 1$. Note that $H^{\mathbf{k}}$ is a block-symmetric polynomial. Polynomials $H^{\mathbf{k}}$ are generalizations of so-called *power symmetric polynomials* F_k.

Proposition 7.1. *For $p \in [1, +\infty)$ and for every $\mathbf{k} \in \mathbb{Z}_+^n$ such that $|\mathbf{k}| \geq \lceil p \rceil$, polynomial $H^{\mathbf{k}}$ on $\ell_p(\mathbb{C}^n)$ is continuous and $\|H^{\mathbf{k}}\| \leq 1$.*

Proof. Let $x \in \ell_p(\mathbb{C}^n)$ such that $\|x\|_p \leq 1$. Note that

$$|H^{\mathbf{k}}(x)| \leq \sum_{j=1}^{\infty} \prod_{\substack{s=1 \\ k_s > 0}}^{n} |x_j^{(s)}|^{k_s}.$$

Since $|x_j^{(s)}| \leq \max\limits_{1\leq m\leq n} |x_j^{(m)}|$ for every $s \in \{1, \ldots, n\}$ and $j \in \mathbb{N}$, it follows that

$$\prod_{\substack{s=1\\k_s>0}}^{n} |x_j^{(s)}|^{k_s} \leq \left(\max_{1\leq m\leq n} |x_j^{(m)}|\right)^{|\mathbf{k}|}$$

for every $j \in \mathbb{N}$. Note that

$$\left(\max_{1\leq m\leq n} |x_j^{(m)}|\right)^{|\mathbf{k}|} = \max_{1\leq m\leq n} |x_j^{(m)}|^{|\mathbf{k}|} \leq \sum_{m=1}^{n} |x_j^{(m)}|^{|\mathbf{k}|}.$$

Therefore,

$$|H^{\mathbf{k}}(x)| \leq \sum_{j=1}^{\infty}\sum_{m=1}^{n} |x_j^{(m)}|^{|\mathbf{k}|}.$$

Since $\|x\|_p \leq 1$, it follows that $|x_j^{(m)}| \leq 1$ for every $m \in \{1, \ldots, n\}$ and $j \in \mathbb{N}$. Therefore, $|x_j^{(m)}|^{|\mathbf{k}|} \leq |x_j^{(m)}|^p$. Thus,

$$|H^{\mathbf{k}}(x)| \leq \sum_{j=1}^{\infty}\sum_{m=1}^{n} |x_j^{(m)}|^p = \|x\|_p^p \leq 1.$$

Therefore,

$$\|H^{\mathbf{k}}\| = \sup_{\|x\|_p\leq 1} |H_k(x)| \leq 1.$$

Hence, $H^{\mathbf{k}}$ is bounded and, consequently, it is continuous. □

For $m \in \mathbb{N}$ let $c_{00}^{(m)}(\mathbb{C}^n)$ be the space of all sequences $x = (x_1, \ldots, x_m, 0, \ldots)$, where $x_1, \ldots, x_m \in \mathbb{C}^n$ and $0 = (0, \ldots, 0) \in \mathbb{C}^n$. Note that $c_{00}^{(m)}(\mathbb{C}^n)$ is isomorphic to $(\mathbb{C}^n)^m$. Let $c_{00}(\mathbb{C}^n) = \bigcup_{m=1}^{\infty} c_{00}^{(m)}(\mathbb{C}^n)$. Note that $c_{00}(\mathbb{C}^n)$ is a dense subspace in $\ell_p(\mathbb{C}^n)$. Also note that $H^{\mathbf{k}}$ is well-defined on $c_{00}(\mathbb{C}^n)$ for every $\mathbf{k} \in \mathbb{Z}_+^n$.

For arbitrary $x = (x_1, \ldots, x_m, 0, \ldots), y = (y_1, \ldots, y_s, 0, \ldots) \in c_{00}(\mathbb{C}^n)$ we set

$$x \oplus y = (x_1, \ldots, x_m, y_1, \ldots, y_s, 0, \ldots).$$

For $x^{(1)}, \ldots, x^{(r)} \in c_{00}(\mathbb{C}^n)$ let

$$\bigoplus_{j=1}^{r} x^{(j)} = x^{(1)} \oplus \ldots \oplus x^{(r)}.$$

Note that

$$\left\|\bigoplus_{j=1}^{r} x^{(j)}\right\|_p^p = \sum_{j=1}^{r} \|x^{(j)}\|_p^p. \tag{7.6}$$

Also note that for every $\mathbf{k} \in \mathbb{Z}_+^n$, such that $|\mathbf{k}| \geq 1$,

$$H^{\mathbf{k}}\left(\bigoplus_{j=1}^{r} x^{(j)}\right) = \sum_{j=1}^{r} H^{\mathbf{k}}(x^{(j)}). \tag{7.7}$$

For every $m \in \mathbb{N}$ and $j \in \{1, \ldots, m\}$ we set

$$\alpha_{mj} = \frac{1}{m^{1/m}} \exp(2\pi i j/m). \tag{7.8}$$

Also we set $\alpha_{01} = 0$. For $\mathbf{l} = (l_1, \ldots, l_n) \in \mathbb{Z}_+^n$ let

$$a_{\mathbf{l}} = \bigoplus_{j_1=1}^{\widehat{l_1}} \cdots \bigoplus_{j_n=1}^{\widehat{l_n}} \left((\alpha_{l_1 j_1}, \ldots, \alpha_{l_n j_n}), (0, \ldots, 0), \ldots\right), \tag{7.9}$$

where $\widehat{l_j} = \max\{1, l_j\}$ for $j \in \{1, \ldots, n\}$.

Let us define a partial order on $\mathbb{Z}_+^n$ by the following way. For $\mathbf{k}, \mathbf{l} \in \mathbb{Z}_+^n$ we set $\mathbf{k} \succeq \mathbf{l}$ if and only if there exists $\mathbf{m} \in \mathbb{Z}_+^n$ such that $k_s = m_s l_s$ for every $s \in \{1, \ldots, n\}$. We write $\mathbf{k} \succ \mathbf{l}$, if $\mathbf{k} \succeq \mathbf{l}$ and $\mathbf{k} \neq \mathbf{l}$.

Proposition 7.2. *For $\mathbf{k} \in \mathbb{Z}_+^n$ such that $|\mathbf{k}| \geq 1$ and for arbitrary $\mathbf{l} \in \mathbb{Z}_+^n$*

$$H^{\mathbf{k}}(a_{\mathbf{l}}) = \begin{cases} \prod\limits_{\substack{s=1 \\ k_s>0}}^{n} \dfrac{1}{l_s^{k_s/l_s - 1}} \prod\limits_{\substack{s=1 \\ k_s=0}}^{n} \widehat{l_s}, & \textit{if} \quad \mathbf{k} \succeq \mathbf{l}, \\ 0, & \textit{otherwise.} \end{cases}$$

where, by the definition, the product of an empty set of multipliers is equal to 1. *In particular,* $H^{\mathbf{k}}(a_{\mathbf{k}}) = 1$.

Proof. By (7.7) and (7.9),

$$H^{\mathbf{k}}(a_{\mathbf{l}}) = \sum_{j_1=1}^{\widehat{l_1}} \cdots \sum_{j_n=1}^{\widehat{l_n}} H^{\mathbf{k}}((\alpha_{l_1 j_1}, \ldots, \alpha_{l_n j_n}), (0, \ldots, 0), \ldots).$$

By the definition of $H^{\mathbf{k}}$,

$$H^{\mathbf{k}}((\alpha_{l_1 j_1}, \ldots, \alpha_{l_n j_n}), (0, \ldots, 0), \ldots) = \prod_{\substack{s=1 \\ k_s>0}}^{n} (\alpha_{l_s j_s})^{k_s}.$$

Therefore,

$$H^{\mathbf{k}}(a_{\mathbf{l}}) = \sum_{j_1=1}^{\widehat{l_1}} \cdots \sum_{j_n=1}^{\widehat{l_n}} \prod_{\substack{s=1 \\ k_s>0}}^{n} (\alpha_{l_s j_s})^{k_s} =$$

$$= \prod_{\substack{s=1\\k_s>0}}^{n} \sum_{j_s=1}^{\widehat{l_s}} (\alpha_{l_s j_s})^{k_s} \prod_{\substack{s=1\\k_s=0}}^{n} \sum_{j_s=1}^{\widehat{l_s}} 1 = \prod_{\substack{s=1\\k_s>0}}^{n} \sum_{j_s=1}^{\widehat{l_s}} (\alpha_{l_s j_s})^{k_s} \prod_{\substack{s=1\\k_s=0}}^{n} \widehat{l_s}. \tag{7.10}$$

Let $\mathbf{k} \succeq \mathbf{l}$. Then there exists $\mathbf{m} \in \mathbb{Z}_+^n$ such that $k_s = m_s l_s$ for every $s \in \{1,\dots,n\}$. For $s \in \{1,\dots,n\}$ such that $k_s > 0$ we have that $l_s > 0$ too. Consequently, for such s we have $\widehat{l_s} = l_s$, and, by (7.8),

$$\sum_{j_s=1}^{\widehat{l_s}} (\alpha_{l_s j_s})^{k_s} = \sum_{j_s=1}^{l_s} \left(\frac{1}{l_s^{1/l_s}} \exp(2\pi i j_s / l_s) \right)^{m_s l_s} =$$

$$= \frac{1}{l_s^{m_s}} \sum_{j_s=1}^{l_s} \exp(2\pi i j_s m_s) = \frac{1}{l_s^{m_s}} \sum_{j_s=1}^{l_s} 1 = \frac{1}{l_s^{m_s-1}} = \frac{1}{l_s^{k_s/l_s-1}}.$$

Therefore, by (7.10),

$$H^{\mathbf{k}}(a_{\mathbf{l}}) = \prod_{\substack{s=1\\k_s>0}}^{n} \frac{1}{l_s^{k_s/l_s-1}} \prod_{\substack{s=1\\k_s=0}}^{n} \widehat{l_s}.$$

In the case $\mathbf{k} = \mathbf{l}$ we have

$$H^{\mathbf{k}}(a_{\mathbf{k}}) = \prod_{\substack{s=1\\k_s>0}}^{n} \frac{1}{k_s^{k_s/k_s-1}} \prod_{\substack{s=1\\k_s=0}}^{n} \widehat{k_s} = 1.$$

Let $\mathbf{k} \not\succeq \mathbf{l}$. Then we have two cases.

Case 1: there exists $s \in \{1,\dots,n\}$ such that $k_s > l_s = 0$. Then

$$\sum_{j_s=1}^{\widehat{l_s}} (\alpha_{l_s j_s})^{k_s} = (\alpha_{01})^{k_s} = 0,$$

therefore, $H^{\mathbf{k}}(a_{\mathbf{l}}) = 0$.

Case 2: there exists $s \in \{1,\dots,n\}$ such that $l_s > k_s > 0$. Then

$$\sum_{j_s=1}^{\widehat{l_s}} (\alpha_{l_s j_s})^{k_s} = \sum_{j_s=1}^{l_s} \left(\frac{1}{l_s^{1/l_s}} \exp(2\pi i j_s / l_s) \right)^{k_s} = \frac{1}{l_s^{k_s/l_s}} \sum_{j_s=1}^{l_s} \exp(2\pi i j_s / l_s)^{k_s}.$$

It is known that

$$\sum_{j=1}^{q} \exp(2\pi i j / q)^r = 0$$

for every $q \in \{2, 3, \dots\}$ and $r \in \{1, \dots, q-1\}$. Therefore,

$$\sum_{j_s=1}^{l_s} \exp(2\pi i j_s/l_s)^{k_s} = 0$$

and, consequently, $H^{\mathbf{k}}(a_{\mathbf{l}}) = 0$. □

Let us prove the following auxiliary proposition.

Proposition 7.3. *A function* $g : (0, +\infty) \to \mathbb{R}$, $g(x) = (c_1^x + \dots + c_m^x)^{1/x}$, *where* $m \in \mathbb{N}$ *and* $c_1, \dots, c_m > 0$, *is strictly decreasing.*

Proof. Let us prove that $g'(x) < 0$ for every $x \in (0, +\infty)$. Note that

$$g(x) = \exp\left(\frac{1}{x}\ln(c_1^x + \dots + c_m^x)\right).$$

Therefore,

$$g'(x) = g(x)\left(-\frac{1}{x^2}\ln(c_1^x + \dots + c_m^x) + \frac{1}{x}\frac{c_1^x \ln c_1 + \dots + c_m^x \ln c_m}{c_1^x + \dots + c_m^x}\right) =$$
$$= -\frac{g(x)((c_1^x + \dots + c_m^x)\ln(c_1^x + \dots + c_m^x) - x(c_1^x \ln c_1 + \dots + c_m^x \ln c_m))}{x^2(c_1^x + \dots + c_m^x)} =$$
$$= -\frac{g(x)(c_1^x(\ln(c_1^x + \dots + c_m^x) - \ln c_1^x) + \dots + c_m^x(\ln(c_1^x + \dots + c_m^x) - \ln c_m^x))}{x^2(c_1^x + \dots + c_m^x)}.$$

Since

$$\frac{g(x)}{x^2(c_1^x + \dots + c_m^x)} > 0$$

and $\ln(c_1^x + \dots + c_m^x) > \ln c_j^x$ for every $j \in \{1, \dots, m\}$, it follows that $g'(x) < 0$. □

Corollary 7.1. *For every* $x \in \ell_p(\mathbb{C}^n)$ *and for every* $q \geq p$

$$\|x\|_p \geq \|x\|_q.$$

For an arbitrary non-empty finite set $M \subset \mathbb{Z}_+^n$ let us define a mapping $\pi_M : c_{00}(\mathbb{C}^n) \to \mathbb{C}^{|M|}$, where $|M|$ is the cardinality of M, by

$$\pi_M(x) = (H^{\mathbf{k}}(x))_{\mathbf{k}\in M}, \tag{7.11}$$

where $(H^{\mathbf{k}}(x))_{\mathbf{k}\in M}$ is an $|M|$-dimensional vector of values of $H^{\mathbf{k}}$ on x, indexed by $\mathbf{k} \in M$. The space $\mathbb{C}^{|M|}$ we endow with norm $\|\xi\|_\infty = \max_{\mathbf{k}\in M} |\xi_{\mathbf{k}}|$, where $\xi = (\xi_{\mathbf{k}})_{\mathbf{k}\in M} \in \mathbb{C}^{|M|}$.

Theorem 7.1. *Let M be a finite non-empty subset of $\mathbb{Z}_+^n$ such that $|\mathbf{k}| \geq 1$ for every $\mathbf{k} \in M$. Then*

(i) there exists $m \in \mathbb{N}$ such that for every $\xi = (\xi_{\mathbf{k}})_{\mathbf{k}\in M} \in \mathbb{C}^{|M|}$ there exists $x_\xi \in c_{00}^{(m)}(\mathbb{C}^n)$ such that $\pi_M(x_\xi) = \xi$;

(ii) there exists a constant $\rho_M > 0$ such that if $\|\xi\|_\infty < 1$, then $\|x_\xi\|_p < \rho_M$ for every $p \in [1, +\infty)$.

Proof. *(i)* Let $\xi = (\xi_{\mathbf{k}})_{\mathbf{k}\in M} \in \mathbb{C}^{|M|}$. For every $\mathbf{k} \in M$ let us define $\eta_{\mathbf{k}} \in \mathbb{C}$ and $b_{\mathbf{k}} \in c_{00}(\mathbb{C}^n)$ by the following way. For minimal elements $\mathbf{k}$ of the partially ordered set $(M, \preceq)$ let $\eta_{\mathbf{k}} = \xi_{\mathbf{k}}$ and $b_{\mathbf{k}} = \sqrt[|\mathbf{k}|]{\eta_{\mathbf{k}}}\, a_{\mathbf{k}}$, where $a_{\mathbf{k}}$ is defined by (7.9) and

$$\sqrt[|\mathbf{k}|]{\eta_{\mathbf{k}}} = \begin{cases} \sqrt[|\mathbf{k}|]{|\eta_{\mathbf{k}}|}\, e^{i \arg \eta_{\mathbf{k}}/|\mathbf{k}|}, & \text{if} \quad \eta_{\mathbf{k}} \neq 0, \\ 0, & \text{if} \quad \eta_{\mathbf{k}} = 0. \end{cases}$$

For $\mathbf{k} \in M$, which are not minimal elements of $(M, \preceq)$, we define $\eta_{\mathbf{k}}$ and $b_{\mathbf{k}}$ inductively by

$$\eta_{\mathbf{k}} = \xi_{\mathbf{k}} - \sum_{\substack{\mathbf{l}\in M \\ \mathbf{l}\prec\mathbf{k}}} H^{\mathbf{k}}(b_{\mathbf{l}}) \tag{7.12}$$

and

$$b_{\mathbf{k}} = \sqrt[|\mathbf{k}|]{\eta_{\mathbf{k}}}\, a_{\mathbf{k}}. \tag{7.13}$$

We set

$$x_\xi = \bigoplus_{\mathbf{l}\in M} b_{\mathbf{l}}.$$

Note that $x_\xi \in c_{00}^{(m)}(\mathbb{C}^n)$, where

$$m = \sum_{\mathbf{k}\in M} \min\{j \in \mathbb{N} : \; a_{\mathbf{k}} \in c_{00}^{(j)}(\mathbb{C}^n)\}.$$

For $\mathbf{k} \in M$, by (7.7), $H^{\mathbf{k}}(x_\xi) = \sum_{\mathbf{l}\in M} H^{\mathbf{k}}(b_{\mathbf{l}})$. Since $H^{\mathbf{k}}$ is a $|\mathbf{k}|$-homogeneous polynomial,

$$H^{\mathbf{k}}(b_{\mathbf{l}}) = \left(\sqrt[|\mathbf{l}|]{\eta_{\mathbf{l}}}\right)^{|\mathbf{k}|} H^{\mathbf{k}}(a_{\mathbf{l}}). \tag{7.14}$$

By Proposition 7.2, $H^{\mathbf{k}}(a_{\mathbf{l}})$ is not equal to zero only for $\mathbf{l} \in M$ such that $\mathbf{l} \preceq \mathbf{k}$. Therefore,

$$H^{\mathbf{k}}(x_\xi) = H^{\mathbf{k}}(b_{\mathbf{k}}) + \sum_{\substack{\mathbf{l}\in M \\ \mathbf{l}\prec\mathbf{k}}} H^{\mathbf{k}}(b_{\mathbf{l}}).$$

By Proposition 7.2, $H^{\mathbf{k}}(a_{\mathbf{k}}) = 1$, therefore, by (7.14), $H^{\mathbf{k}}(b_{\mathbf{k}}) = \eta_{\mathbf{k}}$. Hence,

$$H^{\mathbf{k}}(x_\xi) = \eta_{\mathbf{k}} + \sum_{\substack{\mathbf{l}\in M \\ \mathbf{l}\prec\mathbf{k}}} H^{\mathbf{k}}(b_{\mathbf{l}}).$$

Taking into account (7.12), we have $H^{\mathbf{k}}(x_\xi) = \xi_{\mathbf{k}}$. Hence, $\pi_M(x_\xi) = \xi$.

(ii) Let $\xi = (\xi_{\mathbf{k}})_{\mathbf{k}\in M} \in \mathbb{C}^{|M|}$ be such that $\|\xi\|_\infty < 1$. For $\mathbf{k} \in M$ let

$$\langle \mathbf{k} \rangle = \max\{s \in \mathbb{N} : \exists l^{(1)}, \ldots, l^{(s)} \in M \text{ such that } l^{(1)} \prec \ldots \prec l^{(s)} = \mathbf{k}\}$$

Note that for minimal elements $\mathbf{k} \in M$ we have $\langle \mathbf{k} \rangle = 1$.

Let

$$C = \max\{1, \max_{\mathbf{k}\in M} \|a_{\mathbf{k}}\|_1\}. \tag{7.15}$$

Let $r = \max_{\mathbf{k}\in M}\langle \mathbf{k} \rangle$, and for every $j \in \{1, \ldots, r\}$ let $\mu_j = \prod_{s=1}^{j}(1 + m_s)$, where

$$m_s = |\{\mathbf{k} \in M : \langle \mathbf{k} \rangle = s\}|$$

Also we set $\mu_0 = 1$. Note that for every $j \in \{1, \ldots, r\}$

$$\begin{aligned}\mu_j = \mu_{j-1}(1 + m_j) = \mu_{j-1} + \mu_{j-1}m_j = \mu_{j-2} + \mu_{j-2}m_{j-1} + \mu_{j-1}m_j = \ldots = \\ = \mu_0 + \mu_0 m_1 + \mu_1 m_2 + \ldots + \mu_{j-1}m_j.\end{aligned} \tag{7.16}$$

Let us prove that for every $\mathbf{k} \in M$

$$\|b_{\mathbf{k}}\|_1 < \mu_{\langle \mathbf{k} \rangle - 1} C^{\langle \mathbf{k} \rangle}. \tag{7.17}$$

We proceed by induction on $\langle \mathbf{k} \rangle$. In the case $\langle \mathbf{k} \rangle = 1$ we have $\eta_{\mathbf{k}} = \xi_{\mathbf{k}}$, therefore, $\|b_{\mathbf{k}}\|_1 = \sqrt[|\mathbf{k}|]{|\xi_{\mathbf{k}}|}\|a_{\mathbf{k}}\|_1$. Since $|\xi_{\mathbf{k}}| < 1$, it follows that $\|b_{\mathbf{k}}\|_1 < \|a_{\mathbf{k}}\|_1 \le C = \mu_0 C$. If $r = 1$, then (7.17) is proved. Let $r \ge 2$ and $j \in \{2, \ldots, r\}$. Suppose the inequality (7.17) holds for every $\mathbf{k} \in M$ such that $\langle \mathbf{k} \rangle \in \{1, \ldots, j-1\}$. Let us prove (7.17) for $\mathbf{k} \in M$ such that $\langle \mathbf{k} \rangle = j$. By (7.13) and (7.15),

$$\|b_{\mathbf{k}}\|_1 \le \sqrt[|\mathbf{k}|]{|\eta_{\mathbf{k}}|}\|a_{\mathbf{k}}\|_1 \le \sqrt[|\mathbf{k}|]{|\eta_{\mathbf{k}}|}C. \tag{7.18}$$

By (7.12),

$$|\eta_{\mathbf{k}}| \le |\xi_{\mathbf{k}}| + \sum_{\substack{\mathbf{l}\in M \\ \mathbf{l}\prec\mathbf{k}}} |H^{\mathbf{k}}(b_{\mathbf{l}})|.$$

Since $H^{\mathbf{k}}$ is a $|\mathbf{k}|$-homogeneous polynomial on the space $\ell_1(\mathbb{C}^n)$ and $\|H^{\mathbf{k}}\| \le 1$,

$$|H^{\mathbf{k}}(b_{\mathbf{l}})| \le \|H^{\mathbf{k}}\|\|b_{\mathbf{l}}\|_1^{|\mathbf{k}|} \le \|b_{\mathbf{l}}\|_1^{|\mathbf{k}|}.$$

Therefore, taking into account $|\xi_{\mathbf{k}}| < 1$, we have

$$|\xi_{\mathbf{k}}| + \sum_{\substack{\mathbf{l}\in M \\ \mathbf{l}\prec\mathbf{k}}} |H^{\mathbf{k}}(b_{\mathbf{l}})| < 1 + \sum_{\substack{\mathbf{l}\in M \\ \mathbf{l}\prec\mathbf{k}}} \|b_{\mathbf{l}}\|_1^{|\mathbf{k}|}.$$

Therefore,

$$\sqrt[|\mathbf{k}|]{|\eta_{\mathbf{k}}|} < \left(1 + \sum_{\substack{\mathbf{l}\in M \\ \mathbf{l}\prec\mathbf{k}}} \|b_{\mathbf{l}}\|_1^{|\mathbf{k}|}\right)^{1/|\mathbf{k}|}. \tag{7.19}$$

By Proposition 7.3,

$$\left(1 + \sum_{\substack{\mathbf{l}\in M \\ \mathbf{l}\prec\mathbf{k}}} \|b_{\mathbf{l}}\|_1^{|\mathbf{k}|}\right)^{1/|\mathbf{k}|} \le 1 + \sum_{\substack{\mathbf{l}\in M \\ \mathbf{l}\prec\mathbf{k}}} \|b_{\mathbf{l}}\|_1.$$

Note that if $\mathbf{l} \prec \mathbf{k}$, then $\langle \mathbf{l} \rangle < \langle \mathbf{k} \rangle$. Therefore,

$$\sum_{\substack{\mathbf{l}\in M \\ \mathbf{l}\prec\mathbf{k}}} \|b_{\mathbf{l}}\|_1 \le \sum_{\substack{\mathbf{l}\in M \\ \langle\mathbf{l}\rangle<\langle\mathbf{k}\rangle}} \|b_{\mathbf{l}}\|_1.$$

Since $\langle \mathbf{k} \rangle = j$,

$$\sum_{\substack{\mathbf{l}\in M \\ \langle\mathbf{l}\rangle<\langle\mathbf{k}\rangle}} \|b_{\mathbf{l}}\|_1 = \sum_{s=1}^{j-1} \sum_{\substack{\mathbf{l}\in M \\ \langle\mathbf{l}\rangle=s}} \|b_{\mathbf{l}}\|_1.$$

By the induction hypothesis, if $\langle \mathbf{l} \rangle = s$, where $s \in \{1, \ldots, j-1\}$, then $\|b_{\mathbf{l}}\|_1 < \mu_{s-1} C^s$. Therefore,

$$\sum_{\substack{\mathbf{l}\in M \\ \langle\mathbf{l}\rangle=s}} \|b_{\mathbf{l}}\|_1 < \sum_{\substack{\mathbf{l}\in M \\ \langle\mathbf{l}\rangle=s}} \mu_{s-1} C^s = \mu_{s-1} C^s \sum_{\substack{\mathbf{l}\in M \\ \langle\mathbf{l}\rangle=s}} 1 = \mu_{s-1} m_s C^s.$$

Since $C \ge 1$, it follows that $C^s \le C^{j-1}$ for every $s \in \{1, \ldots, j-1\}$, therefore,

$$1 + \sum_{s=1}^{j-1} \mu_{s-1} m_s C^s \le 1 + C^{j-1} \sum_{s=1}^{j-1} \mu_{s-1} m_s \le \left(1 + \sum_{s=1}^{j-1} \mu_{s-1} m_s\right) C^{j-1}.$$

Since $\mu_0 = 1$, by (7.16),

$$1 + \sum_{s=1}^{j-1} \mu_{s-1} m_s = \mu_{j-1}. \tag{7.20}$$

By (7.19) – (7.20),

$$\sqrt[|\mathbf{k}|]{|\eta_{\mathbf{k}}|} < \mu_{j-1} C^{j-1}. \tag{7.21}$$

By (7.18) and (7.21), $\|b_{\mathbf{k}}\|_1 \le \mu_{j-1} C^j$. Hence, the inequality (7.17) holds for every $\mathbf{k} \in M$.

By (7.6) and by Proposition 7.3,

$$\|x_\xi\|_1 \le \sum_{\mathbf{l}\in M} \|b_{\mathbf{l}}\|_1.$$

By (7.17),

$$\sum_{\mathbf{l}\in M} \|b_{\mathbf{l}}\|_1 = \sum_{j=1}^{r} \sum_{\substack{\mathbf{l}\in M\\ \langle \mathbf{l}\rangle = j}} \|b_{\mathbf{l}}\|_1 < \sum_{j=1}^{r} \sum_{\substack{\mathbf{l}\in M\\ \langle \mathbf{l}\rangle = j}} \mu_{j-1} C^j = \sum_{j=1}^{r} \mu_{j-1} C^j \sum_{\substack{\mathbf{l}\in M\\ \langle \mathbf{l}\rangle = j}} 1 = \sum_{j=1}^{r} \mu_{j-1} m_j C^j$$
$$\le \left(\sum_{j=1}^{r} \mu_{j-1} m_j\right) C^r < \left(\mu_0 + \sum_{j=1}^{r} \mu_{j-1} m_j\right) C^r = \mu_r C^r.$$

Set $\rho_M = \mu_r C^r$. We have that $\|x_\xi\|_1 < \rho_M$ if $\|\xi\|_\infty < 1$. By Corollary 7.1, $\|x_\xi\|_p \le \|x_\xi\|_1 \le \rho_M$ for every $p \in [1, +\infty)$. □

Corollary 7.2. *Let $M = \{\mathbf{k}^{(1)}, \ldots, \mathbf{k}^{(s)}\} \subset \mathbb{Z}_+^n$ such that $|\mathbf{k}^{(j)}| \ge 1$ for every $j \in \{1, \ldots, s\}$. Then there exists $m \in \mathbb{N}$ such that for every $m' \ge m$ polynomials $H^{\mathbf{k}^{(1)}}, \ldots, H^{\mathbf{k}^{(s)}}$ are algebraically independent on $c_{00}^{(m')}(\mathbb{C}^n)$.*

Proof. By Theorem 7.1, there exists $m \in \mathbb{N}$ such that for every $\xi = (\xi_1, \ldots, \xi_s) \in \mathbb{C}^s$ there exists $x_\xi \in c_{00}^{(m)}(\mathbb{C}^n)$ such that

$$H^{\mathbf{k}^{(j)}}(x_\xi) = \xi_j \tag{7.22}$$

for every $j \in \{1, \ldots, s\}$. Let us show that $H^{\mathbf{k}^{(1)}}, \ldots, H^{\mathbf{k}^{(s)}}$ are algebraically independent on $c_{00}^{(m')}(\mathbb{C}^n)$ for every $m' \ge m$. Let $Q : \mathbb{C}^s \to \mathbb{C}$ be a polynomial such that

$$Q(H^{\mathbf{k}^{(1)}}(x), \ldots, H^{\mathbf{k}^{(s)}}(x)) = 0$$

for every $x \in c_{00}^{(m')}(\mathbb{C}^n)$. Set $x = x_\xi$. Taking into account (7.22), we have $Q(\xi_1, \ldots, \xi_s) = 0$ for arbitrary $\xi_1, \ldots, \xi_s \in \mathbb{C}$, i. e. $Q \equiv 0$. Hence, $H^{\mathbf{k}^{(1)}}, \ldots, H^{\mathbf{k}^{(s)}}$ are algebraically independent. □

Algebraic basis of the algebra $\mathcal{P}_s(\ell_1(\mathbb{C}^n))$.

Theorem 7.2. *Every N-homogeneous polynomial $P \in \mathcal{P}_s(c_{00}^{(m)}(\mathbb{C}^n))$, where m is an arbitrary positive integer, can be represented as an algebraic combination of polynomials $H^{\mathbf{k}}$, where $\mathbf{k} \in \mathbb{Z}_+^n$ such that $1 \le |\mathbf{k}| \le N$.*

Proof. We proceed by induction on m. In the case $m = 1$ for $x = (x_1, 0, \ldots) \in c_{00}^{(1)}(\mathbb{C}^n)$ we have

$$P(x) = \sum_{\substack{\mathbf{k}\in\mathbb{Z}_+^n \\ |\mathbf{k}|=N}} \alpha_{\mathbf{k}} (x_1^{(1)})^{k_1} \cdots (x_1^{(n)})^{k_n} = \sum_{\substack{\mathbf{k}\in\mathbb{Z}_+^n \\ |\mathbf{k}|=N}} \alpha_{\mathbf{k}} H^{\mathbf{k}}(x),$$

where $\alpha_{\mathbf{k}} \in \mathbb{C}^n$. Suppose the statement holds for $m-1$ and prove it for m. Let $P \in \mathcal{P}_s(c_{00}^{(m)}(\mathbb{C}^n))$ and $x = (x_1, \ldots, x_m, 0, \ldots) \in c_{00}^{(m)}(\mathbb{C}^n)$. Then $P(x)$ can be represented as a sum of terms

$$\beta_{\mathbf{k}} (x_m^{(1)})^{k_1} \cdots (x_m^{(n)})^{k_n} f_{\mathbf{k}}((x_1, \ldots, x_{m-1}, 0, \ldots)),$$

where $\beta_{\mathbf{k}} \in \mathbb{C}$, $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ such that $1 \le |\mathbf{k}| \le N$, and $f_{\mathbf{k}}$ is an $(N-|\mathbf{k}|)$-homogeneous polynomial. Note that $f_{\mathbf{k}} \in \mathcal{P}_s(c_{00}^{(m-1)}(\mathbb{C}^n))$, therefore, by the induction hypothesis, $f_{\mathbf{k}}((x_1, \ldots, x_{m-1}, 0, \ldots))$ can be represented as an algebraic combination of $H^{\mathbf{l}}((x_1, \ldots, x_{m-1}, 0, \ldots))$, where $\mathbf{l} \in \mathbb{Z}_+^n$ such that $1 \le |\mathbf{l}| \le N - |\mathbf{k}|$. Note that

$$H^{\mathbf{l}}((x_1, \ldots, x_{m-1}, 0, \ldots)) = H^{\mathbf{l}}(x) - (x_m^{(1)})^{l_1} \cdots (x_m^{(n)})^{l_n}.$$

Therefore, $P(x)$ can be represented as an algebraic combination of $H^{\mathbf{l}}(x)$ and $x_m^{(1)}$, $\ldots$, $x_m^{(n)}$. Since P and $H^{\mathbf{l}}$ are symmetric, it follows that together with term

$$\gamma_{r_1,\ldots,r_n,t_1,\ldots,t_s} (x_m^{(1)})^{r_1} \cdots (x_m^{(n)})^{r_n} (H^{\mathbf{l}_1}(x))^{t_1} \cdots (H^{\mathbf{l}_s}(x))^{t_s},$$

where $\gamma_{r_1,\ldots,r_n,t_1,\ldots,t_s} \in \mathbb{C}$, $\mathbf{l}_1, \ldots, \mathbf{l}_s \in \mathbb{Z}_+^n$ and $r_1, \ldots, r_n, t_1, \ldots, t_s \in \mathbb{Z}_+$, the sum must contain terms

$$\gamma_{r_1,\ldots,r_n,t_1,\ldots,t_s} (x_j^{(1)})^{r_1} \cdots (x_j^{(n)})^{r_n} (H^{\mathbf{l}_1}(x))^{t_1} \cdots (H^{\mathbf{l}_s}(x))^{t_s},$$

where $j \in \{1, \ldots, m-1\}$. Therefore, $P(x)$ can be represented as a sum of terms

$$\gamma_{r_1,\ldots,r_n,t_1,\ldots,t_s} \left(\frac{1}{m} \sum_{j=1}^{m} (x_j^{(1)})^{r_1} \cdots (x_j^{(n)})^{r_n} \right) (H^{\mathbf{l}_1}(x))^{t_1} \cdots (H^{\mathbf{l}_s}(x))^{t_s}.$$

Since $\sum_{j=1}^{m} (x_j^{(1)})^{r_1} \cdots (x_j^{(n)})^{r_n} = H^{\mathbf{r}}(x)$, where $\mathbf{r} = (r_1, \ldots, r_n)$, it follows that P is an algebraic combination of polynomials $H^{\mathbf{k}}$, where $\mathbf{k} \in \mathbb{Z}_+^n$ such that $1 \le |\mathbf{k}| \le N$. □

Theorem 7.3. *Let $P : c_{00}(\mathbb{C}^n) \to \mathbb{C}$ be a symmetric N-homogeneous polynomial. Let $M_N = \{\mathbf{k} \in \mathbb{Z}_+^n : 1 \le |\mathbf{k}| \le N\}$. There exists a polynomial $q : \mathbb{C}^{|M_N|} \to \mathbb{C}$ such that $P = q \circ \pi_{M_N}$, where the mapping π_{M_N} is defined by* (7.11).

Proof. By Corollary 7.2, there exists $m \in \mathbb{N}$ such that for every $m' \geq m$ polynomials $H^{\mathbf{k}}$, where $\mathbf{k} \in M$, are algebraically independent. Therefore, the representation, given by Theorem 7.2 for the restriction of P to $c_{00}^{(m')}(\mathbb{C}^n)$, is unique. Thus, for every $m' \geq m$ there exists a unique polynomial $q_{m'} : \mathbb{C}^{|M_N|} \to \mathbb{C}$ such that $P(x) = (q_{m'} \circ \pi_{M_N})(x)$ for every $x \in c_{00}^{(m')}(\mathbb{C}^n)$. Since $c_{00}^{(m')}(\mathbb{C}^n) \supset c_{00}^{(m)}(\mathbb{C}^n)$, it follows that q_m is the restriction of $q_{m'}$ to $\pi_{M_N}(c_{00}^{(m)}(\mathbb{C}^n))$. By Theorem 7.1, $\pi_{M_N}(c_{00}^{(m)}(\mathbb{C}^n)) = \mathbb{C}^{|M_N|}$, therefore, $q_{m'} \equiv q_m$. Let $q = q_m$. Then $P(x) = (q \circ \pi_{M_N})(x)$ for every $x \in c_{00}(\mathbb{C}^n)$. □

Theorem 7.4. *Polynomials $H^{\mathbf{k}}$, where $\mathbf{k} \in \mathbb{Z}_+^n$, form an algebraic basis of the algebra $\mathcal{P}_s(\ell_1(\mathbb{C}^n))$.*

Proof. Let us prove that every symmetric continuous polynomial on $\ell_1(\mathbb{C}^n)$ can be uniquely represented as an algebraic combination of polynomials $H^{\mathbf{k}}$. It suffices to prove the statement only for homogeneous polynomials. Let $P : \ell_1(\mathbb{C}^n) \to \mathbb{C}$ be a symmetric continuous N-homogeneous polynomial. By Theorem 7.3, the restriction of P to $c_{00}(\mathbb{C}^n)$ can be uniquely represented as an algebraic combination of polynomials $H^{\mathbf{k}}$, where $\mathbf{k} \in \mathbb{Z}_+^n$ such that $1 \leq |\mathbf{k}| \leq N$. Since $c_{00}(\mathbb{C}^n)$ is dense in $\ell_1(\mathbb{C}^n)$ and polynomials $H^{\mathbf{k}}$ are well-defined and continuous on $\ell_1(\mathbb{C}^n)$, it follows that given representation can be extended to $\ell_1(\mathbb{C}^n)$. □

Algebraic basis of the algebra $\mathcal{P}_s(\ell_p(\mathbb{C}^n))$. Let $p \in (1, +\infty)$. Now we describe an algebraic basis of the algebra $\mathcal{P}_s(\ell_p(\mathbb{C}^n))$. Let us prove a complex analog of [25, Lemma 2].

Lemma 7.1. *Let $K \subset \mathbb{C}^m$, $\varkappa : K \to \mathbb{C}^{m-1}$ be an orthogonal projection:*

$$\varkappa((x_1, x_2, \ldots, x_m)) = (x_2, \ldots, x_m).$$

Let $K_1 = \varkappa(K)$, $\operatorname{int} K_1 \neq \emptyset$ and for every open set $U \subset K_1$ a set $\varkappa^{-1}(U)$ is unbounded. If polynomial $Q(x_1, \ldots, x_m)$ is bounded on K, then Q does not depend on x_1.

Proof. Suppose Q depends on x_1. Then

$$Q(x_1, \ldots, x_m) = \sum_{j=0}^{k} q_j(x_2, \ldots, x_m) x_1^j,$$

where $1 \leq k \leq \deg Q$ and $q_k \not\equiv 0$. Note that $q_k \not\equiv 0$ on $\operatorname{int} K_1$, therefore, there exists point $a \in \operatorname{int} K_1$ such that $q_k(a) \neq 0$. Since $\operatorname{int} K_1$ is open and q_k is continuous, there exists $r > 0$ such that $B(a, r) \subset \operatorname{int} K_1$ and $\inf_{b \in B(a,r)} |q_k(b)| > 0$, where

$B(a,r)$ is an open ball with center a and radius r in the space $\mathbb{C}^{m-1}$. Note that for $(x_1,\dots,x_m) \in \varkappa^{-1}(B(a,r))$,

$$|Q(x_1,\dots,x_m)| \geq |q_k(x_2,\dots,x_m)||x_1|^k - \sum_{j=0}^{k-1} |q_j(x_2,\dots,x_m)||x_1|^j \geq$$
$$\geq c|x_1|^k - \sum_{j=0}^{k-1} d_j|x_1|^j, \tag{7.23}$$

where $c = \inf_{b\in B(a,r)} |q_k(b)|$ and $d_j = \sup_{b\in B(a,r)} |q_j(b)|$ for $j \in \{0,\dots,k-1\}$. Note that for the polynomial $cx_1^k + \sum_{j=0}^{k-1} d_j x_1^j$ there exists $R > 0$ such that if $|x_1| > R$, then $c|x_1|^k > 2\sum_{j=0}^{k-1} d_j|x_1|^j$, i. e. $\sum_{j=0}^{k-1} d_j|x_1|^j < \frac{1}{2}c|x_1|^k$. Therefore, if $|x_1| > R$, then

$$c|x_1|^k - \sum_{j=0}^{k-1} d_j|x_1|^j > c|x_1|^k - \frac{1}{2}c|x_1|^k = \frac{1}{2}c|x_1|^k. \tag{7.24}$$

Since $\varkappa^{-1}(B(a,r))$ is unbounded, there exists a sequence $((x_1^{(n)},\dots,x_m^{(n)}))_{n\in\mathbb{N}} \subset \varkappa^{-1}(B(a,r))$ such that $x_1^{(n)} \to \infty$ as $n \to +\infty$. Taking into account (7.23) and (7.24), we have

$$|Q(x_1^{(n)},\dots,x_m^{(n)})| > \frac{1}{2}c|x_1^{(n)}|^k \to +\infty$$

as $n \to +\infty$, which contradicts the boundedness of Q on K. □

For $\mathbf{k} = (k_1,\dots,k_n) \in \mathbb{Z}_+^n$ let $\mathcal{V}(\mathbf{k}) = \{s \in \{1,\dots,n\} : \ k_s \neq 0\}$ and $\nu(\mathbf{k}) = |\mathcal{V}(\mathbf{k})|$.

Lemma 7.2. *For* $\mathbf{k},\mathbf{l} \in \mathbb{Z}_+^n$ *if* $\mathbf{l} \succ \mathbf{k}$ *and* $\nu(\mathbf{l}) \geq \nu(\mathbf{k})$, *then* $|\mathbf{l}| > |\mathbf{k}|$.

Proof. Since $\mathbf{l} \succ \mathbf{k}$, there exists $\mathbf{m} \in \mathbb{Z}_+^n$ such that $(l_1,\dots,l_n) = (m_1k_1,\dots,m_nk_n)$, and $\mathbf{l} \neq \mathbf{k}$. Therefore, if $k_s = 0$ for some $s \in \{1,\dots,n\}$, then $l_s = 0$ too. It means that $\mathcal{V}(\mathbf{l}) \subset \mathcal{V}(\mathbf{k})$. On the other hand, $\nu(\mathbf{l}) \geq \nu(\mathbf{k})$. Therefore, $\mathcal{V}(\mathbf{l}) = \mathcal{V}(\mathbf{k})$, i. e. for $s \in \{1,\dots,n\}$ we have that $l_s \neq 0$ if and only if $k_s \neq 0$. Therefore, for every $s \in \mathcal{V}(\mathbf{l})$ we have that $m_s \geq 1$. Since $\mathbf{l} \neq \mathbf{k}$, there exists $s_0 \in \mathcal{V}(\mathbf{l})$ such that $m_{s_0} \geq 2$. Therefore,

$$|\mathbf{l}| = m_1k_1 + \ldots + m_nk_n > k_1 + \ldots + k_n = |\mathbf{k}|.$$

□

For $N \in \mathbb{N}$ and $J \in \{1,\dots,n\}$ let

$$M_N^{(J)} = \{\mathbf{l} \in \mathbb{Z}_+^n :\ 1 \leq |\mathbf{l}| < \lceil p \rceil, \nu(\mathbf{l}) \geq J\} \cup \{\mathbf{l} \in \mathbb{Z}_+^n :\ \lceil p \rceil \leq |\mathbf{l}| \leq N\}.$$

By Theorem 7.1, for $M = M_N^{(1)}$ there exists $\rho = \rho_M > 0$ such that $\pi_M(V_\rho)$ contains the open unit ball of the space $\mathbb{C}^{|M|}$ with norm $\|\cdot\|_\infty$, where

$$V_\rho = \{x \in c_{00}(\mathbb{C}^n) :\ \|x\|_p < \rho\}. \tag{7.25}$$

Proposition 7.4. *For $J \in \{1, \dots, N\}$ let $q((\xi_\mathbf{l})_{\mathbf{l}\in M_N^{(J)}})$ be a polynomial on $\mathbb{C}^{|M_N^{(J)}|}$. If q is bounded on $\pi_{M_N^{(J)}}(V_\rho)$, then q does not depend on $\xi_\mathbf{k}$ such that $\nu(\mathbf{k}) = J$ and $1 \le |\mathbf{k}| < \lceil p \rceil$.*

Proof. Let $\mathbf{k} \in \mathbb{Z}_+^n$ such that $\nu(\mathbf{k}) = J$ and $1 \le |\mathbf{k}| < \lceil p \rceil$. Let $K = \pi_{M_N^{(J)}}(V_\rho)$, $K_1 = \pi_{M_N^{(J)}\setminus\{\mathbf{k}\}}(V_\rho)$ and $\varkappa : K \to K_1$ be an orthogonal projection, defined by

$$\varkappa : (\xi_\mathbf{l})_{\mathbf{l}\in M_N^{(J)}} \mapsto (\xi_\mathbf{l})_{\mathbf{l}\in M_N^{(J)}\setminus\{\mathbf{k}\}}.$$

Let us show that for every ball

$$B(u, r) = \{\xi \in \mathbb{C}^{|M_N^{(J)}\setminus\{\mathbf{k}\}|} :\ \|\xi - u\|_\infty < r\}$$

with center $u = (u_\mathbf{l})_{\mathbf{l}\in M_N^{(J)}\setminus\{\mathbf{k}\}} \in \mathbb{C}^{|M_N^{(J)}\setminus\{\mathbf{k}\}|}$ and radius $r > 0$ such that $B(u, r) \subset \pi_{M_N^{(J)}\setminus\{\mathbf{k}\}}(V_\rho)$, a set $\varkappa^{-1}(B(u, r))$ is unbounded. Since $u \in \pi_{M_N^{(J)}\setminus\{\mathbf{k}\}}(V_\rho)$, there exists $x_u \in V_\rho$ such that $\pi_{M_N^{(J)}\setminus\{\mathbf{k}\}}(x_u) = u$. For $m \in \mathbb{N}$ we set

$$x_m = \bigoplus_{j=1}^{m} \frac{1}{j^{1/|\mathbf{k}|}} a_\mathbf{k},$$

where $a_\mathbf{k}$ is defined by (7.9). Choose ε such that

$$0 < \varepsilon < \min\left\{1, \frac{\rho - \|x_u\|_p}{\|a_\mathbf{k}\|_p \zeta(p/|\mathbf{k}|)^{1/p}}, \frac{r}{\|a_\mathbf{k}\|_1^N \zeta(1 + 1/|\mathbf{k}|)}\right\},$$

where $\zeta(\cdot)$ is a Riemann zeta-function. Let $x_{m,\varepsilon} = (\varepsilon x_m) \oplus x_u$. Let us show that $x_{m,\varepsilon} \in V_\rho$. By (7.6),

$$\|x_m\|_p^p = \sum_{j=1}^{m} \left\| \frac{1}{j^{1/|\mathbf{k}|}} a_\mathbf{k} \right\|_p^p = \sum_{j=1}^{m} \frac{1}{j^{p/|\mathbf{k}|}} \|a_\mathbf{k}\|_p^p = \|a_\mathbf{k}\|_p^p \sum_{j=1}^{m} \frac{1}{j^{p/|\mathbf{k}|}} < \|a_\mathbf{k}\|_p^p \zeta(p/|\mathbf{k}|).$$

Therefore, $\|x_m\|_p < \|a_\mathbf{k}\|_p \zeta(p/|\mathbf{k}|)^{1/p}$. By the triangle inequality,

$$\|x_{m,\varepsilon}\|_p \le \varepsilon \|x_m\|_p + \|x_u\|_p < \varepsilon \|a_\mathbf{k}\|_p \zeta(p/|\mathbf{k}|)^{1/p} + \|x_u\|_p.$$

Since

$$\varepsilon < \frac{\rho - \|x_u\|_p}{\|a_\mathbf{k}\|_p \zeta(p/|\mathbf{k}|)^{1/p}},$$

it follows that $\|x_{m,\varepsilon}\|_p < \rho$. Hence, $x_{m,\varepsilon} \in V_\rho$.

Note that for arbitrary $\mathbf{l} \in \mathbb{Z}_+^n$ such that $|\mathbf{l}| \geq 1$, by (7.7),

$$H^{\mathbf{l}}(x_m) = \sum_{j=1}^{m} \frac{1}{j^{|\mathbf{l}|/|\mathbf{k}|}} H^{\mathbf{l}}(a_{\mathbf{k}}) = H^{\mathbf{l}}(a_{\mathbf{k}}) \sum_{j=1}^{m} \frac{1}{j^{|\mathbf{l}|/|\mathbf{k}|}}$$

and

$$H^{\mathbf{l}}(x_{m,\varepsilon}) = \varepsilon^{|\mathbf{l}|} H^{\mathbf{l}}(x_m) + H^{\mathbf{l}}(x_u) = \varepsilon^{|\mathbf{l}|} H^{\mathbf{l}}(a_{\mathbf{k}}) \sum_{j=1}^{m} \frac{1}{j^{|\mathbf{l}|/|\mathbf{k}|}} + H^{\mathbf{l}}(x_u). \tag{7.26}$$

Let us show that $\pi_{M_N^{(J)} \setminus \{\mathbf{k}\}}(x_{m,\varepsilon}) \in B(u,r)$. For $\mathbf{l} \in M_N^{(J)} \setminus \{\mathbf{k}\}$ such that $\mathbf{l} \not\succ \mathbf{k}$, by Proposition 7.2, $H^{\mathbf{l}}(a_{\mathbf{k}}) = 0$, therefore, by (7.26),

$$H^{\mathbf{l}}(x_{m,\varepsilon}) = H^{\mathbf{l}}(x_u) = u_{\mathbf{l}}.$$

Let $\mathbf{l} \in M_N^{(J)} \setminus \{\mathbf{k}\}$ be such that $\mathbf{l} \succ \mathbf{k}$. If $\lceil p \rceil \leq |\mathbf{l}| \leq N$, then $|\mathbf{l}| > |\mathbf{k}|$, since $|\mathbf{k}| < \lceil p \rceil$. If $1 \leq |\mathbf{l}| < \lceil p \rceil$ and $\nu(\mathbf{l}) \geq J$, then $|\mathbf{l}| > |\mathbf{k}|$ by Lemma 7.2. Hence, $|\mathbf{l}| > |\mathbf{k}|$ in both cases. By (7.26),

$$|H^{\mathbf{l}}(x_{m,\varepsilon}) - u_{\mathbf{l}}| \leq \varepsilon^{|\mathbf{l}|} |H^{\mathbf{l}}(a_{\mathbf{k}})| \sum_{j=1}^{m} \frac{1}{j^{|\mathbf{l}|/|\mathbf{k}|}}.$$

Since $\varepsilon < 1$, it follows that $\varepsilon^{|\mathbf{l}|} \leq \varepsilon$. Since $\|H^{\mathbf{l}}\| \leq 1$, it follows that $|H^{\mathbf{l}}(a_{\mathbf{k}})| \leq \|a_{\mathbf{k}}\|_1^{|\mathbf{l}|}$. Taking into account $\|a_{\mathbf{k}}\|_p \geq 1$ and $|\mathbf{l}| \leq N$, we have that $|H^{\mathbf{l}}(a_{\mathbf{k}})| \leq \|a_{\mathbf{k}}\|_1^N$. Since $|\mathbf{l}|$ and $|\mathbf{k}|$ are integer numbers and $|\mathbf{l}| > |\mathbf{k}|$, it follows that $|\mathbf{l}| \geq |\mathbf{k}| + 1$, therefore,

$$\sum_{j=1}^{m} \frac{1}{j^{|\mathbf{l}|/|\mathbf{k}|}} \leq \sum_{j=1}^{m} \frac{1}{j^{1+1/|\mathbf{k}|}} < \zeta(1 + 1/|\mathbf{k}|).$$

Hence,

$$|H^{\mathbf{l}}(x_{m,\varepsilon}) - u_{\mathbf{l}}| < \varepsilon \|a_{\mathbf{k}}\|_1^N \zeta(1 + 1/|\mathbf{k}|).$$

Since

$$\varepsilon < \frac{r}{\|a_{\mathbf{k}}\|_1^N \zeta(1 + 1/|\mathbf{k}|)},$$

it follows that $|H^{\mathbf{l}}(x_{m,\varepsilon}) - u_{\mathbf{l}}| < r$, therefore, $\pi_{M_N^{(J)} \setminus \{\mathbf{k}\}}(x_{m,\varepsilon}) \in B(u,r)$.

By Proposition 7.2, $H^{\mathbf{k}}(a_{\mathbf{k}}) = 1$, therefore, by (7.26),

$$H^{\mathbf{k}}(x_{m,\varepsilon}) = \varepsilon^{|\mathbf{l}|} \sum_{j=1}^{m} \frac{1}{j} + H^{\mathbf{k}}(x_u) \to \infty$$

as $m \to +\infty$. Hence, $\varkappa^{-1}(B(u,r))$ is unbounded. By Lemma 7.1, q does not depend on $\xi_{\mathbf{k}}$. □

Theorem 7.5. *Let $P \in \mathcal{P}_s(\ell_p(\mathbb{C}^n))$ be an N-homogeneous polynomial. If $N < \lceil p \rceil$, then $P \equiv 0$. Otherwise, there exists a unique polynomial $q : \mathbb{C}^{|M_{p,N}|} \to \mathbb{C}$ such that $P = q \circ \pi^{(p)}_{M_{p,N}}$, where $M_{p,N} = \{\mathbf{k} \in \mathbb{Z}^n_+ : \ \lceil p \rceil \le |\mathbf{k}| \le N\}$ and $\pi^{(p)}_{M_{p,N}} : \ell_p(\mathbb{C}^n) \to \mathbb{C}^{|M_{p,N}|}$ is defined by $\pi^{(p)}_{M_{p,N}}(x) = (H^{\mathbf{k}}(x))_{\mathbf{k} \in M_{p,N}}$.*

Proof. Let P_0 be the restriction of P to $c_{00}(\mathbb{C}^n)$. Note that P_0 is a continuous symmetric

N-homogeneous polynomial. By Theorem 7.3, there exists a unique polynomial $q : \mathbb{C}^{|M_N|} \to \mathbb{C}$, where $M_N = M_N^{(1)}$ such that $P_0 = q \circ \pi_{M_N}$. Since P_0 is continuous, P_0 is bounded on V_ρ, defined by (7.25). Therefore, q is bounded on $\pi_{M_N}(V_\rho)$.

Let us prove that q does not depend on arguments $\xi_{\mathbf{k}}$ such that $1 \le |\mathbf{k}| < \lceil p \rceil$ by induction on $\nu(\mathbf{k})$. By Proposition 7.4, for $J = 1$ we have that $q((\xi_{\mathbf{k}})_{\mathbf{k} \in M_N})$ does not depend on arguments $\xi_{\mathbf{k}}$ such that $\nu(\mathbf{k}) = 1$ and $1 \le |\mathbf{k}| < \lceil p \rceil$. Suppose the statement holds for $\nu(\mathbf{k}) \in \{1, \dots, J-1\}$, where $J \in \{2, \dots, n\}$, i. e. $q((\xi_{\mathbf{k}})_{\mathbf{k} \in M_N})$ does not depend on arguments $\xi_{\mathbf{k}}$ such that $1 \le \nu(\mathbf{k}) \le J-1$ and $1 \le |\mathbf{k}| < \lceil p \rceil$. Then the restriction of q to $\mathbb{C}^{|M_N^{(J)}|}$, by Proposition 7.4, does not depend on $\xi_{\mathbf{k}}$ such that $\nu(\mathbf{k}) = J$ and $1 \le |\mathbf{k}| < \lceil p \rceil$. Hence, q does not depend on $\xi_{\mathbf{k}}$ such that $1 \le |\mathbf{k}| < \lceil p \rceil$.

Since polynomials $H^{\mathbf{k}}$, where $\mathbf{k} \in M_{p,N}$, are well-defined and continuous on $\ell_p(\mathbb{C}^n)$, and $c_{00}(\mathbb{C}^n)$ is dense in $\ell_p(\mathbb{C}^n)$, it follows that $P = q \circ \pi^{(p)}_{M_{p,N}}$. Note that in the case $N < \lceil p \rceil$ we have $M_{p,N} = \emptyset$ and, therefore, $P \equiv 0$. □

Corollary 7.3. *Polynomials $H^{\mathbf{k}}$, where $\mathbf{k} \in \{\mathbf{l} \in \mathbb{Z}^n_+ : \ |\mathbf{l}| \ge \lceil p \rceil\} \cup \{0\}$, form an algebraic basis of the algebra $\mathcal{P}_s(\ell_p(\mathbb{C}^n))$.*

The finite-dimensional case and algebraic dependencies. Let us consider more detailed the case $\ell_1^m(\mathbb{C}^n) = \ell_1(\mathbb{N}_m, \mathbb{C}^n)$, where $\mathbb{N}_m = \{1, \dots, m\}$. In other words, $\ell_1^m(\mathbb{C}^n)$ is an sm-dimensional complex space consisting of sequences of length m of vectors in $\mathbb{C}^n$.

Note that every function on $\ell_1^m(\mathbb{C}^n)$ depends on nm independent variables. We say that a function on $\ell_1^m(\mathbb{C}^n)$ is *totally symmetric* if it is invariant with respect to all possible permutations of these variables. Clearly that every totally symmetric function is block-symmetric. There are exactly nm totally symmetric algebraically independent polynomials $\ell_1^m(\mathbb{C}^n)$. If we restrict the basis (7.5) to $\ell_1^m(\mathbb{C}^n)$, we obtain

$$\sum_{l=1}^{m} \frac{(l+1)(l+2)\cdots(l+n-1)}{(n-1)!}$$

generators of $\mathcal{P}_{vs}(\ell_1^m(\mathbb{C}^n))$. From classic results of invariant theory (see Lemma 5 of [41]) there are at least N algebraic dependencies between these generators, where

$$N = \sum_{l=1}^{m} \frac{(l+1)(l+2)\cdots(l+n-1)}{(n-1)!} - nm.$$

Thus, in the finite-dimensional case, generating elements of the algebra of block-symmetric polynomials on $\ell_1^m(\mathbb{C}^n)$ are always algebraically dependent if $n > 1$.

We say that a system of generators $\tau_{vs}(\ell_1^m(\mathbb{C}^n))$ of $\mathcal{P}_{vs}(\ell_1^m(\mathbb{C}^n))$ is *reasonable* if it contains nm totally symmetric algebraically independent polynomials.

The following theorem provides algebraic dependencies between the generating elements of $\mathcal{P}_{vs}\left(\ell_1^m(\mathbb{C}^n)\right)$.

Theorem 7.6. *Let $\tau_{vs}(\ell_1^m(\mathbb{C}^n))$ be a reasonable system of the generating elements of $\mathcal{P}_{vs}\left(\ell_1^m(\mathbb{C}^n)\right)$. Then for every nonsymmetric polynomials ξ in $\tau_{vs}(\ell_1^m(\mathbb{C}^n))$ of the form (7.5) there are symmetric polynomials a_s such that*

$$\xi^{(m!)^{r-1}} - a_1\xi^{(m!)^{r-1}-1} + \ldots + (-1)^{(m!)^{r-1}-1} a_{(m!)^{r-1}-1}\xi + (-1)^{(m!)^{r-1}} a_{(m!)^{r-1}} \equiv 0,$$

where r is the number of nonzero elements $k_i, i = 1, \ldots, n$ in the multi-index $\mathbf{k}$ of (7.5).

Proof. Let $\xi_1 = \xi(x)$ be some nonsymmetric generating elements of $\tau_{vs}(\ell_1^m(\mathbb{C}^n))$. Let us fix $x^{(1)} = (x_1^{(1)}, x_2^{(1)}, \ldots, x_m^{(1)})$. For other elements $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, \ldots, x_m^{(i)})$, $2 \le i \le n$ we apply all possible permutations. The number of such permutations equals $(m!)^{r-1}$, where r is the number of nonzero elements $k_i, i = 1, \ldots, n$ in the multi-index $\mathbf{k}$ in the representation (7.5) of $\xi = \xi(x)$. Then we get new polynomials $\xi_2, \ldots, \xi_{(m!)^{r-1}}$. It is easy to see that these polynomials satisfy the identity

$$(\xi - \xi_1)(\xi - \xi_2)\ldots(\xi - \xi_{(m!)^{r-1}}) =$$
$$= \xi^{(m!)^{r-1}} - a_1\xi^{(m!)^{r-1}-1} + \ldots + (-1)^{(m!)^{r-1}-1} a_{(m!)^{r-1}-1}\xi + (-1)^{(m!)^{r-1}} a_{(m!)^{r-1}} = 0.$$

According to Vieta's formulas we can find $a_1, a_2, \ldots, a_{(m!)^{r-1}}$ by

$$a_1 = \sum_{i=1}^{(m!)^{r-1}} \xi_i, \ a_2 = \sum_{i,j=1}^{(m!)^{r-1}} \xi_i\xi_j. \ \ldots, \ a_{(m!)^{r-1}} = \xi_1\xi_2\ldots\xi_{(m!)^{r-1}}.$$

Thus the coefficients $a_i, 1 \le i \le (m!)^{r-1}$ are symmetric polynomials. □

In the case $\ell_1^2(\mathbb{C}^2)$ we have next algebraic dependencies.

Example 7.1. *Let*

$$\left(\begin{pmatrix} x_1^{(1)} \\ x_1^{(2)} \end{pmatrix}, \begin{pmatrix} x_2^{(1)} \\ x_2^{(2)} \end{pmatrix} \right) \in \ell_1^2(\mathbb{C}^2).$$

For the generating elements $H^{\mathbf{k}}(x)$ the following identity holds (see [19]):

$$\eta_5^2 - \eta_1\eta_2\eta_5 + \frac{1}{2}\eta_3\eta_2^2 + \frac{1}{2}\eta_4\eta_1^2 - \eta_3\eta_4 \equiv 0,$$

where

$$\eta_1 = H^{1,0} = x_1^{(1)} + x_2^{(1)},\ \eta_2 = H^{0,1} = x_1^{(2)} + x_2^{(2)},\ \eta_3 = H^{2,0} = (x_1^{(1)})^2 + (x_2^{(1)})^2,$$
$$\eta_4 = H^{0,2} = (x_1^{(2)})^2 + (x_2^{(2)})^2, \quad \eta_5 = H^{1,1} = x_1^{(1)}x_1^{(2)} + x_2^{(1)}x_2^{(2)}.$$

7.2 Analogues of the Newton formulas for the block-symmetric polynomials on $\ell_p(\mathbb{C}^s)$

In the case of the space $\ell_1(\mathbb{C}^s)$ there exist another important algebraic basis (see [22, 28, 41]):

$$R^{\mathbf{k}}(x) = R^{k_1,k_2,\ldots,k_s}(x) = \sum_{\substack{i_1^j<\cdots<i_{k_j}^j,\\ 1\le j\le s}}^{\infty} \prod_{j=1}^{s} x_{i_1^j}^{(j)} \ldots x_{i_{k_j}^j}^{(j)}. \tag{7.27}$$

In [28] these polynomials are referred to as the Elementary McMahon Symmetric Functions.

Let $\mathcal{H}(x)(t)$ and $\mathcal{R}(x)(t)$ be the generating functions for $H^{\mathbf{k}}(x)$ and $R^{\mathbf{k}}(x)$ according (see [28])

$$\mathcal{H}(x)(t) = \sum_{|\mathbf{k}|=1}^{\infty} \prod_{i=1}^{s} t_i^{k_i} H^{\mathbf{k}}(x), \quad \mathcal{R}(x)(t) = \sum_{|\mathbf{k}|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} R^{\mathbf{k}}(x), \quad R^{\mathbf{0}} = 1.$$

As shown in [22] and [28],

$$\mathcal{R}(x)(t) = \sum_{|\mathbf{k}|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} R^{\mathbf{k}}(x) = \prod_{i=1}^{\infty}(1 + x_i^{(1)}t_1 + \cdots + x_i^{(s)}t_s).$$

For the generating function $\mathcal{R}(x)(t)$ we have the following relation:

$$\mathcal{R}(x)(t) = \exp\left(- \sum_{|\mathbf{k}|=1}^{\infty} \prod_{i=1}^{\infty} t_i^{k_i} \frac{(|\mathbf{k}|-1)! H^{\mathbf{k}}(-x)}{\mathbf{k}!} \right), \tag{7.28}$$

where $x = (x^{(1)}, \ldots, x^{(s)}) \in \ell_1(\mathbb{C}^s)$ and $t = (t_1, \ldots, t_s)$.

Furthermore, from [28], another algebraic basis of homogeneous polynomials $E^{\mathbf{k}}(x)$ is derived from the generating function:

$$\mathcal{E}(x)(t) = \sum_{|\mathbf{k}|=0}^{\infty} \prod_{i=1}^{s} t_i^{k_i} E^{\mathbf{k}}(x) = \prod_{i=1}^{\infty} \frac{1}{1 - x_i^{(1)} t_1 - \cdots - x_i^{(s)} t_s}, \quad E^{\mathbf{0}} = 1.$$

These polynomials are known as the Complete Homogeneous MacMahon Symmetric Functions.

For $\mathcal{R}(x)(t)$ and $\mathcal{E}(x)(t)$ we have

$$\mathcal{R}(x)(t) = \frac{1}{\mathcal{E}(-x)(t)}. \tag{7.29}$$

Conversely, each polynomial $F_n(t_1x^{(1)} + t_2x^{(2)} + \cdots + t_sx^{(s)})$ and $G_n(t_1x^{(1)} + t_2x^{(2)} + \cdots + t_sx^{(s)})$ can be expressed as a linear combination of block-symmetric polynomials $H^{\mathbf{k}}(x)$ and $R^{\mathbf{k}}(x)$, respectively. This follows directly from explicit calculations,

$$G_n(t_1x^{(1)} + t_2x^{(2)} + \cdots + t_sx^{(s)}) = \sum_{|\mathbf{k}|=n} t_1^{k_1} t_2^{k_2} \cdots t_s^{k_s} R^{\mathbf{k}}(x) \tag{7.30}$$

$$F_n(t_1x^{(1)} + t_2x^{(2)} + \cdots + t_sx^{(s)}) = \sum_{|\mathbf{k}|=n} \frac{|\mathbf{k}|!}{\mathbf{k}!} t_1^{k_1} t_2^{k_2} \cdots t_s^{k_s} H^{\mathbf{k}}(x), \tag{7.31}$$

where $x = (x^{(1)}, \ldots, x^{(s)})$.

Combining (7.5) and (7.27), we can get an analog of Newton's formula for block-symmetric polynomials on $\ell_1(\mathbb{C}^s)$ (see [14, 16])

Theorem 7.7. *The following formula is true for the algebraic bases of block-symmetric polynomials on $\ell_1(\mathbb{C}^s)$.*

$$\begin{aligned} nR^{k_1,k_2,\ldots,k_s} = &\sum_{\substack{|\mathbf{q}|=1 \\ k_r \geq q_r}} H^{q_1,q_2,\ldots,q_s} R^{k_1-q_1,k_2-q_2,\ldots,k_s-q_s} - \\ &- \sum_{\substack{|\mathbf{q}|=2 \\ k_r \geq q_r}} \frac{2!}{q_1!q_2!\ldots q_s!} H^{q_1,q_2,\ldots,q_s} R^{k_1-q_1,k_2-q_2,\ldots,k_s-q_s} + \ldots + \\ &+ (-1)^{n-2} \sum_{\substack{|\mathbf{q}|=n-1 \\ k_r \geq q_r}} \frac{(n-1)!}{q_1!q_2!\ldots q_s!} H^{q_1,q_2,\ldots,q_s} R^{k_1-q_1,k_2-q_2,\ldots,k_s-q_s} + \\ &+ (-1)^{n-1} \frac{n!}{k_1!k_2!\ldots k_s!} H^{k_1,k_2,\ldots,k_s}, \end{aligned} \tag{7.32}$$

where $|\mathbf{k}| = k_1 + \ldots + k_s = n$, $\mathbf{q} = (q_1, q_2, \ldots, q_s)$, $R^{0,\ldots,0} \equiv 1$ and if $k_r < q_r$ for some $r = 1, \ldots, s$, then $R^{k_1-q_1,k_2-q_2,\ldots,k_s-q_s} \equiv 0$.

Proof. Let us consider the polynomial $P(t_1x^{(1)} + t_2x^{(2)} + \ldots + t_sx^{(s)})$, which is symmetric on the space ℓ_1 with respect to simultaneously permutations of $t_1x_i^{(1)} + t_2x_i^{(2)} + \ldots + t_sx_i^{(s)}$, $i \geq 1$. Let us denote by $\widetilde{tx} = t_1x^{(1)} + t_2x^{(2)} + \ldots + t_sx^{(s)}$. For the algebraic bases $F_k(\widetilde{tx})$ and $G_k(\widetilde{tx})$ of this polynomial the Newton formula (7.2) holds

$$nG_n(\widetilde{tx}) = F_1(\widetilde{tx})G_{n-1}(\widetilde{tx}) - F_2(\widetilde{tx})G_{n-2}(\widetilde{tx}) + \\ +F_3(\widetilde{tx})G_{n-3}(\widetilde{tx}) - \ldots + (-1)^{n-2}F_{n-1}(\widetilde{tx})G_1(\widetilde{tx}) + (-1)^{n-1}F_n(\widetilde{tx}). \tag{7.33}$$

According to (7.30) and (7.31) each of polynomials $F_m(\widetilde{tx})$ and $G_m(\widetilde{tx})$ can be represented as a linear combination of polynomials $H^{k_1,k_2,\ldots,k_s}(x)$ and $R^{k_1,k_2,\ldots,k_s}(x)$ respectively. So each term from (7.33) can be represented by polynomials $H^{k_1,k_2,\ldots,k_s}$ and $R^{p_1,p_2,\ldots,p_s}$. Then we obtain

$$F_1(\widetilde{tx})G_{n-1}(\widetilde{tx}) = \left(\sum_{k_1+k_2+\ldots+k_s=1} \frac{1!}{k_1!k_2!\ldots k_s!} t_1^{k_1}t_2^{k_2}\ldots t_s^{k_s} H^{k_1,k_2,\ldots,k_s}(x)\right) \times \\ \times \left(\sum_{p_1+p_2+\ldots+p_s=n-1} t_1^{p_1}t_2^{p_2}\ldots t_s^{p_s} R^{p_1,p_2,\ldots,p_s}(x)\right) = \\ = \sum_{\substack{k_1+k_2+\ldots+k_s=1\\ p_1+p_2+\ldots+p_s=n-1}} \frac{1!}{k_1!k_2!\ldots k_s!} t_1^{k_1+p_1}t_2^{k_2+p_2}\ldots t_s^{k_s+p_s} H^{k_1,k_2,\ldots,k_s}(x)R^{p_1,p_2,\ldots,p_s}(x),$$

etc.,

$$F_r(\widetilde{tx})G_{n-r}(\widetilde{tx}) = \left(\sum_{k_1+k_2+\ldots+k_s=r} \frac{r!}{k_1!k_2!\ldots k_s!} t_1^{k_1}t_2^{k_2}\ldots t_s^{k_s} H^{k_1,k_2,\ldots,k_s}(x)\right) \times \\ \times \left(\sum_{p_1+p_2+\ldots+p_s=n-r} t_1^{p_1}t_2^{p_2}\ldots t_s^{p_s} R^{p_1,p_2,\ldots,p_s}(x)\right) = \\ = \sum_{\substack{k_1+k_2+\ldots+k_s=r\\ p_1+p_2+\ldots+p_s=n-r}} \frac{r!}{k_1!k_2!\ldots k_s!} t_1^{k_1+p_1}t_2^{k_2+p_2}\ldots t_s^{k_s+p_s} H^{k_1,k_2,\ldots,k_s}(x)R^{p_1,p_2,\ldots,p_s}(x).$$

If we substitute this equalities and equalities (7.30), (7.31) into (7.33) and equate multipliers at the all powers of $t_i, i = 1, \ldots, s$ we obtain the required formula. □

Note that equation (7.32) is invertible and so we have

$$\frac{n!}{k_1!\ldots k_s!}H^{k_1,\ldots,k_s} = \sum_{\substack{|\mathbf{q}|=n-1\\ k_r\geq q_r}} \frac{(n-1)!}{q_1!\ldots q_s!}H^{q_1,\ldots,q_s}R^{k_1-q_1,\ldots,k_s-q_s} + \ldots +$$

$$+(-1)^{n-1}\sum_{\substack{|\mathbf{q}|=2\\ k_r\geq q_r}}\frac{2!}{q_1!\dots q_s!}H^{q_1,\dots,q_s}R^{k_1-q_1,\dots,k_s-q_s}+$$
$$+(-1)^{n}\sum_{\substack{|\mathbf{q}|=1\\ k_r\geq q_r}}H^{q_1,\dots,q_s}R^{k_1-q_1,\dots,k_s-q_s}+(-1)^{n+1}nR^{k_1,\dots,k_s}.$$

Let us rewrite formula (7.32) using multi-index notations. We denote by $\mathbf{k}! = k_1!k_2!\dots k_s!$ and by $\mathbf{k}-\mathbf{q} = (k_1-q_1, k_2-q_2, \dots k_s-q_s)$. Also, we say that $\mathbf{k}\geq\mathbf{q}$ if and only if $k_1\geq q_1$, $k_2\geq q_2$, ... $k_s\geq q_S$. Then (7.32) can be expressed by

$$nR^{\mathbf{k}}=\sum_{\substack{|\mathbf{q}|=1\\ \mathbf{k}\geq\mathbf{q}}}H^{\mathbf{q}}R^{\mathbf{k}-\mathbf{q}}-\sum_{\substack{|\mathbf{q}|=2\\ \mathbf{k}\geq\mathbf{q}}}\frac{|q|!}{\mathbf{q}!}H^{\mathbf{q}}R^{\mathbf{k}-\mathbf{q}}+\dots+(-1)^{n-2}\sum_{\substack{|\mathbf{q}|=n-1\\ \mathbf{k}\geq\mathbf{q}}}\frac{|\mathbf{q}|!}{\mathbf{q}!}H^{\mathbf{q}}R^{\mathbf{k}-\mathbf{q}}+$$
$$+(-1)^{n-1}\frac{|\mathbf{k}|!}{\mathbf{k}!}H^{\mathbf{k}}=\sum_{j=1}^{n}(-1)^{j-1}\sum_{\substack{|\mathbf{q}|=j\\ \mathbf{k}\geq\mathbf{q}}}\frac{|\mathbf{q}|!}{\mathbf{q}!}H^{\mathbf{q}}R^{\mathbf{k}-\mathbf{q}},\quad\text{where}\quad n=|\mathbf{k}|. \tag{7.34}$$

Comparing formula (7.32) with the classical Newton formula, we can see that they coincide if $s=1$.

Let us turn out to the space $\ell_p(\mathbb{C}^s)$. Taking into account formula (7.5) we can see that by definition, $H^{\mathbf{k}}=0$ in $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$ if $|\mathbf{k}|<\lceil p\rceil$. So, using (7.34), we can define *elementary block-symmetric polynomials on* $\ell_p(\mathbb{C}^s)$ by

$$nR^{\mathbf{k}}=\sum_{j=\lceil p\rceil}^{n-\lceil p\rceil}(-1)^{j-1}\sum_{\substack{|\mathbf{q}|=j\\ \mathbf{k}\geq\mathbf{q}}}\frac{|\mathbf{q}|!}{\mathbf{q}!}H^{\mathbf{q}}R^{\mathbf{k}-\mathbf{q}},\quad\text{where}\quad n=|\mathbf{k}|\geq\lceil p\rceil. \tag{7.35}$$

Theorem 7.8. *Elementary block-symmetric polynomials on* $\ell_p(\mathbb{C}^s)$ *defined by* (7.35) *form an algebraic basis of* n*-homogeneous polynomials* $n\geq\lceil p\rceil$ *in* $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$.

Proof. It is easy to see that equation (7.35) is invertible. So we have a bijection between polynomials $H^{\mathbf{q}}$ and $R^{\mathbf{q}}$. Since $\{H^{\mathbf{q}}\}_{|\mathbf{q}|\geq\lceil p\rceil}$ is an algebraic basis in $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$, so the set $\{R^{\mathbf{q}}_n\}_{n\geq\lceil p\rceil}$ is an algebraic basis in $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$ too. □

Let ω be the isomorphism of $\mathcal{P}_{vs}(\ell_1(\mathbb{C}^s))$ to itself defined so that $\omega(H^{\mathbf{k}})=-H^{\mathbf{k}}$ for every multi-index $\mathbf{k}$. In other words, if $P\in\mathcal{P}_{vs}(\ell_1(\mathbb{C}^s))$ is of the form

$$P(x)=Q(H^{\mathbf{k}},H^{\mathbf{m}},\dots,H^{\mathbf{r}})$$

for some polynomial Q of several variables, then

$$\omega(P)(x)=Q\big(\omega(H^{\mathbf{k}}),\omega(H^{\mathbf{m}}),\dots,\omega(H^{\mathbf{r}})\big).$$

Clearly that $\omega(H^{\mathbf{k}})(x)=(-1)^{|\mathbf{k}|+1}(H^{\mathbf{k}})(-x)$ for every multi-index $\mathbf{k}$, and $\omega^2(P)=P$ for every $P\in\mathcal{P}_{vs}(\ell_1(\mathbb{C}^s))$.

Proposition 7.5. *For every multi-index* $\mathbf{k}$,

$$\omega(R^{\mathbf{k}}) = E^{\mathbf{k}} \quad and \quad \omega(E^{\mathbf{k}}) = R^{\mathbf{k}}.$$

Proof. According to relations (7.28) and (7.29) we have

$$\mathcal{R}(x)(t) = \exp\left(-\sum_{|\mathbf{k}|=1}^{\infty}\prod_{i=1}^{\infty} t_i^{k_i}\frac{(|\mathbf{k}|-1)!H^{\mathbf{k}}(-x)}{\mathbf{k}!}\right) =$$

$$= \exp\left(\sum_{|\mathbf{k}|=1}^{\infty}\prod_{i=1}^{\infty} t_i^{k_i}\frac{(|\mathbf{k}|-1)!(-1)^{|\mathbf{k}|+1}H^{\mathbf{k}}(x)}{\mathbf{k}!}\right) =$$

$$= \exp\left(\sum_{|\mathbf{k}|=1}^{\infty}\prod_{i=1}^{\infty} t_i^{k_i}\frac{(|\mathbf{k}|-1)!\omega\big(H^{\mathbf{k}}\big)(x)}{\mathbf{k}!}\right) = \omega\big(\mathcal{E}\big)(x)(t).$$

By the definition of $E^{\mathbf{k}}$, we can see that $\omega(E^{\mathbf{k}}) = R^{\mathbf{k}}$. Thus, $E^{\mathbf{k}} = \omega^2(E^{\mathbf{k}}) = \omega(R^{\mathbf{k}})$. □

Using Proposition 7.5 and equation (7.34), we can prove an analog of Newton's formula (7.3).

Theorem 7.9. *For every multi-index* $\mathbf{k}$,

$$nE^{\mathbf{k}} = \sum_{j=1}^{|\mathbf{k}|}\sum_{\substack{|\mathbf{q}|=j\\ \mathbf{k}\geq\mathbf{q}}}\frac{|\mathbf{q}|!}{\mathbf{q}!}H^{\mathbf{q}}E^{\mathbf{k}-\mathbf{q}}.$$

Proof. Applying the isomorphism ω to equation (7.34), we have

$$nE^{\mathbf{k}} = n\omega(R^{\mathbf{k}}) = \sum_{j=1}^{|\mathbf{k}|}(-1)^{j-1}\sum_{\substack{|\mathbf{q}|=j\\ \mathbf{k}\geq\mathbf{q}}}\frac{|\mathbf{q}|!}{\mathbf{q}!}\omega(H^{\mathbf{q}}R^{\mathbf{k}-\mathbf{q}}) =$$

$$= \sum_{j=1}^{|\mathbf{k}|}(-1)^{j-1}\sum_{\substack{|\mathbf{q}|=j\\ \mathbf{k}\geq\mathbf{q}}}\frac{|\mathbf{q}|!}{\mathbf{q}!}(-1)^{|\mathbf{q}|+1}H^{\mathbf{q}}E^{\mathbf{k}-\mathbf{q}} =$$

$$= \sum_{j=1}^{|\mathbf{k}|}(-1)^{2j}\sum_{\substack{|\mathbf{q}|=j\\ \mathbf{k}\geq\mathbf{q}}}\frac{|\mathbf{q}|!}{\mathbf{q}!}H^{\mathbf{q}}E^{\mathbf{k}-\mathbf{q}} = \sum_{j=1}^{|\mathbf{k}|}\sum_{\substack{|\mathbf{q}|=j\\ \mathbf{k}\geq\mathbf{q}}}\frac{|\mathbf{q}|!}{\mathbf{q}!}H^{\mathbf{q}}E^{\mathbf{k}-\mathbf{q}}.$$

□

7.3 Analogues of Waring-Girard formulas for the block-symmetric polynomials

Let we denote by $\mathbf{q}^m = (q_1^m, q_2^m, \ldots, q_s^m)$ the multi-index with non-negative integer entries
$q_1^m, q_2^m, \ldots, q_s^m$, $|\mathbf{q}^m| = q_1^m + q_2^m + \cdots + q_s^m$, $\mathbf{q}^m! = q_1^m! q_2^m! \cdots q_s^m!$.

$$\|\lambda^{\mathbf{q}^i}\|_1 = \sum_{|\mathbf{q}^i|=i} \lambda_i^{\mathbf{q}^i}, \quad \|\lambda^{\mathbf{q}_r}\|_2 = \sum_{j=1}^{n} \sum_{|\mathbf{q}^j|=j} q_r^j \lambda_j^{\mathbf{q}^j}, \quad \|\lambda_{p,n}^{\mathbf{q}_r}\|_2 = \sum_{j=p}^{n} \sum_{|\mathbf{q}^j|=j} q_r^j \lambda_j^{\mathbf{q}^j}.$$

The analogues of Waring-Girard formulas for the block-symmetric polynomials was constructed in [17]. The next theorem will be true:

Theorem 7.10. *For every $\lambda_i, \lambda_i^{\mathbf{q}^i}, k_j, q_j^i \in \mathbb{Z}_+$, $i \in \{1, \ldots, n\}, j \in \{1, \ldots, s\}$ we have*

$$R^{\mathbf{k}} = \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}} \tag{7.36}$$

and

$$E^{\mathbf{k}} = \sum_{|\lambda|_1=n} \frac{1}{z_n^\lambda} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}}$$

Proof. From formulas (7.30) and (7.31) we have that

$$F_n(t_1 x^{(1)} + t_2 x^{(2)} + \ldots + t_s x^{(s)}) = \sum_{|\mathbf{q}^n|=n} \frac{|\mathbf{q}^n|!}{\mathbf{q}^n!} t_1^{q_1^n} t_2^{q_2^n} \ldots t_s^{q_s^n} H^{\mathbf{q}^n}(x_1, x_2, \ldots, x_s)$$

and

$$G_n(t_1 x^{(1)} + t_2 x^{(2)} + \ldots + t_s x^{(s)}) = \sum_{|\mathbf{k}|=n} t_1^{k_1} t_2^{k_2} \ldots t_s^{k_s} R^{\mathbf{k}}(x_1, x_2, \ldots, x_s) \tag{7.37}$$

From direct calculations we obtain

$$\left(F_n(t_1 x^{(1)} + t_2 x^{(2)} + \ldots + t_s x^{(s)})\right)^{\lambda_n} =$$

$$= \left(\sum_{|\mathbf{q}^n|=n} \frac{|\mathbf{q}^n|!}{\mathbf{q}^n!} t_1^{q_1^n} t_2^{q_2^n} \ldots t_s^{q_s^n} H^{\mathbf{q}^n}(x_1, x_2, \ldots, x_s)\right)^{\lambda_n} =$$

$$= \sum_{\|\lambda^{\mathbf{q}^n}\|_1=\lambda_n} t_1^{\sum\limits_{|\mathbf{q}^n|=n} q_1^n \lambda_n^{\mathbf{q}^n}} \dots t_s^{\sum\limits_{|\mathbf{q}^n|=n} q_s^n \lambda_n^{\mathbf{q}^n}} \frac{\lambda_n!}{\prod\limits_{|\mathbf{q}^n|=n} \lambda_n^{\mathbf{q}^n}} \prod_{|\mathbf{q}^n|=n} \left(\frac{|\mathbf{q}^n|!}{\mathbf{q}^n!}\right)^{\lambda_n^{\mathbf{q}^n}} \left(H^{\mathbf{q}^n}\right)^{\lambda_n^{\mathbf{q}^n}}. \quad (7.38)$$

If we put (7.37) and (7.38) to the formula (7.4) we obtain

$$\sum_{|\mathbf{k}|=n} t_1^{k_1} t_2^{k_2} \dots t_s^{k_s} R^{\mathbf{k}}(x_1, x_2, \dots, x_s) =$$

$$= \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda \lambda!} \left(F_1(t_1 x^{(1)} + \dots + t_s x^{(s)})\right)^{\lambda_1} \cdot \ldots \cdot \left(F_n(t_1 x^{(1)} + \dots + t_s x^{(s)})\right)^{\lambda_n} =$$

$$= \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda \lambda!} \times$$

$$\times \left(\sum_{\|\lambda^{\mathbf{q}^1}\|_1=\lambda_1} t_1^{\sum\limits_{|\mathbf{q}^1|=1} q_1^1 \lambda_1^{\mathbf{q}^1}} \dots t_s^{\sum\limits_{|\mathbf{q}^1|=1} q_s^1 \lambda_1^{\mathbf{q}^1}} \frac{\lambda_1!}{\prod\limits_{|\mathbf{q}^1|=1} \lambda_1^{\mathbf{q}^1}} \prod_{|\mathbf{q}^1|=1} \left(\frac{|\mathbf{q}^1|!}{\mathbf{q}^1!}\right)^{\lambda_1^{\mathbf{q}^1}} \left(H^{\mathbf{q}^1}\right)^{\lambda_1^{\mathbf{q}^1}} \right) \times \dots \times$$

$$\times \left(\sum_{\|\lambda^{\mathbf{q}^n}\|_1=\lambda_n} t_1^{\sum\limits_{|\mathbf{q}^n|=n} q_1^n \lambda_n^{\mathbf{q}^n}} \dots t_s^{\sum\limits_{|\mathbf{q}^n|=n} q_s^n \lambda_n^{\mathbf{q}^n}} \frac{\lambda_n!}{\prod\limits_{|\mathbf{q}^n|=n} \lambda_n^{\mathbf{q}^n}} \prod_{|\mathbf{q}^n|=n} \left(\frac{|\mathbf{q}^n|!}{\mathbf{q}^n!}\right)^{\lambda_n^{\mathbf{q}^n}} \left(H^{\mathbf{q}^n}\right)^{\lambda_n^{\mathbf{q}^n}} \right) =$$

$$= \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda \lambda!} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i \\ 1\le i\le n}} t_1^{\sum\limits_{r=1}^{n} \sum\limits_{|\mathbf{q}^r|=r} q_1^r \lambda_r^{\mathbf{q}^r}} \dots t_s^{\sum\limits_{r=1}^{n} \sum\limits_{|\mathbf{q}^r|=r} q_s^r \lambda_r^{\mathbf{q}^r}} \times$$

$$\times \prod_{i=1}^{n} \lambda_i! \prod_{|\mathbf{q}^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}} =$$

$$= \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i \\ 1\le i\le n}} t_1^{\|\lambda^{\mathbf{q}_1}\|_2} \dots t_s^{\|\lambda^{\mathbf{q}_s}\|_2} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}}. \quad (7.39)$$

If equate multipliers at the all powers of $t_i, 1 \le i \le s$ we obtain the required formula (7.36).

By applying the isomorphism ω to equation (7.36), we have

$$E^{\mathbf{k}} = \omega(R^{\mathbf{k}}) =$$

$$= \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda} \times \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(\omega(H^{\mathbf{q}^i})\right)^{\lambda_i^{\mathbf{q}^i}} =$$

$$= \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} (-1)^{|\mathbf{q}^i|\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}} =$$

$$= \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} (-1)^{2\lambda_1+3\lambda_2+\ldots+(n+1)\lambda_n} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!} H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}} =$$

$$= \sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|_1+2|\lambda|}}{z_n^\lambda} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}} =$$

$$= \sum_{|\lambda|_1=n} \frac{1}{z_n^\lambda} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}}.$$

□

Formula (7.4) is useful in combinatorics. It is well-known combinatorial identity

$$\sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda \lambda!} = 0.$$

It can be obtained if we compute $G_n(e_1)$ using (7.4), where $e_1 = (1, 0, 0, \ldots)$. In the case $\ell_1(\mathbb{C}^2)$ we obtained some new combinatorial identity

$$\sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ r=1,2}} \prod_{i=1}^{n} \prod_{|q^i|=i} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \frac{1}{\lambda_i^{\mathbf{q}^i}} = 0.$$

In the case $\ell_1(\mathbb{C}^s)$ let $e_1 = (1, 0, \ldots, 0, \ldots)$, where $1 = (\underbrace{1, \ldots, 1}_{s})$ and $0 = (\underbrace{0, \ldots, 0}_{s})$. Then $H^{\mathbf{k}}(e_1) = 1$ and $R^{\mathbf{k}}(e_1) = 0$. By the similar way we obtain next combinatorial identity:

$$\sum_{|\lambda|_1=n} \frac{(-1)^{n+|\lambda|}}{z_n^\lambda} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ 1\le i\le n \\ \|\lambda^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=1}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} = 0.$$

In the case of spaces $\ell_p(\mathbb{C}^s)$, where p is a positive integer number we can obtain an analogy of the Waring-Girard formulas. If p is not integer, then we can take $\lceil p \rceil$ instead of p.

Proposition 7.6. *For every $\lambda_i, \lambda_i^{\mathbf{q}^i}, k_j, q_j^i \in \mathbb{Z}_+$, $i \in \{p, \ldots, n\}, j \in \{1, \ldots, s\}$ we have*

$$R_{(p)}^{\mathbf{k}} = \sum_{|\lambda_{p,n}|_1=n} \frac{(-1)^{n+\lambda_{p,n}}}{z_{p,n}^{\lambda_{p,n}}} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ p\le i\le n \\ \|\lambda_{p,n}^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=p}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}}, \tag{7.40}$$

$$E_{(p)}^{\mathbf{k}} = \sum_{|\lambda_{p,n}|_1=n} \frac{1}{z_{p,n}^{\lambda_{p,n}}} \sum_{\substack{\|\lambda^{\mathbf{q}^i}\|_1=\lambda_i,\ p\le i\le n \\ \|\lambda_{p,n}^{\mathbf{q}_r}\|_2=k_r,\ 1\le r\le s}} \prod_{i=p}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!}\right)^{\lambda_i^{\mathbf{q}^i}} \left(H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}} \tag{7.41}$$

Proof. If we put $F_i = 0$, for all $1 \le i \le p$ to the formulas (7.39) we obtain

$$\sum_{|\mathbf{k}|=n} t_1^{k_1} t_2^{k_2} \ldots t_s^{k_s} R^{\mathbf{k}}(x_1, x_2, \ldots, x_s) = \sum_{|\lambda_{p,n}|_1=n} \frac{(-1)^{n+|\lambda_{p,n}|}}{z_{p,n}^{\lambda_{p,n}} \lambda_{p,n}!} \times$$

$$\times \left(F_p(t_1 x^{(1)} + \ldots + t_s x^{(s)})\right)^{\lambda_p} \times \ldots \times \left(F_n(t_1 x^{(1)} + \ldots + t_s x^{(s)})\right)^{\lambda_n} =$$

$$= \sum_{|\lambda_{p,n}|_1=n} \frac{(-1)^{n+|\lambda_{p,n}|}}{z_{p,n}^{\lambda_{p,n}} \lambda_{p,n}!} \times$$

$$\times \left(\sum_{\|\lambda^{\mathbf{q}^p}\|_1=\lambda^{\mathbf{q}^p}} t_1^{\sum\limits_{|\mathbf{q}^p|=p} q_1^p \lambda_p^{\mathbf{q}^p}} \ldots t_s^{\sum\limits_{|\mathbf{q}^p|=p} q_s^p \lambda_p^{\mathbf{q}^p}} \frac{\lambda_p!}{\prod\limits_{|\mathbf{q}^p|=p} \lambda_p^{\mathbf{q}^p}} \prod_{|\mathbf{q}^p|=p} \left(\frac{|\mathbf{q}^p|!}{\mathbf{q}^p!} H^{\mathbf{q}^p}\right)^{\lambda_p^{\mathbf{q}^p}} \right) \times \ldots \times$$

$$\times \left(\sum_{\|\lambda^{\mathbf{q}^n}\|_1=\lambda^{\mathbf{q}^n}} t_1^{\sum\limits_{|\mathbf{q}^n|=n} q_1^n \lambda_n^{\mathbf{q}^n}} \ldots t_s^{\sum\limits_{|\mathbf{q}^n|=n} q_s^n \lambda_n^{\mathbf{q}^n}} \frac{\lambda_n!}{\prod\limits_{|\mathbf{q}^n|=n} \lambda_n^{\mathbf{q}^n}} \prod_{|\mathbf{q}^n|=n} \left(\frac{|\mathbf{q}^n|!}{\mathbf{q}^n!}\right)^{\lambda_n^{\mathbf{q}^n}} \left(H^{\mathbf{q}^n}\right)^{\lambda_n^{\mathbf{q}^n}} \right) =$$

$$\sum_{|\lambda_{p,n}|_1=n} \frac{(-1)^{n+|\lambda_{p,n}|}}{z_{p,n}^{\lambda_{p,n}}} \sum_{\|\lambda^{\mathbf{q}^n}\|_1=\lambda_i,\ p\le i\le n} t_1^{\|\lambda_{p,n}^{\mathbf{q}_1}\|_2} \ldots t_s^{\|\lambda_{p,n}^{\mathbf{q}_s}\|_2} \prod_{i=p}^{n} \prod_{|q^i|=i} \frac{1}{\lambda_i^{\mathbf{q}^i}} \left(\frac{|\mathbf{q}^i|!}{\mathbf{q}^i!} H^{\mathbf{q}^i}\right)^{\lambda_i^{\mathbf{q}^i}}.$$

If equate multipliers at the all powers of t_i, $1 \le i \le s$ we obtain the required formula (7.40). By applying the isomorphism ω to equation (7.40) as in Theorem 7.10 we obtain (7.41). □

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Anatolii ROMANYUK, Kateryna POZHARSKA, and Sergii YANCHENKO

8 Characteristics of linear and nonlinear approximation of classes of periodic functions of many variables in Lebesgue subspaces

In this chapter, we present some important results that were obtained during recent years of research on approximation characteristics of classes of periodic functions of many variables, determined by certain restrictions on a dominant mixed smoothness (derivative): the Nikol'skii classes $H^{\boldsymbol{r}}_p(\mathbb{T}^d)$, the Besov classes $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$ and the Sobolev classes $W^{\boldsymbol{r}}_{p,\alpha}(\mathbb{T}^d)$ in the metric of the Lebesgue subspaces $B_{q,1}(\mathbb{T}^d)$. Besides, we mention also a number of other previously obtained results in the Lebesgue spaces $L_q(\mathbb{T}^d)$. The considered classes of functions are analogues of function classes from the well-known spaces of S. L. Sobolev (W-spaces), S. M. Nikol'skii (H-spaces) and O. V. Besov (B-spaces).

Spaces of differentiable functions of many variables became the object of comprehensive mathematical study in the early 1930s, when S. L. Sobolev, while solving problems of mathematical physics, laid the foundations of the theory of embeddings of these spaces as a separate direction of function analysis and function theory. The next important contribution to the theory of function spaces was made in the 1950s by S. M. Nikols'kii, who proposed a new classification of functions of many variables, the key role in which plays the condition of belonging of the function (or its derivatives) to the well-known Hölder-Zygmund spaces. As a result of this classification, the spaces appeared that were later called the H-spaces. In the late 1950s, O. V. Besov introduced into consideration the B-spaces, a definition of which involves an additional, in comparison with the H-spaces, parameter, with which it is possible to finely take into account the differential-difference properties of functions.

From the point of view of approximation theory, interest in the mentioned function spaces (more precisely, classes from these spaces) began to grow since the 1960s. The foundation of systematic research was laid in Babenko's papers [4, 5],

Anatolii ROMANYUK, Institute of Mathematics of the National Academy of Sciences of Ukraine, 3 Tereschenkivska str., 01024, Kyiv, Ukraine, e-mail: romanyuk@imath.kiev.ua

Kateryna POZHARSKA, Institute of Mathematics of the National Academy of Sciences of Ukraine, 3 Tereschenkivska str., 01024, Kyiv, Ukraine; Chemnitz University of Technology, 39 Reichenhainer str., 09126, Chemnitz, Germany, e-mail: pozharska.k@gmail.com

Sergii YANCHENKO, Institute of Mathematics of the National Academy of Sciences of Ukraine, 3 Tereschenkivska str., 01024, Kyiv, Ukraine, e-mail: romanyuk@imath.kiev.ua

in which an approach to optimal approximation of classes of functions from the Sobolev W-spaces was proposed. K. I. Babenko showed that the finite-dimensional aggregates of the best (in a certain sense) approximation of the classes $W^r_{2,r}$ in the space L_2 are trigonometric polynomials with "numbers" of harmonics from a so-called "hyperbolic cross". This fact gave a powerful impact on the study of approximation properties of the classes of periodic functions of many variables $H^{\boldsymbol{r}}_p(\mathbb{T}^d)$ and $W^{\boldsymbol{r}}_{p,\alpha}(\mathbb{T}^d)$ in the works of many well-known specialists in the field of function theory, such as B. S. Mityagin, S. O. Telyakovs'kii, Ya. S. Bugrov, N. S. Nikol'ska, V. E. Mayorov, V. N. Temlyakov, E. M. Galeev, E. S. Belins'kii, Dinh Dũng, etc.

Summarizing the above, let us pay attention to two circumstances that initially motivated the study of the approximation characteristics of the classes $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$. First, as a result of studies related to the approximation problems of the classes $W^{\boldsymbol{r}}_{p,\alpha}(\mathbb{T}^d)$ and $H^{\boldsymbol{r}}_p(\mathbb{T}^d)$, it was found that in the case of functions of many variables, trigonometric polynomials with "numbers" of harmonics from hyperbolic crosses have the same approximation capabilities as ordinary trigonometric polynomials in the case of approximation of functions of one variable. Secondly, and this is probably very important, it was found that in the multidimensional case the classes $W^{\boldsymbol{r}}_{p,\alpha}(\mathbb{T}^d)$ and $H^{\boldsymbol{r}}_p(\mathbb{T}^d)$ differ in terms of order estimates of their approximation by certain approximation aggregates. Moreover, it was found that in some cases, when the best, in the sense of order values, approximation of the classes $W^{\boldsymbol{r}}_{p,\alpha}(\mathbb{T}^d)$ is provided by trigonometric polynomials with "numbers" of harmonics from hyperbolic crosses, the best, in the same sense, approximation for the $H^{\boldsymbol{r}}_p(\mathbb{T}^d)$ classes is provided already by trigonometric polynomials with "numbers" of harmonics from modified, in a certain way, hyperbolic crosses.

In addition, let us justify a choice of the metric in which the investigated approximation characteristics are estimated, namely $B_{q,1}(\mathbb{T}^d)$, $1 \le q \le \infty$. In some cases, the questions regarding values and, accordingly, methods for finding exact-order estimates remain open for the number of important approximation characteristics in the so-called "limiting" cases (for extreme values 1 and ∞ of the parameters from the definition of the classes of functions or the metric in which the approximation error is estimated). Besides, it was shown that in the vast majority of situations in the multidimensional case, in contrast to the one-dimensional, the corresponding approximation characteristics in the spaces $B_{q,1}(\mathbb{T}^d)$ and $L_q(\mathbb{T}^d)$, $q \in \{1, \infty\}$, have different orders. Also, significant progress was made in finding estimates of the approximation characteristics of the mentioned Sobolev and Nikol'skii–Besov classes of functions in comparison with the corresponding estimates known at that time in the L_q-space.

Note that a prominent role among the extreme problems of the theory of approximation of functions have problems related to linear and nonlinear approximation of function classes. An increasing interest in nonlinear approximation in recent decades (in particular, in the best orthogonal trigonometric approximations and the best M-term trigonometric approximations) is due, first of all, to the fact that in many cases nonlinear approximation methods turned out to be more effective compared to linear ones.

Thus, in view of the above, there is a natural interest in studying the approximation characteristics of the classes $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$, $H^{\boldsymbol{r}}_p(\mathbb{T}^d)$ and $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}(\mathbb{T}^d)$ in the space $B_{q,1}(\mathbb{T}^d)$ for $1 \le q, p < \infty$.

The statements below cover some results of the works [32,34,35,51,53] and provide answers to a number of questions caused by the above circumstances, which naturally arose in the studies of approximation characteristics of the mentioned classes of functions. It is also important that when solving individual approximation problems on the Besov classes, a number of effects were discovered that did not appear on the Nikol'skii and Sobolev classes, as well as those that are associated with the metric of the space $B_{q,1}(\mathbb{T}^d)$.

8.1 Definition of function classes and spaces $B_{q,1}$

Let $\mathbb{R}^d$ be a d-dimensional space with elements $\boldsymbol{x} = (x_1, \ldots, x_d)$ and $(\boldsymbol{x}, \boldsymbol{y}) = x_1 y_1 + \ldots + x_d y_d$. By $L_p(\mathbb{T}^d)$, $\mathbb{T}^d := \prod_{j=1}^d [0, 2\pi)$, we denote the space of 2π-periodic in each variable functions $f(\boldsymbol{x})$, for which

$$\|f\|_p := \|f\|_{L_p(\mathbb{T}^d)} := \left((2\pi)^{-d} \int\limits_{\mathbb{T}^d} |f(\boldsymbol{x})|^p \, d\boldsymbol{x} \right)^{1/p}, \quad 1 \le p < \infty,$$

$$\|f\|_\infty := \|f\|_{L_\infty(\mathbb{T}^d)} := \operatorname*{ess\,sup}_{\boldsymbol{x} \in \mathbb{T}^d} |f(\boldsymbol{x})|.$$

We put

$$L^0_p(\mathbb{T}^d) := \left\{ f \in L_p(\mathbb{T}^d) \colon \int_0^{2\pi} f(\boldsymbol{x}) dx_j = 0 \text{ almost everywhere}, \ j = 1, \ldots, d \right\}.$$

For a function $f \in L^0_p(\mathbb{T}^d)$, $1 \le p \le \infty$, we consider its first difference in the jth variable with step $h \in \mathbb{R}$:

$$\Delta_{h,j} f(\boldsymbol{x}) = f(x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_d) - f(\boldsymbol{x})$$

and define its lth difference, $l \in \mathbb{N}$, by

$$\Delta^l_{h,j} f(\boldsymbol{x}) = \overbrace{\Delta_{h,j} \ldots \Delta_{h,j}}^{l} f(\boldsymbol{x}).$$

Assume that the vectors $\boldsymbol{k} \in \mathbb{N}^d$ and $\boldsymbol{h} \in \mathbb{R}^d$ are given. Then the mixed difference of order $\boldsymbol{k}$ with a vector step $\boldsymbol{h}$ is defined by the equality

$$\Delta_{\boldsymbol{h}}^{\boldsymbol{k}} f(\boldsymbol{x}) = \Delta_{h_1,1}^{k_1} \Delta_{h_2,2}^{k_2} \dots \Delta_{h_d,d}^{k_d} f(\boldsymbol{x}).$$

The spaces $B_{p,\theta}^{\boldsymbol{r}}(\mathbb{T}^d)$, $1 \le p, \theta \le \infty$, where $\boldsymbol{r} \in \mathbb{R}^d$ is a given vector with the elements $r_j > 0$, $j = 1, \dots, d$, are defined as follows:

$$B_{p,\theta}^{\boldsymbol{r}}(\mathbb{T}^d) := \{f \in L_p^0(\mathbb{T}^d) \colon \ \|f\|_{B_{p,\theta}^{\boldsymbol{r}}} < \infty\},$$

and the respective norm is specified by

$$\|f\|_{B_{p,\theta}^{\boldsymbol{r}}(\mathbb{T}^d)} := \left(\int\limits_{\mathbb{T}^d} \|\Delta_{\boldsymbol{h}}^{\boldsymbol{k}} f\|_p^\theta \prod_{j=1}^{d} \frac{dh_j}{h_j^{1+r_j\theta}} \right)^{1/\theta}, \quad 1 \le \theta < \infty, \tag{8.1}$$

$$\|f\|_{B_{p,\infty}^{\boldsymbol{r}}(\mathbb{T}^d)} \equiv \|f\|_{H_p^{\boldsymbol{r}}(\mathbb{T}^d)} := \sup_{\boldsymbol{h}} \|\Delta_{\boldsymbol{h}}^{\boldsymbol{k}} f\|_p \prod_{j=1}^{d} h_j^{-r_j}. \tag{8.2}$$

Here we assume that the components of the vectors $\boldsymbol{k}$ and $\boldsymbol{r}$ satisfy the conditions $k_j > r_j$, $j = 1, \dots, d$. Note that the values of (8.1) and (8.2) are independent of $\boldsymbol{k}$.

In this form, the definition of the spaces $B_{p,\theta}^{\boldsymbol{r}}(\mathbb{T}^d)$, $1 \le \theta < \infty$, was given by V. N. Temlyakov [58] and, for $H_p^{\boldsymbol{r}}(\mathbb{T}^d) \equiv B_{p,\infty}^{\boldsymbol{r}}(\mathbb{T}^d)$, by S. M. Nikol'skii and P. I. Lizorkin [25]. These spaces belong to the scale of spaces of mixed smoothness, introduced by S. M. Nikol'skii [30] and T. I. Amanov [3]. In addition, they are generalizations of the well-known isotropic Besov spaces [12], and the Nikol'skii spaces [31] for the case $\theta = \infty$.

In the following considerations, we will use the definition of the norm of functions from the Nikol'skii–Besov spaces $B_{p,\theta}^{\boldsymbol{r}}(\mathbb{T}^d)$ in another forms, that are equivalent to (8.1) and (8.2), namely, indirectly through the so-called decomposition representation of the elements of these spaces. The decomposition representation and the corresponding rationing of functions from the Nikol'skii–Besov spaces were first obtained by V. N. Temlyakov [58, Ch.2, §1], S. M. Nikol'skii and P. I. Lizorkin [25]. This norm is more convenient for calculations.

For all vectors $\boldsymbol{s} \in \mathbb{N}^d$, we set

$$\rho(\boldsymbol{s}) := \{\boldsymbol{k} \in \mathbb{Z}^d \colon \ 2^{s_j-1} \le |k_j| < 2^{s_j}, \ j = 1, \dots, d\}. \tag{8.3}$$

For $f \in L_p^0(\mathbb{T}^d)$, let

$$\delta_{\boldsymbol{s}}(f) := \delta_{\boldsymbol{s}}(f, \boldsymbol{x}) := \sum_{\boldsymbol{k} \in \rho(\boldsymbol{s})} \widehat{f}(\boldsymbol{k}) e^{i(\boldsymbol{k}, \boldsymbol{x})}, \tag{8.4}$$

where

$$\widehat{f}(\boldsymbol{k}) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(\boldsymbol{t}) e^{-i(\boldsymbol{k},\boldsymbol{t})} d\boldsymbol{t}$$

are the Fourier coefficients of the function f.

First, let $1 < p < \infty$. Then we can define the norm of the spaces $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$, $\boldsymbol{r} \in \mathbb{R}^d$, $r_j > 0$, $j = 1, \dots, d$, as follows (see, e.g., [25]):

$$\|f\|_{B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)} \asymp \left(\sum_{\boldsymbol{s}\in\mathbb{N}^d} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|\delta_{\boldsymbol{s}}(f)\|_p^\theta \right)^{\frac{1}{\theta}}, \quad 1 \le \theta < \infty, \tag{8.5}$$

$$\|f\|_{B^{\boldsymbol{r}}_{p,\infty}(\mathbb{T}^d)} \equiv \|f\|_{H^{\boldsymbol{r}}_p(\mathbb{T}^d)} \asymp \sup_{\boldsymbol{s}\in\mathbb{N}^d} 2^{(\boldsymbol{s},\boldsymbol{r})} \|\delta_{\boldsymbol{s}}(f)\|_p. \tag{8.6}$$

Here and in what follows, for positive quantities a and b we will use the notation $a \asymp b$ (the order equality), that means that there exist positive constants C_1 and C_2, which do not depend on one essential parameter in the quantities a and b, such that $C_1 a \le b$ (we write $a \ll b$, i.e., order inequality) and $C_2 a \ge b$ (we write $a \gg b$). The main results of this chapter will be formulated in terms of the order relations. All of the constants C_i, $i = 1, 2, \dots$, appearing in the chapter, could depend only on the parameters from the definition of the class, metrics where we measure the approximation error, and from the dimension of the space $\mathbb{R}^d$. In some cases this dependence will be indicated in an explicit way, sometimes it will be clear from the context.

Note that after the corresponding modification of the "blocks" $\delta_{\boldsymbol{s}}(f)$, the given above norm representation for the spaces $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$ can also be generalized to the extreme values $p = 1$ and $p = \infty$ (see, e.g., [25, Remark 2.1]).

Let $V_l(t)$, $t \in \mathbb{R}$, $l \in \mathbb{N}$, denotes the de la Vallée-Poussin kernel of the form

$$V_l(t) = 1 + 2\sum_{k=1}^{l} \cos kt + 2 \sum_{k=l+1}^{2l-1} \left(1 - \frac{k-l}{l}\right) \cos kt\,,$$

where for $l = 1$ we assume that the third term equals to zero.

We associate each vector $\boldsymbol{s} \in \mathbb{N}^d$ with the polynomial

$$A_{\boldsymbol{s}}(\boldsymbol{x}) = \prod_{j=1}^{d} (V_{2^{s_j}}(x_j) - V_{2^{s_j-1}}(x_j)),$$

and for $f \in L^0_p(\mathbb{T}^d)$, $1 \le p \le \infty$, set

$$A_{\boldsymbol{s}}(f) := A_{\boldsymbol{s}}(f, \boldsymbol{x}) := (f * A_{\boldsymbol{s}})(\boldsymbol{x}),$$

where $*$ denotes the operation of convolution. Thus, for $1 \leq p \leq \infty$, the norm of the spaces $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$, $\boldsymbol{r} \in \mathbb{R}^d$, $r_j > 0$, $j = 1, \ldots, d$, can be defined as follows:

$$\|f\|_{B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)} \asymp \left(\sum_{\boldsymbol{s} \in \mathbb{N}^d} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|A_{\boldsymbol{s}}(f)\|_p^\theta \right)^{\frac{1}{\theta}}, \quad 1 \leq \theta < \infty, \tag{8.7}$$

$$\|f\|_{B^{\boldsymbol{r}}_{p,\infty}(\mathbb{T}^d)} \equiv \|f\|_{H^{\boldsymbol{r}}_p(\mathbb{T}^d)} \asymp \sup_{s \in \mathbb{N}^d} 2^{(\boldsymbol{s},\boldsymbol{r})} \|A_{\boldsymbol{s}}(f)\|_p. \tag{8.8}$$

We will keep the notation $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$ also for the respective classes (the unit balls in the spaces $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$). For the classes, we will use the respective definition of the norm from either (8.5), (8.6) or (8.7), (8.8) depending on the value of the parameter p. This should not create any confusion, since we study the asymptotic characteristics of the classes of functions. These definitions of the norm, as noted, are equivalent to (8.1) and (8.2).

Further we formulate a definition of the Sobolev classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}(\mathbb{T}^d)$, that are also investigated in this chapter.

Let $F_{\boldsymbol{r}}(\boldsymbol{x}, \boldsymbol{\alpha})$ are multivariate analogs of the Bernoulli kernels, i.e.,

$$F_{\boldsymbol{r}}(\boldsymbol{x}, \boldsymbol{\alpha}) = 2^d \sum_{\boldsymbol{k}} \prod_{j=1}^{d} k_j^{-r_j} \cos\left(k_j x_j - \frac{\alpha_j \pi}{2}\right), \quad r_j > 0, \alpha_j \in \mathbb{R},$$

and we sum over the vectors $\boldsymbol{k} = (k_1, \ldots, k_d)$, such that $k_j > 0$, $j = 1, \ldots, d$. Then by $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}(\mathbb{T}^d)$ we denote the class of functions f of the form

$$f(\boldsymbol{x}) = \varphi(\boldsymbol{x}) * F_{\boldsymbol{r}}(\boldsymbol{x}, \boldsymbol{\alpha}) = (2\pi)^{-d} \int_{\mathbb{T}^d} \varphi(\boldsymbol{y}) F_{\boldsymbol{r}}(\boldsymbol{x} - \boldsymbol{y}, \boldsymbol{\alpha}) d\boldsymbol{y},$$
$$\varphi \in L_p^0(\mathbb{T}^d), \quad \|\varphi\|_p \leq 1.$$

For forerunners in the investigation of different approximation characteristics of the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}(\mathbb{T}^d)$, $H^{\boldsymbol{r}}_p(\mathbb{T}^d)$ and $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$, we refer to [18, 37, 58, 60, 66, 70], and the references therein. Also we recall that for the introduced classes the following embeddings hold:

$$\begin{aligned} B^{\boldsymbol{r}}_{p,p}(\mathbb{T}^d) \subset W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}(\mathbb{T}^d) \subset B^{\boldsymbol{r}}_{p,2}(\mathbb{T}^d), &\quad 1 < p \leq 2; \\ B^{\boldsymbol{r}}_{p,2}(\mathbb{T}^d) \subset W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}(\mathbb{T}^d) \subset B^{\boldsymbol{r}}_{p,p}(\mathbb{T}^d), &\quad 2 \leq p < \infty; \\ W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}(\mathbb{T}^d) \subset B^{\boldsymbol{r}}_{p,\infty}(\mathbb{T}^d) \equiv H^{\boldsymbol{r}}_p(\mathbb{T}^d), &\quad 1 \leq p \leq \infty. \end{aligned} \tag{8.9}$$

In particular, in the case $p = 2$ we have $W^{\boldsymbol{r}}_{2,\boldsymbol{\alpha}}(\mathbb{T}^d) \subset B^{\boldsymbol{r}}_{2,2}(\mathbb{T}^d) \subset W^{\boldsymbol{r}}_{2,\boldsymbol{\alpha}}(\mathbb{T}^d)$.

Note that with the growth of the parameter θ, the classes $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$ expand, i.e.,

$$B^{\boldsymbol{r}}_{p,1}(\mathbb{T}^d) \subset B^{\boldsymbol{r}}_{p,\theta_1}(\mathbb{T}^d) \subset B^{\boldsymbol{r}}_{p,\theta_2}(\mathbb{T}^d) \subset B^{\boldsymbol{r}}_{p,\infty}(\mathbb{T}^d) \equiv H^{\boldsymbol{r}}_p(\mathbb{T}^d), \quad 1 \leq \theta_1 < \theta_2 \leq \infty.$$

Further, for simplicity, in place of $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}(\mathbb{T}^d)$, $H^{\boldsymbol{r}}_p(\mathbb{T}^d)$ and $B^{\boldsymbol{r}}_{p,\theta}(\mathbb{T}^d)$ we will use the notation $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$, $H^{\boldsymbol{r}}_p$ and $B^{\boldsymbol{r}}_{p,\theta}$. This should not lead to misunderstanding because only these classes of functions are considered in the current chapter. Also we write L_p and L^0_p in place of $L_p(\mathbb{T}^d)$ and $L^0_p(\mathbb{T}^d)$, respectively.

We assume that coordinates of the vector $\boldsymbol{r} \in \mathbb{R}^d$, as the parameter of the defined classes, are ordered such that $0 < r_1 = r_2 = \cdots = r_\nu < r_{\nu+1} \le \cdots \le r_d$, and also that $\boldsymbol{\gamma} \in \mathbb{R}^d$ is a vector with the coordinates $\gamma_j = r_j/r_1$, $j = 1, \ldots, d$. Besides, $\boldsymbol{\gamma}' \in \mathbb{R}^d$, where $\gamma'_j = \gamma_j = 1$ if $j = 1, \ldots, \nu$ and $1 < \gamma'_j < \gamma_j$ if $j = \nu + 1, \ldots, d$.

In what follows, we define the norm $\|\cdot\|_{B_{q,1}} := \|\cdot\|_{B_{q,1}(\mathbb{T}^d)}$ in the subspaces $B_{q,1} \subset L^0_q$, $1 \le q \le \infty$.

For trigonometric polynomials t with respect to the trigonometric system

$$\{e^{i(\boldsymbol{k},\boldsymbol{x})}\}_{\boldsymbol{k}\in\mathbb{Z}^d},$$

the norm $\|t\|_{B_{q,1}}$ is defined by the formula

$$\|t\|_{B_{q,1}} := \sum_{\boldsymbol{s}\in\mathbb{N}^d} \|A_{\boldsymbol{s}}(t)\|_q, \quad 1 \le q \le \infty$$

(the sum contains a finite number of terms).

We define the norm for the functions $f \in L^0_q$ such that $\sum_{\boldsymbol{s}\in\mathbb{N}^d} \|A_{\boldsymbol{s}}(f)\|_q$ is convergent:

$$\|f\|_{B_{q,1}} := \sum_{\boldsymbol{s}\in\mathbb{N}^d} \|A_{\boldsymbol{s}}(f)\|_q, \quad 1 \le q \le \infty.$$

Note, that in the case $1 < q < \infty$ it holds

$$\|f\|_{B_{q,1}} \asymp \sum_{\boldsymbol{s}\in\mathbb{N}^d} \|\delta_{\boldsymbol{s}}(f)\|_q. \tag{8.10}$$

For $f \in B_{q,1}$, $1 \le q \le \infty$, it holds

$$\|f\|_q \ll \|f\|_{B_{q,1}}; \quad \|f\|_{B_{1,1}} \ll \|f\|_{B_{q,1}} \ll \|f\|_{B_{\infty,1}}. \tag{8.11}$$

The space $B_{q,1}$ is sometimes called the “zero” Besov space (when the smoothness parameter $\boldsymbol{r}$ equals $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^d$). Besides, it was shown in [14] that the space $B_{\infty,1}$ can replace the space L_∞ without changing the associated approximation numbers for the embeddings of the periodic isotropic Sobolev spaces and Sobolev spaces of functions with dominating mixed smoothness.

Remark 8.1. *Such spaces and, respectively, norms, can be considered in a more general case, namely $B_{p,q}(\mathbb{T}^d)$, $1 \le p, q \le \infty$ (see, e.g., [24]).*

8.2 Approximations by step hyperbolic Fourier sums and best approximations

First, we give definitions of the approximation characteristics, which are investigated in this section.

For $n \in \mathbb{N}$, $\boldsymbol{s} \in \mathbb{N}^d$ and $\boldsymbol{\gamma} \in \mathbb{R}^d$, $\gamma_j > 0$, $j = 1, \ldots, d$, we put

$$Q_n^{\gamma} := \bigcup_{(\boldsymbol{s},\boldsymbol{\gamma})<n} \rho(\boldsymbol{s}),$$

where $\rho(\boldsymbol{s})$ is defined by (8.3). The set Q_n^{γ} is called the step hyperbolic cross. In the case $\boldsymbol{\gamma} = (1, \ldots, 1) := \mathbf{1} \in \mathbb{N}^d$, we use the notation $Q_n^{\mathbf{1}}$.

We consider the set of trigonometric polynomials

$$T(Q_n^{\gamma}) := \left\{ t\colon t(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in Q_n^{\gamma}} c_{\boldsymbol{k}} e^{i(\boldsymbol{k},\boldsymbol{x})},\ c_{\boldsymbol{k}} \in \mathbb{C},\ \boldsymbol{x} \in \mathbb{R}^d \right\}$$

and for $f \in L_1^0$ define

$$S_{Q_n^{\gamma}}(f) := S_{Q_n^{\gamma}}(f, \boldsymbol{x}) := \sum_{\boldsymbol{k}\in Q_n^{\gamma}} \widehat{f}(\boldsymbol{k})\, e^{i(\boldsymbol{k},\boldsymbol{x})}, \quad \boldsymbol{x} \in \mathbb{R}^d,$$

where, recall, $\widehat{f}(\boldsymbol{k})$ are the Fourier coefficients of the function f.

The polynomial $S_{Q_n^{\gamma}}(f)$ is called the step hyperbolic Fourier sum of the function f. According to (8.4), it can be written in the form

$$S_{Q_n^{\gamma}}(f) = \sum_{(\boldsymbol{s},\boldsymbol{\gamma})<n} \delta_{\boldsymbol{s}}(f). \tag{8.12}$$

In connection to the above defined sets of polynomials, we will consider the following approximation characteristics.

Let $\mathscr{X}$ be a d-dimensional, $d \geq 1$, function space defined on $\mathbb{T}^d$ equipped with the norm $\|\cdot\|_{\mathscr{X}}$. Then, for $f \in \mathscr{X}$, by

$$E_{Q_n^{\gamma}}(f)_{\mathscr{X}} := \inf_{t\in T(Q_n^{\gamma})} \|f - t\|_{\mathscr{X}}$$

we define a quantity of the best approximation of the function f by polynomials from the set $T(Q_n^{\gamma})$.

Respectively, for the function class $F \subset \mathscr{X}$, we set

$$E_{Q_n^{\gamma}}(F)_{\mathscr{X}} := \sup_{f\in F} E_{Q_n^{\gamma}}(f)_{\mathscr{X}}. \tag{8.13}$$

Along with the quantity (8.13), we investigate the approximation characteristic

$$\mathscr{E}_{Q_n^{\gamma}}(F)_{\mathscr{X}} := \sup_{f\in F} \|f - S_{Q_n^{\gamma}}(f)\|_{\mathscr{X}}. \tag{8.14}$$

An investigation of the approximation properties of (8.13) and (8.14) on the Sobolev classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ and Nikol'skii–Besov classes $B^{\boldsymbol{r}}_{p,\theta}$ in the case $\mathscr{X} = L_q$ has a rich history. We refer to the monographs [18, 37, 58, 60, 66].

The aim of this section is to get estimates for the quantities (8.13) and (8.14) in the case when F is the class $B^{\boldsymbol{r}}_{p,\theta}$ or $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ and $\mathscr{X} = B_{q,1}$.

First, we note the existing connection between the quantities $E_{Q_n^{\gamma}}(f)_{B_{q,1}}$ and $\mathscr{E}_{Q_n^{\gamma}}(f)_{B_{q,1}}$, namely, the following relation:

$$E_{Q_n^{\gamma}}(f)_{B_{q,1}} \asymp \mathscr{E}_{Q_n^{\gamma}}(f)_{B_{q,1}}, \quad 1 < q < \infty. \tag{8.15}$$

In comments to the obtained results we will refer to the uni-variate case. Let us provide respective modifications of the quantities (8.13) and (8.14).

For $F \subset \mathscr{X}$ we set

$$E_{2^n}(F)_{\mathscr{X}} := \sup_{f\in F} \inf_{t\in T(2^n)} \|f - t\|_{\mathscr{X}},$$

$$T(2^n) := \left\{ t\colon\ t(x) = \sum_{k=-2^n}^{2^n} c_k e^{ikx}, \quad c_k \in \mathbb{C}, \quad x \in \mathbb{R} \right\}.$$

According to the definition (8.14), we put

$$\mathscr{E}_{2^n}(F)_{\mathscr{X}} := \sup_{f\in F} \|f - S_{2^n}(f)\|_{\mathscr{X}},$$

where

$$S_{2^n}(f) := S_{2^n}(f,x) := \sum_{k=-2^n}^{2^n} \widehat{f}(k) e^{ikx} = \sum_{s=1}^{n} \delta_s(f,x), \quad x \in \mathbb{R}. \tag{8.16}$$

Let us formulate an auxiliary statement that we will use multiple times in the proofs.

Lemma A ([58], Lemma B, p. 11). *Let $\boldsymbol{s} \in \mathbb{N}^d$, $\boldsymbol{\gamma} \in \mathbb{R}^d$, $\gamma_j > 0$, $j = 1, \dots, d$. Then for $\alpha > 0$, the following estimate holds:*

$$\sum_{(s,\gamma)\ge l} 2^{-\alpha(s,\gamma)} \asymp 2^{-\alpha l} l^{d-1}.$$

If $\boldsymbol{\gamma}' \in \mathbb{R}^d$ is such that $\gamma_j = \gamma'_j = 1$ for $j = 1, \dots, \nu$ and $1 < \gamma'_j < \gamma_j$ for $j = \nu + 1, \dots, d$, then for $\alpha > 0$, it holds:

$$\sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\ge l} 2^{-\alpha(\boldsymbol{s},\boldsymbol{\gamma})} \asymp 2^{-\alpha l} l^{\nu-1}.$$

Theorem A. *Let*

$$t(\boldsymbol{x}) = \sum_{|k_j| \le n_j} c_{\boldsymbol{k}} e^{i(\boldsymbol{k},\boldsymbol{x})},$$

where $\boldsymbol{x} \in \mathbb{R}^d$, $\boldsymbol{k} \in \mathbb{Z}^d$, $\boldsymbol{n} \in \mathbb{N}^d$, $c_{\boldsymbol{k}} \in \mathbb{C}$. *Then for* $1 \le p < q \le \infty$ *the following inequality holds:*

$$\|t\|_q \le 2^d \prod_{j=1}^{d} n_j^{\frac{1}{p}-\frac{1}{q}} \|t\|_p. \tag{8.17}$$

Inequality (8.17) was obtained by S. M. Nikol'skii [31] and is referred to as the inequality for different metrics.

Now we move to a step-by-step proof of the results. For the details on parameters contained in the definitions of the function classes (e.g., $\boldsymbol{r}, \gamma, \gamma', \nu$) we refer to Section 8.1.

We begin with estimates for the Nikol'skii–Besov classes $B^{\boldsymbol{r}}_{p,\theta}$ from [32] (see Remark 8.10 for the general statement). First, let us get the exact order estimates for the quantities $\mathscr{E}_{Q_n^\gamma}(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}}$ and $E_{Q_n^\gamma}(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}}$ in the case $1 < p < q < \infty$.

Theorem 8.1. *Let* $d \ge 2$, $1 < p < q < \infty$, $1 \le \theta \le \infty$. *Then for* $r_1 > 1/p - 1/q$ *the following estimates hold*

$$\mathscr{E}_{Q_n^\gamma}(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}} \asymp E_{Q_n^\gamma}(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}} \asymp 2^{-n\left(r_1-\frac{1}{p}+\frac{1}{q}\right)} n^{(\nu-1)\left(1-\frac{1}{\theta}\right)}. \tag{8.18}$$

Proof. To get the upper estimates in (8.18), it is sufficient, due to (8.15), to respectively estimate the quantity $\mathscr{E}_{Q_n^\gamma}(B^{\boldsymbol{r}}_{p,\theta})_{B_{p,1}}$.

Let f be a function from the class $B^{\boldsymbol{r}}_{p,\theta}$. Then, from the equivalent representation (8.10) of the norm in the space $B_{q,1}$, we have

$$\mathscr{E}_{Q_n^\gamma}(f)_{B_{q,1}} = \Bigg\| f - \sum_{(\boldsymbol{s},\gamma)<n} \delta_{\boldsymbol{s}}(f) \Bigg\|_{B_{q,1}} = \Bigg\| \sum_{(\boldsymbol{s},\gamma)\ge n} \delta_{\boldsymbol{s}}(f) \Bigg\|_{B_{q,1}} \asymp$$

$$\asymp \sum_{\boldsymbol{s}\in\mathbb{N}^d} \Bigg\| \delta_{\boldsymbol{s}} \Bigg(\sum_{\substack{\boldsymbol{s}'\in\mathbb{N}^d \\ (\boldsymbol{s}',\gamma)\ge n}} \delta_{\boldsymbol{s}'}(f) \Bigg) \Bigg\|_q \le \sum_{(\boldsymbol{s},\gamma)\ge n} \|\delta_{\boldsymbol{s}}(f)\|_q := J_1. \tag{8.19}$$

To further estimate the quantity J_1, let us distinguish three cases depending on the value of the parameter θ.

Case $\theta = 1$. Then, taking into account Theorem A and the fact that $(\boldsymbol{s}, \mathbf{1}) \le (\boldsymbol{s}, \boldsymbol{\gamma})$, we can write

$$J_1 \le \sum_{(\boldsymbol{s},\gamma)\ge n} 2^{-(\boldsymbol{s},\boldsymbol{\gamma})r_1} 2^{(\boldsymbol{s},\mathbf{1})\left(\frac{1}{p}-\frac{1}{q}\right)} 2^{(\boldsymbol{s},\boldsymbol{r})} \|\delta_{\boldsymbol{s}}(f)\|_p \le$$

$$\leq \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n} 2^{-(\boldsymbol{s},\boldsymbol{\gamma})\left(r_1-\frac{1}{p}+\frac{1}{q}\right)} 2^{(\boldsymbol{s},\boldsymbol{r})}\|\delta_{\boldsymbol{s}}(f)\|_p \leq$$

$$\leq 2^{-n\left(r_1-\frac{1}{p}+\frac{1}{q}\right)} \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n} 2^{(\boldsymbol{s},\boldsymbol{r})}\|\delta_{\boldsymbol{s}}(f)\|_p \ll 2^{-n\left(r_1-\frac{1}{p}+\frac{1}{q}\right)}\|f\|_{B^{\boldsymbol{r}}_{p,1}} \leq 2^{-n\left(r_1-\frac{1}{p}+\frac{1}{q}\right)}.$$

Case $1<\theta<\infty$. Using the Hölder inequality with the power θ to J_1, we get

$$J_1 \leq \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n} 2^{-\left((\boldsymbol{s},\boldsymbol{r})-(\boldsymbol{s},\mathbf{1})\left(\frac{1}{p}-\frac{1}{q}\right)\right)} 2^{(\boldsymbol{s},\boldsymbol{r})}\|\delta_{\boldsymbol{s}}(f)\|_p \leq$$

$$\leq \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n} 2^{(\boldsymbol{s},\boldsymbol{r})\theta}\|\delta_{\boldsymbol{s}}(f)\|_p^\theta\right)^{\frac{1}{\theta}} \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n} 2^{-\left((\boldsymbol{s},\boldsymbol{r})-(\boldsymbol{s},\mathbf{1})\left(\frac{1}{p}-\frac{1}{q}\right)\right)\theta'}\right)^{\frac{1}{\theta'}} \ll$$

$$\ll \|f\|_{B^{\boldsymbol{r}}_{p,\theta}} \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n} 2^{-\left(\boldsymbol{s},\boldsymbol{r}-\mathbf{1}\left(\frac{1}{p}+\frac{1}{q}\right)\right)\theta'}\right)^{\frac{1}{\theta'}} \leq \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n} 2^{-(\boldsymbol{s},\widetilde{\boldsymbol{\gamma}})\left(r_1-\frac{1}{p}+\frac{1}{q}\right)\theta'}\right)^{\frac{1}{\theta'}}, \quad (8.20)$$

where $\widetilde{\boldsymbol{\gamma}} \in \mathbb{R}^d$ is such that $\widetilde{\gamma}_j = (r_j-1/p+1/q)/(r_1-1/p+1/q)$, $j=1,\dots,d$.

One can easily check that $\widetilde{\gamma}_j=\gamma_j=1$ for $j=1,\dots,\nu$ and $1<\gamma_j<\widetilde{\gamma}_j$ for $j=\nu+1,\dots,d$. From this, using Lemma A for the last sum in (8.20), we obtain

$$J_1 \ll 2^{-n\left(r_1-\frac{1}{p}+\frac{1}{q}\right)} n^{(\nu-1)\left(1-\frac{1}{\theta}\right)}.$$

Case $\theta=\infty$. Taking into account that for $f\in B^{\boldsymbol{r}}_{p,\infty}$, $1<p<\infty$, it holds (see (8.6))

$$\|\delta_{\boldsymbol{s}}(f)\|_p \ll 2^{-(\boldsymbol{s},\boldsymbol{r})}, \quad \boldsymbol{s}\in\mathbb{N}^d, \quad (8.21)$$

using Lemma A, we have that

$$J_1 \ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n} 2^{-\left((\boldsymbol{s},\boldsymbol{r})-(\boldsymbol{s},\mathbf{1})\left(\frac{1}{p}-\frac{1}{q}\right)\right)} = \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n} 2^{-\left(\boldsymbol{s},\boldsymbol{r}-\mathbf{1}\left(\frac{1}{p}+\frac{1}{q}\right)\right)} =$$

$$= \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n} 2^{-(\boldsymbol{s},\widetilde{\boldsymbol{\gamma}})\left(r_1-\frac{1}{p}+\frac{1}{q}\right)} \ll 2^{-n\left(r_1-\frac{1}{p}+\frac{1}{q}\right)} n^{\nu-1}. \quad (8.22)$$

Combining (8.19)–(8.22), we obtain the upper estimate for $\mathscr{E}_{Q_n^\gamma}(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}}$, and hence for $E_{Q_n^\gamma}(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}}$.

It is sufficient to get the lower estimate in (8.18) for the case $\nu=d$. Below we construct a function that realizes the obtained upper order estimate.

Let

$$g(\boldsymbol{x}) = C_3\, 2^{-n\left(r_1+1-\frac{1}{p}\right)} n^{-\frac{d-1}{\theta}} d_n(\boldsymbol{x}), \quad C_3>0, \quad (8.23)$$

where

$$d_n(\boldsymbol{x}) = \sum_{(\boldsymbol{s},\boldsymbol{1})=n} \sum_{\boldsymbol{k}\in\rho(\boldsymbol{s})} e^{i(\boldsymbol{k},\boldsymbol{x})}.$$

Let us show that $g \in B^{\boldsymbol{r}}_{p,\theta}$ for certain choice of the constant C_3, $\boldsymbol{r} = (r_1, \dots, r_1) \in \mathbb{R}^d$, $r_1 > 0$, $1 < p < \infty$, $1 \le \theta \le \infty$.

Let first $1 \le \theta < \infty$. Then we can write

$$\|g\|_{B^{\boldsymbol{r}}_{p,\theta}} \asymp \left(\sum_{(\boldsymbol{s},\boldsymbol{1})=n} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|\delta_{\boldsymbol{s}}(g)\|_p^\theta \right)^{\frac{1}{\theta}} \asymp$$

$$\asymp 2^{-n\left(r_1+1-\frac{1}{p}\right)} n^{-\frac{d-1}{\theta}} \left(\sum_{(\boldsymbol{s},\boldsymbol{1})=n} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|\delta_{\boldsymbol{s}}(d_n)\|_p^\theta \right)^{\frac{1}{\theta}} =$$

$$= 2^{-n\left(1-\frac{1}{p}\right)} n^{-\frac{d-1}{\theta}} \left(\sum_{(\boldsymbol{s},\boldsymbol{1})=n} \|\delta_{\boldsymbol{s}}(d_n)\|_p^\theta \right)^{\frac{1}{\theta}}. \tag{8.24}$$

Then we use the known relation

$$\left\| \sum_{k=-m}^{m} e^{ikx} \right\|_p \asymp m^{1-\frac{1}{p}}, \quad 1 < p < \infty,\ x \in \mathbb{R}$$

(see, e.g., [60, Ch. 1, §1]), and get

$$\|\delta_{\boldsymbol{s}}(d_n)\|_p \asymp 2^{(\boldsymbol{s},\boldsymbol{1})(1-\frac{1}{p})}. \tag{8.25}$$

Substituting (8.25) into (8.24), we continue the estimation of $\|g\|_{B^{\boldsymbol{r}}_{p,\theta}}$ as follows

$$\|g\|_{B^{\boldsymbol{r}}_{p,\theta}} \asymp 2^{-n\left(1-\frac{1}{p}\right)} n^{-\frac{d-1}{\theta}} \left(\sum_{(\boldsymbol{s},\boldsymbol{1})=n} 2^{(\boldsymbol{s},\boldsymbol{1})\left(1-\frac{1}{p}\right)\theta} \right)^{\frac{1}{\theta}} =$$

$$= n^{-\frac{d-1}{\theta}} \left(\sum_{(\boldsymbol{s},\boldsymbol{1})=n} 1 \right)^{\frac{1}{\theta}} \asymp n^{-\frac{d-1}{\theta}} n^{\frac{d-1}{\theta}} = 1.$$

Let further $\theta = \infty$. Then the function g takes the form

$$g(\boldsymbol{x}) = C_4\, 2^{-n\left(r_1+1-\frac{1}{p}\right)} d_n(\boldsymbol{x})$$

and therefore, in view of (8.5) and (8.25), we get

$$\|g\|_{B^{\boldsymbol{r}}_{p,\infty}} \asymp \sup_{\boldsymbol{s}\in\mathbb{N}^d} 2^{(\boldsymbol{s},\boldsymbol{r})}\|\delta_{\boldsymbol{s}}(g)\|_p \asymp \sup_{\boldsymbol{s}:(\boldsymbol{s},\boldsymbol{1})=n} 2^{(\boldsymbol{s},\boldsymbol{r})}2^{-n\left(r_1+1-\frac{1}{p}\right)}\|\delta_{\boldsymbol{s}}(d_n)\|_p \asymp 1.$$

We make a conclusion that for an appropriate choice of the constant $C_4 > 0$, the function $g \in B^{\boldsymbol{r}}_{p,\theta}$.

Then, taking into account that $S_{Q^{\boldsymbol{1}}_n}(g) = 0$ and the relation (8.25), we can write

$$E_{Q^{\boldsymbol{1}}_n}(g)_{B_{q,1}} \asymp \mathscr{E}_{Q^{\boldsymbol{1}}_n}(g)_{B_{q,1}} = \|g\|_{B_{q,1}} \asymp 2^{-n\left(r_1+1-\frac{1}{p}\right)}n^{-\frac{d-1}{\theta}} \sum_{(\boldsymbol{s},\boldsymbol{1})=n} \|\delta_{\boldsymbol{s}}(d_n)\|_q \asymp$$

$$\asymp 2^{-n\left(r_1+1-\frac{1}{p}\right)}n^{-\frac{d-1}{\theta}} \sum_{(\boldsymbol{s},\boldsymbol{1})=n} 2^{(\boldsymbol{s},\boldsymbol{1})\left(1-\frac{1}{q}\right)} \asymp 2^{-n\left(r_1-\frac{1}{p}+\frac{1}{q}\right)}n^{(d-1)\left(1-\frac{1}{\theta}\right)}.$$

Theorem 8.1 is proved. □

In what follows, we separately consider the case $p = q$, since in this situation the orders of the corresponding approximation quantities for $Q^{\boldsymbol{\gamma}}_n$ and $Q^{\boldsymbol{\gamma}'}_n$ with $\gamma'_j = \gamma_j = 1$ if $j = 1, \ldots, \nu$ and $1 < \gamma'_j < \gamma_j$ if $j = \nu + 1, \ldots, d$, differ.

Theorem 8.2. *Let $d \geq 2$, $1 < p < \infty$, $1 \leq \theta \leq \infty$. Then for $r_1 > 0$ it holds*

$$\mathscr{E}_{Q^{\boldsymbol{\gamma}'}_n}(B^{\boldsymbol{r}}_{p,\theta})_{B_{p,1}} \asymp E_{Q^{\boldsymbol{\gamma}'}_n}(B^{\boldsymbol{r}}_{p,\theta})_{B_{p,1}} \asymp 2^{-nr_1}n^{(\nu-1)\left(1-\frac{1}{\theta}\right)}.$$

Proof. We will argue similarly to that in Theorem 8.1. For $f \in B^{\boldsymbol{r}}_{p,\theta}$, we obtain

$$\mathscr{E}_{Q^{\boldsymbol{\gamma}'}_n}(f)_{B_{p,1}} \leq \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n} \|\delta_{\boldsymbol{s}}(f)\|_p := J_2.$$

Further, as above, we distinguish three cases.

For the case $\theta = 1$, we note that $2^{-(\boldsymbol{s},\boldsymbol{\gamma})r_1} \leq 2^{-(\boldsymbol{s},\boldsymbol{\gamma}')r_1}$ and write

$$J_2 = \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n} 2^{-(\boldsymbol{s},\boldsymbol{\gamma})r_1}2^{(\boldsymbol{s},\boldsymbol{r})}\|\delta_{\boldsymbol{s}}(f)\|_p \leq 2^{-nr_1}\|f\|_{B^{\boldsymbol{r}}_{p,1}} \leq 2^{-nr_1}.$$

In the case $1 < \theta < \infty$, by Lemma A, in view of $\boldsymbol{\gamma} = \boldsymbol{r}/r_1$, we obtain

$$J_2 \ll \|f\|_{B^{\boldsymbol{r}}_{p,\theta}} \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n} 2^{-(\boldsymbol{s},\boldsymbol{r})\theta'}\right)^{\frac{1}{\theta'}} \leq \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n} 2^{-(\boldsymbol{s},\boldsymbol{\gamma})r_1\theta'}\right)^{\frac{1}{\theta'}} \ll 2^{-nr_1}n^{(\nu-1)\left(1-\frac{1}{\theta}\right)}.$$

Case $\theta = \infty$. In view of (8.21), Lemma A yields

$$J_2 \ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n} 2^{-(\boldsymbol{s},\boldsymbol{r})} \ll 2^{-nr_1}n^{\nu-1}.$$

For the respective lower estimate, it is sufficient to consider the function $g \in B^{\boldsymbol{r}}_{p,\theta}$ from the proof of Theorem 8.1.

Theorem 8.2 is proved. □

Remark 8.2. *Under the conditions of Theorem 8.2, for the quantities* $\mathscr{E}_{Q_n^\gamma}(B^{\boldsymbol{r}}_{p,\theta})_{B_{p,1}}$ *and* $E_{Q_n^\gamma}(B^{\boldsymbol{r}}_{p,\theta})_{B_{p,1}}$ *it holds*

$$\mathscr{E}_{Q_n^\gamma}(B^{\boldsymbol{r}}_{p,\theta})_{B_{p,1}} \asymp E_{Q_n^\gamma}(B^{\boldsymbol{r}}_{p,\theta})_{B_{p,1}} \asymp 2^{-nr_1} n^{(d-1)\left(1-\frac{1}{\theta}\right)}. \tag{8.26}$$

To comment on the obtained results in Theorems 8.1 and 8.2, first, let us note that the corresponding statements, where the error is measured in the norm of L_q, are known. We formulate them for convenience.

Theorem B. *Let* $d \geq 2$. *Then for* $r_1 > 0$ *it holds*

$$\mathscr{E}_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{p,\theta})_p \asymp E_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{p,\theta})_p \asymp \begin{cases} 2^{-nr_1} n^{(\nu-1)\left(\frac{1}{p}-\frac{1}{\theta}\right)}, & 1 < p \leq 2,\ p < \theta \leq \infty, \\ 2^{-nr_1} n^{(\nu-1)\left(\frac{1}{2}-\frac{1}{\theta}\right)}, & 2 < p < \infty,\ 2 < \theta \leq \infty. \end{cases}$$

In the cases either $1 < p \leq 2$ *and* $1 \leq \theta \leq p$, *or* $2 < p < \infty$ *and* $1 \leq \theta \leq 2$, *it holds*

$$\mathscr{E}_{Q_n^\gamma}(B^{\boldsymbol{r}}_{p,\theta})_p \asymp E_{Q_n^\gamma}(B^{\boldsymbol{r}}_{p,\theta})_p \asymp 2^{-nr_1}.$$

In the case $\theta = \infty$, Theorem B was proved in [13] for $p = 2$ and in [29] for $1 < p < \infty$. In the case $1 \leq \theta < \infty$, the corresponding estimates were obtained in [40].

Theorem C. *Let* $d \geq 2$, $1 < p < q < \infty$, $1 \leq \theta \leq \infty$. *Then for* $r_1 > 1/p - 1/q$ *the following relations hold*

$$\mathscr{E}_{Q_n^\gamma}(B^{\boldsymbol{r}}_{p,\theta})_q \asymp E_{Q_n^\gamma}(B^{\boldsymbol{r}}_{p,\theta})_q \asymp 2^{-n\left(r_1-\frac{1}{p}+\frac{1}{q}\right)} n^{(\nu-1)\left(\frac{1}{q}-\frac{1}{\theta}\right)_+},$$

where $a_+ = \max\{a, 0\}$.

In the case $\theta = \infty$, Theorem C was proved in [59] for $1 < p < q \leq 2$ and in [58] for $1 < p < q < \infty$, $q > 2$, respectively. In the case $1 \leq \theta < \infty$, the corresponding estimates were obtained in [40].

Remark 8.3. *Comparing the results of Theorems 8.1 and 8.2 (see also* (8.26)*) and Theorems B and C, we make the conclusion: only in the cases* $\theta = 1$ *or* $\nu = 1$ *(*$d = 1$ *respectively) the corresponding approximation characteristics of the classes* $B^{\boldsymbol{r}}_{p,\theta}$ *in the spaces* $B_{q,1}$ *and* L_q, $1 < p \leq q < \infty$, *coincide in order and, if additionally* $p = q$, *they do not depend on the parameter* p. *In all other cases, their orders differ.*

Returning to Theorem 8.2, we note that it does not cover the limiting cases $p \in \{1, \infty\}$. The next statement concerns these limiting cases, but only for the quantity $E_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{p,\theta})_{B_{p,1}}$ [32].

Theorem 8.3. *Let $d \geq 2$, $p \in \{1, \infty\}$, $1 \leq \theta \leq \infty$. Then for $r_1 > 0$ it holds*

$$E_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{p,\theta})_{B_{p,1}} \asymp 2^{-nr_1} n^{(\nu-1)\left(1-\frac{1}{\theta}\right)}. \tag{8.27}$$

Remark 8.4. *One can show that under the conditions of Theorem 8.3 it holds*

$$E_{Q_n^{\gamma}}(B^{\boldsymbol{r}}_{p,\theta})_{B_{p,1}} \asymp 2^{-nr_1} n^{(d-1)\left(1-\frac{1}{\theta}\right)}. \tag{8.28}$$

To compare the estimate (8.27) with the corresponding results in the space L_p, $p \in \{1, \infty\}$, let us formulate the known statements.

Theorem D. *Let $d \geq 2$, $1 \leq \theta \leq \infty$. Then for $r_1 > 0$ it holds*

$$E_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{1,\theta})_1 \asymp 2^{-nr_1} n^{(\nu-1)\left(1-\frac{1}{\theta}\right)}. \tag{8.29}$$

The estimate (8.29) in the case $\theta = \infty$ was obtained in [61], and in the case $1 \leq \theta < \infty$ in [36].

Theorem E. *Let $d = 2$, $1 \leq \theta \leq \infty$. Then for $\boldsymbol{r} = (r_1, r_1)$, $r_1 > 0$, it holds*

$$E_{Q_n^{\gamma}}(B^{\boldsymbol{r}}_{\infty,\theta})_\infty \asymp 2^{-nr_1} n^{1-\frac{1}{\theta}}. \tag{8.30}$$

The estimate (8.30) in the case $\theta = \infty$ was proved in [61], and in the case $1 \leq \theta < \infty$ in [46].

Remark 8.5. *A question concerning the orders of the quantity $E_{Q_n^{\gamma}}(B^{\boldsymbol{r}}_{\infty,\theta})_\infty$ for $d > 2$ remains open. In particular, for the classes $H^{\boldsymbol{r}}_\infty$ see [18, Open Problem 4.7].*

Remark 8.6. *Comparing Theorem 8.3 for $p = 1$ with Theorem D we conclude that for $d \geq 2$, $1 \leq \theta \leq \infty$ and $r_1 > 0$ it holds*

$$E_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{1,\theta})_{B_{1,1}} \asymp E_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{1,\theta})_1 \asymp 2^{-nr_1} n^{(\nu-1)\left(1-\frac{1}{\theta}\right)}.$$

Similarly, comparing Theorem 8.3 for $p = \infty$ (see also (8.28)) with Theorem E we see that for $d = 2$, $1 \leq \theta \leq \infty$ and $\boldsymbol{r} = (r_1, r_1)$, $r_1 > 0$, it holds

$$E_{Q_n^{\gamma}}(B^{\boldsymbol{r}}_{\infty,\theta})_{B_{\infty,1}} \asymp E_{Q_n^{\gamma}}(B^{\boldsymbol{r}}_{\infty,\theta})_\infty \asymp 2^{-nr_1} n^{1-\frac{1}{\theta}}.$$

Remark 8.7. *The orders of the quantities $\mathscr{E}_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{p,\theta})_{B_{p,1}}$ and $\mathscr{E}_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{p,\theta})_p$, $p \in \{1, \infty\}$, $1 \leq \theta \leq \infty$, $r_1 > 0$ in the case $d \geq 2$ remain unknown.*

In view of (8.27), (8.28) and (8.30), we formulate the conjecture.

Conjecture 8.1. *Let* $d > 2$, $1 \le \theta \le \infty$. *Then for* $r_1 > 0$ *it holds*

$$E_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{\infty,\theta})_\infty \asymp 2^{-nr_1} n^{(\nu-1)\left(1-\frac{1}{\theta}\right)},$$
$$E_{Q_n^{\gamma}}(B^{\boldsymbol{r}}_{\infty,\theta})_\infty \asymp 2^{-nr_1} n^{(d-1)\left(1-\frac{1}{\theta}\right)}.$$

To conclude, we consider one more relation between the parameters p and q [32].

Theorem 8.4. *Let* $d \ge 2$, $1 \le q < p \le \infty$, $1 \le \theta \le \infty$. *Then for* $r_1 > 0$ *it holds*

$$\mathscr{E}_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}} \asymp E_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}} \asymp 2^{-nr_1} n^{(\nu-1)\left(1-\frac{1}{\theta}\right)}. \tag{8.31}$$

Let us recall the known statements for the corresponding approximation characteristics in the space L_q.

Theorem F. *Let* $d \ge 2$, $1 < q < p \le \infty$, $p^* = \min\{2, p\}$, $r_1 > 0$. *Then for* $p^* < \theta \le \infty$ *it holds*

$$\mathscr{E}_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{p,\theta})_q \asymp E_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{p,\theta})_q \asymp 2^{-nr_1} n^{(\nu-1)\left(\frac{1}{p^*}-\frac{1}{\theta}\right)}. \tag{8.32}$$

In the case $1 \le \theta \le p^*$, *it holds*

$$\mathscr{E}_{Q_n^{\gamma}}(B^{\boldsymbol{r}}_{p,\theta})_q \asymp E_{Q_n^{\gamma}}(B^{\boldsymbol{r}}_{p,\theta})_q \asymp 2^{-nr_1}.$$

For the classes $B^{\boldsymbol{r}}_{p,\infty}$ in the case $2 \le q < p \le \infty$ the corresponding upper estimates follow from the upper estimates for the case $1 < q = p < \infty$ (see Theorem B) and the lower estimates follow from [67]. In the case $1 < q < 2 \le p < \infty$, Theorem F was proved in [16,22]. In the case $1 \le q < p \le 2$, the proof of the lower estimates required a new technique, see [63]. For $1 \le \theta < \infty$, the corresponding statement was obtained in [40].

Theorem G. *Let* $d \ge 2$, $1 < p \le \infty$, $1 \le \theta \le \infty$. *Then for* $r_1 > 0$ *it holds*

$$\mathscr{E}_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{p,\theta})_1 \asymp E_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{p,\theta})_1 \asymp 2^{-nr_1} n^{(\nu-1)\left(\frac{1}{p^*}-\frac{1}{\theta}\right)_+}. \tag{8.33}$$

In the case $1 < p \le 2$, Theorem G was proved in [63] for $\theta = \infty$, and in [21,41] for $1 \le \theta < \infty$.

For $2 < p \le \infty$ and $\theta = \infty$, the upper estimates in Theorem G follow from the upper estimates for $\mathscr{E}_{Q_n^{\gamma'}}(H^{\boldsymbol{r}}_2)_2$, see [13]. The lower estimates are nontrivial. They follow from the corresponding lower estimates for the Kolmogorov widths $d_M(H^{\boldsymbol{r}}_\infty, L_1)$, which, as it was observed in [9], are derived from the lower estimates for the entropy numbers $\varepsilon_M(H^{\boldsymbol{r}}_\infty, L_1)$ from [65].

If $2 < p \le \infty$ and $1 \le \theta < \infty$, the upper estimate for the quantity $\mathscr{E}_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{p,\theta})_1$ follows from Theorem B as $p = 2$. It is due to the relation $\mathscr{E}_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{p,\theta})_1 \ll \mathscr{E}_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{2,\theta})_2$, $2 < p \le \infty$.

The lower estimates for the quantity $E_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{p,\theta})_1$, $2 < p \le \infty$, in the case $1 \le \theta \le 2$ were obtained in [41], and for $2 < \theta < \infty$ they follow from the estimate for the Kolmogorov width $d_M(B^{\boldsymbol{r}}_{\infty,\theta}, L_1)$, see [49].

Remark 8.8. *Comparing the estimates* (8.31) *with* (8.32) *and* (8.33)*, we conclude that the corresponding approximation characteristics for the classes* $B^{\boldsymbol{r}}_{p,\theta}$, $r_1 > 0$, $1 \le \theta \le \infty$ *in the spaces* $B_{q,1}$ *and* L_q *for* $1 \le q < p \le \infty$ *and* $d \ge 2$ *coincide in order only either in the case* $\theta = 1$ *or* $\nu = 1$.

Remark 8.9. *It is worth noting, that in the univariate case these approximation characteristics in the spaces* $B_{q,1}$ *and* L_q *coincide in order for all values of the parameters* p, q *and* θ*, see [52].*

Remark 8.10. *Summing up the results from Section 8.2, regarding the classes of* $B^{\boldsymbol{r}}_{p,\theta}$*, we get the following statement.*

Let $d \ge 2$, $1 \le \theta \le \infty$. *Then for* $r > (1/p - 1/q)_+$ *it holds*

$$E_{Q_n^{\gamma^*}}(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}} \asymp \mathscr{E}_{Q_n^{\gamma^*}}(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}} \asymp 2^{-n\left(r_1 - \left(\frac{1}{p} - \frac{1}{q}\right)_+\right)} n^{(\mu^*-1)\left(1-\frac{1}{\theta}\right)},$$

where

- *in the case* $1 < p < q < \infty$ *we put* $\boldsymbol{\gamma}^* = \boldsymbol{\gamma}$, $\mu^* = \nu$;
- *in the cases* $1 < p = q < \infty$ *and* $1 \le q < p \le \infty$ *we put either* $\boldsymbol{\gamma}^* = \boldsymbol{\gamma}'$, $\mu^* = \nu$ *or* $\boldsymbol{\gamma}^* = \boldsymbol{\gamma}$, $\mu^* = d$.

If $p = q \in \{1, \infty\}$, *then*

$$E_{Q_n^{\gamma^*}}(B^{\boldsymbol{r}}_{p,\theta})_{B_{p,1}} \asymp 2^{-nr_1} n^{(\mu^*-1)\left(1-\frac{1}{\theta}\right)}$$

with either $\boldsymbol{\gamma}^* = \boldsymbol{\gamma}'$, $\mu^* = \nu$ *or* $\boldsymbol{\gamma}^* = \boldsymbol{\gamma}$, $\mu^* = d$.

In what follows, we formulate the obtained results for the Sobolev classes [34].

Theorem 8.5. *Let* $d \ge 2$, $1 < p < \infty$, $r_1 > 0$, $\boldsymbol{\alpha} \in \mathbb{R}^d$. *Then it holds*

$$E_{Q_n^{\gamma'}}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{p,1}} \asymp \mathscr{E}_{Q_n^{\gamma'}}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{p,1}} \asymp 2^{-nr_1} n^{(\nu-1)\xi}, \tag{8.34}$$

where $\xi := \max\{1/2, 1/p'\}$, $1/p + 1/p' = 1$.

Remark 8.11. *One can easily check that under the conditions of Theorem 8.5 the following relations hold:*

$$E_{Q_n^{\gamma}}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{p,1}} \asymp \mathscr{E}_{Q_n^{\gamma}}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{p,1}} \asymp 2^{-nr_1} n^{(d-1)\xi}, \tag{8.35}$$

To complement Theorem 8.5, let us formulate the statement that concerns the univariate case.

Theorem 8.6. *Let $d = 1$, $1 < p < \infty$, $r_1 > 0$, $\alpha \in \mathbb{R}$. Then it holds*

$$E_{2^n}(W^{r_1}_{p,\alpha})_{B_{p,1}} \asymp \mathscr{E}_{2^n}(W^{r_1}_{p,\alpha})_{B_{p,1}} \asymp 2^{-nr_1}.$$

As a result of the obtained result, we can make a conclusion that in the univariate case the corresponding approximation characteristics of the classes $W^{r_1}_{p,\alpha}$ in the spaces $B_{p,1}$ and L_p coincide in order.

In the case $d \geq 2$, the situation is different. Let us formulate the statement for the space L_p that corresponds to Theorem 8.5.

Theorem H. *Let $d \geq 2$, $1 < p < \infty$, $r_1 > 0$. Then for $\boldsymbol{\alpha} \in \mathbb{R}^d$ it holds*

$$E_{Q_n^\gamma}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_p \asymp \mathscr{E}_{Q_n^\gamma}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_p \asymp 2^{-nr_1}. \tag{8.36}$$

Recall that in the case where $\boldsymbol{r}$ is a vector with integer coordinates, the estimates (8.36) were established in [26], and for an arbitrary vector $\boldsymbol{r}$ in [29].

In connection with the estimate (8.36), we also note the following relations:

$$E_{Q_n^{\gamma'}}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_p \asymp \mathscr{E}_{Q_n^{\gamma'}}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_p \asymp 2^{-nr_1}. \tag{8.37}$$

Thus, comparing (8.34) and (8.35) with (8.37) and (8.36), we see that for $d \geq 2$ the orders of the corresponding approximation characteristics of the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ are the same only for $\nu = 1$. In addition, in the space $B_{p,1}$ there is a dependence of the estimates of these characteristics both on the dimensionality indices ν or d, and on the value of the parameter p. Moreover, for $\nu \neq d$ the orders of approximations by polynomials from the sets $T(Q_n^\gamma)$ and $T(Q_n^{\gamma'})$ in the space $B_{p,1}$ are different in contrast to the approximations in the L_p-space.

In the following statement, similarly to Theorem 8.3, we complement the estimates (8.34) by considering the case $p = 1$, but only for the best approximations $E_{Q_n^{\gamma'}}(W^{\boldsymbol{r}}_{1,\boldsymbol{\alpha}})_{B_{1,1}}$.

Theorem 8.7. *Let $d \geq 2$, $r_1 > 0$ and $\boldsymbol{\alpha} \in \mathbb{R}^d$. Then it holds*

$$E_{Q_n^{\gamma'}}(W^{\boldsymbol{r}}_{1,\boldsymbol{\alpha}})_{B_{1,1}} \asymp 2^{-nr_1} n^{\nu-1}.$$

Remark 8.12. *Under the conditions of Theorem 8.7, the following estimate holds:*

$$E_{Q_n^\gamma}(W^{\boldsymbol{r}}_{1,\boldsymbol{\alpha}})_{B_{1,1}} \asymp 2^{-nr_1} n^{d-1}.$$

We now present the statement of Theorem 8.7 for the one-dimensional case.

Theorem 8.8. *Let $d = 1$, $r_1 > 0$ and $\alpha \in \mathbb{R}$. Then it holds*

$$E_{2^n}(W^{r_1}_{1,\alpha})_{B_{1,1}} \asymp 2^{-nr_1}.$$

By analyzing the results of Theorems 8.7 and 8.8, we conclude that, for all dimensions $d \geq 1$, the best approximations to the classes $W^{\boldsymbol{r}}_{1,\boldsymbol{\alpha}}$ in the spaces $B_{1,1}$ and L_1 have the same order.

Further, we establish estimates for the quantities (8.13) and (8.14) in the cases where the parameters p and q are different in the investigated classes and spaces, whose metric is used to estimate the error of approximation.

Theorem 8.9. *Let $d \geq 2$, $2 \leq p < q < \infty$, $r_1 > 1/p - 1/q$. Then for $\boldsymbol{\alpha} \in \mathbb{R}^d$ it holds*

$$E_{Q_n^\gamma}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{q,1}} \asymp \mathscr{E}_{Q_n^\gamma}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{q,1}} \asymp 2^{-n\left(r_1-\frac{1}{p}+\frac{1}{q}\right)} n^{(\nu-1)\left(1-\frac{1}{p}\right)}. \tag{8.38}$$

A corresponding to Theorem 8.9 statement for the space L_q is the following.

Theorem I ([20]). *Let $d \geq 2$, $1 < p < q < \infty$, $r_1 > 1/p - 1/q$. Then for $\boldsymbol{\alpha} \in \mathbb{R}^d$ it holds*

$$E_{Q_n^\gamma}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_q \asymp \mathscr{E}_{Q_n^\gamma}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_q \asymp 2^{-n\left(r_1-\frac{1}{p}+\frac{1}{q}\right)}. \tag{8.39}$$

Hence, comparing the estimates (8.38) and (8.39) for $2 \leq p < q < \infty$, we conclude that the corresponding approximation characteristics of the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ in the space L_p have different orders (except the case $\nu = 1$). Moreover, in the space $B_{q,1}$, the obtained estimates depend on the dimension ν. In the one-dimensional case, the situation is different.

The following assertion is true.

Theorem 8.10. *Let $d = 1$, $1 < p < q < \infty$, $r_1 > 1/p - 1/q$, $\alpha \in \mathbb{R}$. Then it holds*

$$E_{2^n}(W^{r_1}_{p,\alpha})_{B_{q,1}} \asymp \mathscr{E}_{2^n}(W^{r_1}_{p,\alpha})_{B_{q,1}} \asymp 2^{-n\left(r_1-\frac{1}{p}+\frac{1}{q}\right)}.$$

Note that for $d = 1$ the considered characteristics of the classes $W^{r_1}_{p,\alpha}$ in the spaces $B_{q,1}$ and L_q coincide in order.

To conclude this part, let us consider one more relation between the parameters p and q.

Theorem 8.11. *Let $d \geq 2$, $1 \leq q \leq 2$, $q < p < \infty$, $r_1 > 0$. Then for $\boldsymbol{\alpha} \in \mathbb{R}^d$ it holds*

$$E_{Q_n^{\gamma'}}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{q,1}} \asymp \mathscr{E}_{Q_n^{\gamma'}}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{q,1}} \asymp 2^{-nr_1} n^{\frac{\nu-1}{2}}. \tag{8.40}$$

A statement that corresponds to Theorem 8.11 in the space L_q is as follows.

Theorem J ([60], Ch. 3, § 3). *Let $d \geq 2$, $1 < q < p < \infty$, $r_1 > 0$. Then for $\boldsymbol{\alpha} \in \mathbb{R}^d$ it holds*

$$E_{Q_n^{\gamma'}}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_q \asymp \mathscr{E}_{Q_n^{\gamma'}}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_q \asymp 2^{-nr_1}. \tag{8.41}$$

Thus, comparing the relations (8.40) and (8.41), we see that the order estimates of the corresponding quantities as $\nu \neq 1$ are different.

In addition to Theorem 8.11, we formulate a result for the one-dimensional case.

Theorem 8.12. *Let $d = 1$, $1 < q < p < \infty$, $r_1 > 0$. Then for $\alpha \in \mathbb{R}$ it holds*

$$E_{2^n}(W^{r_1}_{p,\alpha})_{B_{q,1}} \asymp \mathscr{E}_{2^n}(W^{r_1}_{p,\alpha})_{B_{q,1}} \asymp 2^{-nr_1}.$$

Thus, in the one-dimensional case, for $1 < q < p < \infty$ the corresponding approximation characteristics of the classes $W^{r_1}_{p,\alpha}$ in the spaces $B_{q,1}$ and L_q have the same order.

Remark 8.13. *To conclude, we note that the upper estimates that were obtained for the Sobolev classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ in Theorems 8.5–8.12, in view of the embeddings* (8.9) *and relations* (8.11)*, are corollaries to the respective results for the Nikol'skii–Besov classes $B^{\boldsymbol{r}}_{p,\theta}$. So, in particular, the upper estimate in Theorem 8.5 follows from the estimates of Theorem 8.2, and the estimate in Theorem 8.9, respectively, follows from Theorem 8.1. To establish the lower estimates in both of these theorems, we use the function g, given by the formula* (8.23) *as $\theta = 2$.*

Remark 8.14. *For the results concerning errors of approximation of functions from the classes $B^{\boldsymbol{r}}_{p,\theta}$ and $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ by their step hyperbolic Fourier sums in the space $B_{\infty,1}$ see Section 8.3 (Corollaries 8.1– 8.4 and 8.7).*

8.3 Best orthogonal trigonometric approximations

Let us define the investigated below approximation characteristic and formulate auxiliary statements.

Let $\mathscr{X}$ be a normed space with the norm $\|\cdot\|_{\mathscr{X}}$ and $\Theta_M := \{\boldsymbol{k}^1, \dots, \boldsymbol{k}^M\}$ be a set of vectors $\boldsymbol{k}^j \in \mathbb{Z}^d$, $j = 1, \dots, M$. For the function $f \in \mathscr{X}$ let us denote

$$e^{\perp}_M(f)_{\mathscr{X}} := \inf_{\Theta_M} \|f - S_{\Theta_M}(f)\|_{\mathscr{X}},$$

where

$$S_{\Theta_M}(f) := S_{\Theta_M}(f, \boldsymbol{x}) := \sum_{j=1}^{m} \widehat{f}(\boldsymbol{k}^j) e^{i(\boldsymbol{k}^j, \boldsymbol{x})}, \quad \boldsymbol{x} \in \mathbb{R}^d,$$

and

$$\widehat{f}(\boldsymbol{k}^j) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(\boldsymbol{t}) e^{-i(\boldsymbol{k}^j, \boldsymbol{t})} d\boldsymbol{t}$$

are the Fourier coefficients of the function f that correspond to the set of vectors from Θ_M.

Respectively, for the function class $F \subset \mathscr{X}$ we set

$$e_M^{\perp}(F)_{\mathscr{X}} := \sup_{f \in F} e_M^{\perp}(f)_{\mathscr{X}}. \tag{8.42}$$

The quantity $e_M^{\perp}(F)_{\mathscr{X}}$ is called the best orthogonal trigonometric approximation of the class F in the space $\mathscr{X}$. The history of investigation of the quantity (8.42) for the classes $F = W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ and $F = B^{\boldsymbol{r}}_{p,\theta}$ in the spaces $\mathscr{X} = L_q$ and $\mathscr{X} = B_{q,1}$, $1 < q \le \infty$, is described in the papers [10, 19, 38, 39, 43, 45, 54] and in the monograph [37].

In this section we prove the results that concern the limiting case $B_{\infty,1}$. For the estimates of the best orthogonal trigonometric approximations of the Nikol'skii–Besov and Sobolev classes in case of other parameter values see Section 8.4 (Theorems T, 8.19, 8.20).

Let us formulate the statements that we will use in further argumentations.

Theorem K ([43]). *Let $1 \le p < \infty$, $1 \le \theta \le \infty$, $r_1 > 1/p$. Then for $d \ge 1$ it holds*

$$e_M^{\perp}(B^{\boldsymbol{r}}_{p,\theta})_{\infty} \asymp \left(M^{-1} \log^{\nu-1} M\right)^{r_1 - \frac{1}{p}} (\log^{\nu-1} M)^{1-\frac{1}{\theta}}.$$

Theorem L ([60], Ch. 1, § 3). *Let $d = 1$, $\alpha \in \mathbb{R}$ and $r_1 > 1$. Then it holds*

$$\mathscr{E}_{2^n}(W^{r_1}_{1,\alpha})_{\infty} \asymp 2^{-n(r_1-1)}.$$

Theorem M ([43]). *Let $1 < p < \infty$, $\alpha \in \mathbb{R}$ and $r_1 > 1/p$. Then for $d \ge 1$ it holds*

$$e_M^{\perp}(W^{\boldsymbol{r}}_{p,\alpha})_{\infty} \asymp \left(M^{-1} \log^{\nu-1} M\right)^{r_1 - \frac{1}{p}} \left(\log^{\nu-1} M\right)^{1-\frac{1}{p}}.$$

In what follows we consider first the uni-variate case.

Theorem 8.13. *Let $d = 1$, $1 \le p < \infty$, $1 \le \theta \le \infty$ and $r_1 > 1/p$. Then it holds*

$$e_M^{\perp}(B^{r_1}_{p,\theta})_{B_{\infty,1}} \asymp M^{-r_1 + \frac{1}{p}}. \tag{8.43}$$

Proof. Let us establish an upper estimate in (8.43). First, we note that due to the embedding of $B^{r_1}_{p,\theta} \subset H^{r_1}_p$, $1 \le \theta < \infty$, it suffices to obtain it for $\theta = \infty$, i.e. for the classes $H^{r_1}_p$.

So, let $M \in \mathbb{N}$ and $f \in H^{r_1}_p$. To approximate the function f, we will use a polynomial of the form (8.16), where the number n is related to M as $2^n \le M \le$

2^{n+1}. Then, according to the definition of the norm in the space $B_{\infty,1}$, taking into account the convolution property, we obtain

$$e_M^{\perp}(f)_{B_{\infty,1}} \ll \|f - S_n(f)\|_{B_{\infty,1}} = \left\| \sum_{s=n+1}^{\infty} \delta_s(f) \right\|_{B_{\infty,1}} = \sum_{s=n+1}^{\infty} \left\| A_s * \sum_{s'=s-1}^{s+1} \delta_{s'}(f) \right\|_{\infty}$$
$$\le \sum_{s=n+1}^{\infty} \|A_s\|_1 \left\| \sum_{s'=s-1}^{s+1} \delta_{s'}(f) \right\|_{\infty} := J_3. \tag{8.44}$$

To continue the estimation of the quantity J_3, we note that according to the relation $\|V_{2^s}\|_1 \le C_5$, $C_5 > 0$ (see, e.g., [60, Ch. 1, § 1]) we have

$$\|A_s\|_1 = \|V_{2^s} - V_{2^{s-1}}\|_1 \le \|V_{2^s}\|_1 + \|V_{2^{s-1}}\|_1 \le C_6, \quad C_6 > 0. \tag{8.45}$$

Next, we consider two cases depending on the values of the parameter p. Let $p = 1$ first. Then for the quantity J_3, taking into account (8.45), we obtain the estimate

$$J_3 \ll \sum_{s=n+1}^{\infty} \left\| \sum_{s'=s-1}^{s+1} \delta_{s'}(f) \right\|_{\infty} \le \sum_{s=n+1}^{\infty} \sum_{s'=s-1}^{s+1} \|\delta_{s'}(f)\|_{\infty}$$
$$\ll \sum_{s=n}^{\infty} \|\delta_s(f)\|_{\infty} = \sum_{s=n}^{\infty} \left\| \delta_s \left(\sum_{s'=s-1}^{s+1} A_{s'}(f) \right) \right\|_{\infty} := J_4.$$

Next, taking into account that the norm of the operator $\delta_s \colon \delta_s f = \delta_s(f)$, as an operator from L_1 to L_∞ does not exceed 2^s in order, we continue to estimate the quantity J_4:

$$J_4 \ll \sum_{s=n}^{\infty} 2^s \left\| \sum_{s'=s-1}^{s+1} A_{s'}(f) \right\|_1 \le \sum_{s=n}^{\infty} 2^s \sum_{s'=s-1}^{s+1} \|A_{s'}(f)\|_1 := J_5.$$

Taking into account that for the function $f \in H_1^{r_1}$ the relation (8.8) holds as $p = 1$, i.e., $\|A_{s'}(f)\|_1 \ll 2^{-s' r_1}$, we can write

$$J_5 \ll \sum_{s=n}^{\infty} 2^s \sum_{s'=s-1}^{s+1} 2^{-s' r_1} \ll \sum_{s=n}^{\infty} 2^s 2^{-s r_1} = \sum_{s=n}^{\infty} 2^{-s(r_1-1)} \ll 2^{-n(r_1-1)}. \tag{8.46}$$

Now consider the case $1 < p < \infty$. Using the inequality of different metrics, and also taking into account that for $f \in H_p^{r_1}$ according to (8.6) the condition $\|\delta_{s'}(f)\|_p \ll 2^{-s' r_1}$ is satisfied, we obtain

$$\left\| \sum_{s'=s-1}^{s+1} \delta_{s'}(f) \right\|_{\infty} \le \sum_{s'=s-1}^{s+1} 2^{\frac{s'}{p}} \|\delta_{s'}(f)\|_p \le \sum_{s'=s-1}^{s+1} 2^{\frac{s'}{p}} 2^{-s' r_1} \ll 2^{-s\left(r_1 - \frac{1}{p}\right)}. \tag{8.47}$$

Accordingly, for J_3, using (8.47), we can write the estimate

$$J_3 \leq \sum_{s=n+1}^{\infty} 2^{-s\left(r_1-\frac{1}{p}\right)} \asymp 2^{-n\left(r_1-\frac{1}{p}\right)}. \tag{8.48}$$

So, taking into account (8.44), (8.46), (8.48), as well as the relationship between the numbers M and n, we obtain the estimate

$$e_M^{\perp}(B_{p,\theta}^{r_1})_{B_{\infty,1}} \ll M^{-r_1+\frac{1}{p}}.$$

The estimate above is proved.

Regarding the lower estimate in (8.43), we note that it is a consequence of Theorem K under the condition $\nu = 1$, since according to the relation (8.11) ($\|f\|_\infty \ll \|f\|_{B_{\infty,1}}$), we can write

$$e_M^{\perp}(B_{p,\theta}^{r_1})_{B_{\infty,1}} \gg e_M^{\perp}(B_{p,\theta}^{r_1})_{\infty} \asymp M^{-r_1+\frac{1}{p}}.$$

Theorem 8.13 is proved. □

Corollary 8.1. *Let $d = 1$, $1 \leq p < \infty$, $1 \leq \theta \leq \infty$ and $r_1 > 1/p$. Then*

$$\mathcal{E}_{2^n}(B_{p,\theta}^{r_1})_{B_{\infty,1}} \asymp 2^{-n\left(r_1-\frac{1}{p}\right)}. \tag{8.49}$$

The upper estimate in (8.49) was established in the proof of Theorem 8.13. The corresponding lower estimate is also a consequence of this theorem, since for $2^n \leq M \leq 2^{n+1}$ we have

$$\mathcal{E}_{2^n}(B_{p,\theta}^{r_1})_{B_{\infty,1}} \gg e_M^{\perp}(B_{p,\theta}^{r_1})_{B_{\infty,1}} \asymp 2^{-n\left(r_1-\frac{1}{p}\right)}.$$

Further, we obtain statements similar to Theorem 8.13 and to Corollary 8.1 for the classes $W_{p,\alpha}^{r_1}$.

Theorem 8.14. *Let $d = 1$, $1 \leq p < \infty$, $\alpha \in \mathbb{R}$ and $r_1 > 1/p$. Then*

$$e_M^{\perp}(W_{p,\alpha}^{r_1})_{B_{\infty,1}} \asymp M^{-r_1+\frac{1}{p}}.$$

Proof. The upper estimate follows from Theorem 8.13 according to the embedding $W_{p,\alpha}^{r_1} \subset H_p^{r_1}$. The corresponding lower estimate for $p = 1$ is a consequence of Theorem L under the condition $2^n \leq M \leq 2^{n+1}$, and, in the case $1 < p < \infty$, is a consequence of Theorem M.

Theorem 8.14 is proved. □

Corollary 8.2. *Let $d = 1$, $1 \le p < \infty$, $\alpha \in \mathbb{R}$ and $r_1 > 1/p$. Then it holds*

$$\mathscr{E}_{2^n}(W^{r_1}_{p,\alpha})_{B_{\infty,1}} \asymp 2^{-n\left(r_1 - \frac{1}{p}\right)}.$$

Remark 8.15. *Analyzing Theorem 8.13 and Corollary 8.1 for the classes $B^{r_1}_{p,\theta}$ as well as Theorem 8.14 and Corollary 8.2 for the classes $W^{r_1}_{1,\alpha}$, we conclude that under the respective conditions the following relations hold:*

$$e^{\perp}_M(B^{r_1}_{p,\theta})_{B_{\infty,1}} \asymp e^{\perp}_M(B^{r_1}_{p,\theta})_{\infty}, \qquad e^{\perp}_M(W^{r_1}_{p,\alpha})_{B_{\infty,1}} \asymp e^{\perp}_M(W^{r_1}_{p,\alpha})_{\infty};$$
$$\mathscr{E}_{2^n}(B^{r_1}_{p,\theta})_{B_{\infty,1}} \asymp \mathscr{E}_{2^n}(B^{r_1}_{p,\theta})_{\infty}, \qquad \mathscr{E}_{2^n}(W^{r_1}_{p,\alpha})_{B_{\infty,1}} \asymp \mathscr{E}_{2^n}(W^{r_1}_{p,\alpha})_{\infty}.$$

In the following statements we consider the multidimensional case ($d \ge 2$).

Theorem 8.15. *Let $d \ge 2$, $1 \le p < \infty$, $1 \le \theta \le \infty$, $r_1 > 1/p$. Then it holds*

$$e^{\perp}_M(B^{\boldsymbol{r}}_{p,\theta})_{B_{\infty,1}} \asymp (M^{-1}\log^{\nu-1} M)^{r_1 - \frac{1}{p}} (\log^{\nu-1} M)^{1-\frac{1}{\theta}}. \tag{8.50}$$

Proof. Let us first establish the upper estimate. Let $f \in B^{\boldsymbol{r}}_{p,\theta}$ and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_d)$. Then, setting $\gamma(d) := \gamma_1 + \cdots + \gamma_d$ and choosing the number $n = n(M) \in \mathbb{N}$ from the condition $M \asymp 2^n n^{\nu-1}$ and using the convolution property, we can write

$$e^{\perp}_M(f)_{B_{\infty,1}} \ll \Bigg\| f - \sum_{(\boldsymbol{s},\boldsymbol{\gamma})<n} \delta_{\boldsymbol{s}}(f) \Bigg\|_{B_{\infty,1}} = \Bigg\| \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\ge n} \delta_{\boldsymbol{s}}(f) \Bigg\|_{B_{\infty,1}} =$$
$$= \sum_{\boldsymbol{s}\in\mathbb{N}^d} \Bigg\| A_{\boldsymbol{s}} * \sum_{\substack{\boldsymbol{s}'\in\mathbb{N}^d \\ (\boldsymbol{s}',\boldsymbol{\gamma})\ge n}} \delta_{\boldsymbol{s}'}(f) \Bigg\|_{\infty} \le \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\ge n-\gamma(d)} \Bigg\| A_{\boldsymbol{s}} * \sum_{\substack{\boldsymbol{s}'\in\mathbb{N}^d \\ \|\boldsymbol{s}-\boldsymbol{s}'\|_\infty \le 1}} \delta_{\boldsymbol{s}'}(f) \Bigg\|_{\infty} \le$$
$$\le \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\ge n-\gamma(d)} \|A_{\boldsymbol{s}}\|_1 \Bigg\| \sum_{\substack{\boldsymbol{s}'\in\mathbb{N}^d \\ \|\boldsymbol{s}-\boldsymbol{s}'\|_\infty \le 1}} \delta_{\boldsymbol{s}'}(f) \Bigg\|_{\infty} \ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\ge n-\gamma(d)} \sum_{\substack{\boldsymbol{s}'\in\mathbb{N}^d \\ \|\boldsymbol{s}-\boldsymbol{s}'\|_\infty \le 1}} \|\delta_{\boldsymbol{s}'}(f)\|_\infty \ll$$
$$\ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\ge n-2\gamma(d)} \|\delta_{\boldsymbol{s}}(f)\|_\infty := J_6. \tag{8.51}$$

Next, we first consider the case $p = 1$. Taking into account that the norm of the operator $\delta_{\boldsymbol{s}}\colon \delta_{\boldsymbol{s}} f = \delta_{\boldsymbol{s}}(f)$, as an operator from L_1 to L_∞ does not exceed $2^{(\boldsymbol{s},\mathbf{1})}$ in order, where $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}^d$, for the quantity J_6 we obtain

$$J_6 = \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\ge n-2\gamma(d)} \Bigg\| \delta_{\boldsymbol{s}} \Bigg(\sum_{\|\boldsymbol{s}-\boldsymbol{s}'\|_\infty \le 1} A_{\boldsymbol{s}'}(f) \Bigg) \Bigg\|_\infty \ll$$

$$\ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n-2\gamma(d)} 2^{(\boldsymbol{s},\boldsymbol{1})} \sum_{\|\boldsymbol{s}-\boldsymbol{s}'\|_\infty\leq 1} \|A_{\boldsymbol{s}'}(f)\|_1 \ll$$
$$\ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n-3\gamma(d)} 2^{(\boldsymbol{s},\boldsymbol{1})}\|A_{\boldsymbol{s}}(f)\|_1 := J_7. \tag{8.52}$$

To continue estimating the value of J_7, let us consider several cases.

a) Let $1<\theta<\infty$. Then, applying to J_7 the Hölder inequality with the exponent θ and taking into account (8.7), we can write

$$J_7 \leq \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n-3\gamma(d)} 2^{(\boldsymbol{s},\boldsymbol{r})\theta}\|A_{\boldsymbol{s}}(f)\|_1^\theta\right)^{\frac{1}{\theta}} \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n-3\gamma(d)} 2^{-(\boldsymbol{s},\boldsymbol{r}-\boldsymbol{1})\frac{\theta}{\theta-1}}\right)^{1-\frac{1}{\theta}} \ll$$
$$\ll \|f\|_{B^{\boldsymbol{r}}_{1,\theta}} \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n-3\gamma(d)} 2^{-(\boldsymbol{s},\boldsymbol{r}-\boldsymbol{1})\frac{\theta}{\theta-1}}\right)^{1-\frac{1}{\theta}} \leq \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n-3\gamma(d)} 2^{-(\boldsymbol{s},\tilde{\boldsymbol{\gamma}})(r_1-1)\frac{\theta}{\theta-1}}\right)^{1-\frac{1}{\theta}} := J_8,$$

where $\tilde{\boldsymbol{\gamma}}=(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_d)$ is the vector with coordinates $\tilde{\gamma}_j=(r_j-1)/(r_1-1)$, $j=1,\ldots,d$, and $\boldsymbol{r}-\boldsymbol{1}$ denotes vector with coordinates r_j-1, $j=1,\ldots,d$. It is easy to see that $\tilde{\gamma}_j=\gamma_j=1$, $j=1,\ldots,\nu$, and $1<\gamma_j<\tilde{\gamma}_j$ for $j=\nu+1,\ldots d$, and therefore, using Lemma A for $M\asymp 2^n n^{\nu-1}$, we obtain

$$J_8 \ll 2^{-n(r_1-1)} n^{(\nu-1)\left(1-\frac{1}{\theta}\right)} \asymp (M^{-1}\log^{\nu-1}M)^{r_1-1}(\log^{\nu-1}M)^{1-\frac{1}{\theta}}. \tag{8.53}$$

Combining (8.51)–(8.53) we arrive at the desired upper estimate of the quantity $e^\perp_M(B^{\boldsymbol{r}}_{1,\theta})_{B_{\infty,1}}$ in the case $1<\theta<\infty$.

b) Let $\theta=1$. Then we estimate J_7 as follows:

$$J_7 = \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n-3\gamma(d)} 2^{(\boldsymbol{s},\boldsymbol{r})}\|A_{\boldsymbol{s}}(f)\|_1 2^{-(\boldsymbol{s},\boldsymbol{r})}2^{(\boldsymbol{s},\boldsymbol{1})} \ll$$
$$\ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n-3\gamma(d)} 2^{(\boldsymbol{s},\boldsymbol{r})}\|A_{\boldsymbol{s}}(f)\|_1 2^{-(\boldsymbol{s},\boldsymbol{\gamma})(r_1-1)} \leq \|f\|_{B^{\boldsymbol{r}}_{1,\theta}} \sup_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n-3\gamma(d)} 2^{-(\boldsymbol{s},\boldsymbol{\gamma})(r_1-1)} \ll$$
$$\ll 2^{-n(r_1-1)} \asymp (M^{-1}\log^{\nu-1}M)^{r_1-1}. \tag{8.54}$$

From (8.51), (8.52) and (8.54) we obtain the desired upper estimate of the quantity $e^\perp_M(B^{\boldsymbol{r}}_{1,1})_{B_{\infty,1}}$.

c) In the case of $\theta=\infty$, taking into account that for $f\in B^{\boldsymbol{r}}_{1,\infty}\equiv H^{\boldsymbol{r}}_1$ according to (8.8) the relation $\|A_{\boldsymbol{s}}(f)\|_1\ll 2^{-(\boldsymbol{s},\boldsymbol{r})}$, $\boldsymbol{s}\in\mathbb{N}^d$, is true, we have

$$J_7 \ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n-3\gamma(d)} 2^{(\boldsymbol{s},\boldsymbol{1})}2^{-(\boldsymbol{s},\boldsymbol{r})} \leq \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n-3\gamma(d)} 2^{-(\boldsymbol{s},\tilde{\boldsymbol{\gamma}})(r_1-1)} \ll$$

$$\ll 2^{-n(r_1-1)}n^{\nu-1} \asymp (M^{-1}\log^{\nu-1} M)^{r_1-1}(\log^{\nu-1} M). \tag{8.55}$$

Combining (8.51), (8.52), (8.55) we get the desired upper estimate of the quantity $e^{\perp}_M(B^{\boldsymbol{r}}_{1,\infty})_{B_{\infty,1}}$.

Now consider the case $1 < p < \infty$. Using the inequality of different metrics, we can continue the estimate for J_6 as follows

$$J_6 = \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n-2\gamma(d)} \|\delta_{\boldsymbol{s}}(f)\|_\infty \leq \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\geq n-2\gamma(d)} 2^{\frac{\|\boldsymbol{s}\|_1}{p}} \|\delta_{\boldsymbol{s}}(f)\|_p, \tag{8.56}$$

where $\|\boldsymbol{s}\|_1 := s_1 + \cdots + s_d$.

The further evaluation in (8.56) is carried out in a similar way as in the case of $p = 1$ for J_7, with the difference that for the classes $B^{\boldsymbol{r}}_{1,\theta}$ the decomposition definitions of the norm (8.5) and (8.6) are used.

As a result, we obtain that for $1 < p < \infty$ it holds

$$e^{\perp}_M(B^{\boldsymbol{r}}_{p,\theta})_{B_{\infty,1}} \ll (M^{-1}\log^{\nu-1} M)^{r_1-\frac{1}{p}}(\log^{\nu-1} M)^{1-\frac{1}{\theta}}.$$

The upper estimate on (8.50) is proved.

The lower estimate on (8.50) is a consequence of Theorem K according to the relation (8.11).

Theorem 8.15 is proved. □

Corollary 8.3. *Let $d \geq 2$, $1 \leq p < \infty$, $1 \leq \theta \leq \infty$ and $r_1 > 1/p$. Then it holds*

$$\mathscr{E}_{Q_n^\gamma}(B^{\boldsymbol{r}}_{p,\theta})_{B_{\infty,1}} \asymp 2^{-n\left(r_1-\frac{1}{p}\right)} n^{(\nu-1)\left(1-\frac{1}{\theta}\right)}. \tag{8.57}$$

Let us formulate two statements similar to Theorem 8.15 and Corollary 8.4, but for the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$, the estimates in which are obtained taking into account the embedding (8.9) and Theorem M.

Theorem 8.16. *Let $d \geq 2$, $2 \leq p < \infty$, $\alpha \in \mathbb{R}$ and $r_1 > 1/p$. Then it holds*

$$e^{\perp}_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{\infty,1}} \asymp (M^{-1}\log^{\nu-1} M)^{r_1-\frac{1}{p}}(\log^{\nu-1} M)^{1-\frac{1}{p}}.$$

Corollary 8.4. *Let $d \geq 2$, $2 \leq p < \infty$, $\alpha \in \mathbb{R}$ and $r_1 > 1/p$. Then it holds*

$$\mathscr{E}_{Q_n^\gamma}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{\infty,1}} \asymp 2^{-n\left(r_1-\frac{1}{p}\right)} n^{(\nu-1)\left(1-\frac{1}{p}\right)}.$$

Remark 8.16. *From Theorems 8.15, 8.16 and Corollaries 8.3, 8.4 it follows that, in contrast to the one-dimensional case ($d = 1$) the values of the considered approximation characteristics of the classes $B^{\boldsymbol{r}}_{p,\theta}$ and $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ differ in order. In addition, in the case $d \geq 2$, the estimates that were obtained in Theorem 8.15 and Corollary 8.3 depend on the parameter θ.*

Furthermore, as a consequence of the estimates of the quantity $\mathscr{E}_{Q_n^\gamma}(B^{\boldsymbol{r}}_{1,\theta})_{B_{\infty,1}}$ and the known results of the approximation of these classes in the space L_∞, we establish the orders of the orthowidths $d^\perp_M(B^{\boldsymbol{r}}_{1,\theta}, B_{\infty,1})$. Let us recall the definition of the corresponding approximation characteristic.

Let $\{u_i\}_{i=1}^M$ be an orthonormal system of functions $u_i \in L_\infty$ in the space L_2, $i = 1, \dots, M$. To each function $f \in \mathscr{X}$ we assign an approximation aggregate of the form $\sum_{i=1}^M (f, u_i)u_i$, i.e., an orthogonal projection of the function f onto the subspace generated by the system of functions $\{u_i\}_{i=1}^M$. Here

$$(f, u_i) = (2\pi)^{-d} \int_{\mathbb{T}^d} f(\boldsymbol{x})\overline{u}_i(\boldsymbol{x})d\boldsymbol{x}.$$

For $F \subset \mathscr{X}$ the quantity

$$d^\perp_M(F, \mathscr{X}) = \inf_{\{u_i\}_{i=1}^M \subset L_\infty} \sup_{f\in F} \left\| f - \sum_{i=1}^M (f, u_i)u_i \right\|_{\mathscr{X}}$$

is called the orthowidth (Fourier-width) of the class F in the space $\mathscr{X}$. The orthowidth $d^\perp_M(F, L_q)$ was introduced by V. N. Temlyakov [68] and, in particular, on the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$, $H^{\boldsymbol{r}}_p$ and $B^{\boldsymbol{r}}_{p,\theta}$ in the spaces L_q, $1 \le q \le \infty$, $B_{p,1}$, $p \in \{1, \infty\}$ was studied in the papers [2, 7, 8, 21, 41, 46–48, 50, 54, 64] (see also the monographs [18, 37, 58, 60]). In these works, one can find the history of the study of the corresponding quantities on other functional classes.

Let us formulate one result, which we will use when obtaining the order estimate of the quantity $d^\perp_M(B^{\boldsymbol{r}}_{1,\theta}, B_{\infty,1})$.

Theorem N. *Let $d \ge 1$, $1 \le p < \infty$, $1 \le \theta \le \infty$. Then for $r_1 > 1/p$ it holds*

$$d^\perp_M(B^{\boldsymbol{r}}_{p,\theta}, L_\infty) \asymp (M^{-1}\log^{\nu-1} M)^{r_1-\frac{1}{p}}(\log^{\nu-1} M)^{1-\frac{1}{\theta}}.$$

Note that for $\theta = \infty$, i.e., for the classes $H^{\boldsymbol{r}}_p$, the respective estimate was obtained in [64], and for $1 \le \theta < \infty$ in [46].

The next corollary concerns orthowidths of the classes $B^{\boldsymbol{r}}_{1,\theta}$ in the space $B_{\infty,1}$.

Corollary 8.5. *Let $d \ge 1$, $1 \le \theta \le \infty$. The for $r_1 > 1$ it holds*

$$d^\perp_M(B^{\boldsymbol{r}}_{1,\theta}, B_{\infty,1}) \asymp M^{-r_1+1}(\log^{\nu-1} M)^{r_1-\frac{1}{\theta}}. \tag{8.58}$$

The upper estimate follows from (8.49) and (8.57) for an appropriate choice of the numbers n, and the lower estimate according to the inequality (8.11) is a consequence of Theorem N for $p = 1$.

Note that the estimate (8.58) complements the corresponding result for $B^{\boldsymbol{r}}_{p,\theta}$, $1 < p < \infty$ established in [50].

Finally, we present two corollaries concerning the corresponding approximation characteristics of the Sobolev classes $W^{\boldsymbol{r}}_{1,\mathbf{0}}$. For this, we also need an auxiliary statement.

Theorem O ([64]). *Let $d=2$, $\boldsymbol{r}=(r_1,r_1)$, $r_1>1$. Then it holds*

$$d^{\perp}_{M}(W^{\boldsymbol{r}}_{1,\mathbf{0}}, L_\infty) \asymp M^{-r_1+1}(\log M)^{r_1}.$$

Corollary 8.6. *Let $d=2$, $\boldsymbol{r}=(r_1,r_1)$, $r_1>1$. Then it holds*

$$d^{\perp}_{M}(W^{\boldsymbol{r}}_{1,\mathbf{0}}, B_{\infty,1}) \asymp M^{-r_1+1}(\log M)^{r_1}. \tag{8.59}$$

The estimate from above follows from (8.58) as $\theta=\infty$ due to the embedding $W^{\boldsymbol{r}}_{1,\mathbf{0}} \subset H^{\boldsymbol{r}}_1$. The corresponding estimate from below in (8.59) is obtained from Theorem O according to the relation (8.11).

Corollary 8.7. *Let $d=2$, $\boldsymbol{r}=(r_1,r_1)$, $r_1>1$. Then it holds*

$$\mathscr{E}_{Q^{\mathbf{1}}_n}(W^{\boldsymbol{r}}_{1,\mathbf{0}})_\infty \asymp \mathscr{E}_{Q^{\mathbf{1}}_n}(W^{\boldsymbol{r}}_{1,\mathbf{0}})_{B_{\infty,1}} \asymp 2^{-n(r_1-1)}n. \tag{8.60}$$

Note that it is sufficient to establish the upper estimate in (8.60) for the quantity $\mathscr{E}_{Q^{\mathbf{1}}_n}(W^{\boldsymbol{r}}_{1,\mathbf{0}})_{B_{\infty,1}}$, and the lower estimate for $\mathscr{E}_{Q^{\mathbf{1}}_n}(W^{\boldsymbol{r}}_{1,\mathbf{0}})_\infty$.

Therefore, the upper estimate for the quantity $\mathscr{E}_{Q^{\mathbf{1}}_n}(W^{\boldsymbol{r}}_{1,\mathbf{0}})_{B_{\infty,1}}$ follows from Corollary 8.3 under the condition $d=2$, $\theta=\infty$ according to the embedding $W^{\boldsymbol{r}}_{1,\mathbf{0}} \subset H^{\boldsymbol{r}}_1$. The lower estimate for the quantity $\mathscr{E}_{Q^{\mathbf{1}}_n}(W^{\boldsymbol{r}}_{1,\mathbf{0}})_\infty$ follows from Theorem O under the condition that the number $n\in\mathbb{N}$ is chosen for a given M from the relation $M \asymp 2^n n$.

8.4 Best M-term trigonometric approximations

Let $\mathscr{X}$ be a normed space with the norm $\|\cdot\|_{\mathscr{X}}$ and $\Theta_M := \{\boldsymbol{k}^1,\dots,\boldsymbol{k}^M\}$ be a set of vectors $\boldsymbol{k}^j\in\mathbb{Z}^d$, $j=1,\dots,M$.

We will consider trigonometric polynomials of the form

$$P(\Theta_M) := P(\Theta_m,\boldsymbol{x}) := \sum_{\boldsymbol{k}\in\Theta_M} c_{\boldsymbol{k}} e^{i(\boldsymbol{k},\boldsymbol{x})}, \quad c_{\boldsymbol{k}}\in\mathbb{C},$$

and for the function $f\in\mathscr{X}$ introduce the quantity

$$e_M(f)_{\mathscr{X}} := \inf_{c_{\boldsymbol{k}}}\inf_{\Theta_M}\|f-P(\Theta_M)\|_{\mathscr{X}}.$$

If $F\subset\mathscr{X}$ is a function class, then we set

$$e_M(F)_{\mathscr{X}} := \sup_{f\in F} e_M(f)_{\mathscr{X}}. \tag{8.61}$$

The approximation characteristic $e_M(F)_{\mathscr{X}}$ is called the best M-term trigonometric approximation of the class F in the space $\mathscr{X}$.

The quantity $e_M(f)_{L_2}$ for functions of one variable in a more general case was introduced by S.B. Stechkin [56] when formulating the criterion of absolute convergence of orthogonal series. We note that the quantity (8.61) has a rich investigation history on different function classes. Let us mention several papers that are related to our research. For the introduced Sobolev $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ and Nikol'skii–Besov $B^{\boldsymbol{r}}_{p,\theta}$ classes in the spaces $\mathscr{X} = L_q$, $d \geq 1$, the estimates of the best M-term trigonometric approximations were obtained, in particular, in the papers [10, 11, 17, 24, 42, 43, 62, 69], and in [45] the respective results were applied to estimate the best bilinear approximations. For isotropic Besov classes, the quantity (8.61) was studied in [15, 55], for Nikol'skii–Besov and Sobolev–type classes see, e.g., papers [1, 6, 27, 57], and for the Lizorkin–Triebel and Wiener classes see [23, 28]. A more detailed bibliography can be found in the monographs [18, 37, 58, 70]. We also note the recent papers [33, 52], where the quantity (8.61) was studied for isotropic Nikol'skii–Besov classes in the space $B_{q,1}$.

Immediately from the definitions of the quantities (8.61) and (8.42) the following relation follows:

$$e_M(F)_{\mathscr{X}} \leq e^{\perp}_M(F)_{\mathscr{X}}.$$

To formulate the known statements that we use in the proofs, we first introduce some more additional notation.

Let D is a bounded set in $\mathbb{R}^d$, $d \in \mathbb{N}$, and $\Phi = \{\varphi_n(\boldsymbol{x})\}_{n=1}^{\infty}$ is a system of functions from $L_q(D)$, $1 \leq q \leq \infty$. For $f \in L_q(D)$ we set

$$e_M(f, \Phi)_{L_q(D)} := \inf_{\substack{\{n_j\}=\Lambda\in\mathbb{Z}_+, |\Lambda|=M \\ \{c_j\}\in\mathbb{R}^M}} \left\| f - \sum_{j=1}^{M} c_j \varphi_{n_j} \right\|_{L_q(D)},$$

where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Further, if $K \subset L_q(D)$ is some class of functions, we define

$$e_M(K, \Phi)_{L_q(D)} := \sup_{f\in K} e_M(f, \Phi)_{L_q(D)}. \qquad (8.62)$$

Remark 8.17. *In the case of trigonometric system* $T := \{e^{i(\boldsymbol{k},\boldsymbol{x})}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$, *we will write* (8.62) *as*

$$e_M(K, T)_{L_q(D)} = e_M(K, \{e^{i(\boldsymbol{k},\boldsymbol{x})}\}_{\boldsymbol{k}\in\mathbb{Z}^d})_{L_q(D)} := e_M(K)_q.$$

In what follows for the vector $\boldsymbol{s} \in \mathbb{N}^d$ with even components s_j, $j = 1, \ldots, d$, we denote

$$\rho^+(\boldsymbol{s}) := \{\boldsymbol{k} \in \mathbb{N}^d\colon\ 2^{s_j-1} \leq k_j < 2^{s_j},\ j = 1, \ldots, d\}$$

and for $n \in \mathbb{N}$ set

$$D_n := \left\{ \boldsymbol{s} : (\boldsymbol{s}, \mathbf{1}) = 2 \left[\frac{n}{2} \right] \right\}, \quad \mathcal{Y}_n := \bigcup_{\boldsymbol{s} \in D_n} \rho^+(\boldsymbol{s}),$$

where $[a]$ is the integer part of the number a.

Note that for the number of elements in the sets D_n and $\mathcal{Y}_n$ the following relations hold $|D_n| \asymp n^{d-1}$, $\mathcal{Y}_n \asymp 2^n n^{d-1}$.

Let $\mathcal{T}(Y_n)$ is a set of polynomials of the form

$$t(\boldsymbol{x}) := \sum_{|\boldsymbol{k}| \in \mathcal{Y}_n} c_{\boldsymbol{k}} e^{i(\boldsymbol{k}, \boldsymbol{x})},$$

where $|\boldsymbol{k}| = (|k_1|, \ldots, |k_d|)$.

If $\mathscr{X}$ is a normed space with the norm $\| \cdot \|_{\mathscr{X}}$, by $\mathcal{T}(\mathcal{Y}_n)_{\mathscr{X}}$ we denote the unit ball in the space $\mathcal{T}(\mathcal{Y}_n)$.

In the introduced notation the following statement holds.

Theorem P ([24]). *There exists a constant $C_7(d) > 0$, such that for any set of functions $\Phi = \{\varphi_j\}_{j=1}^l \subset B_{1,1}$, $l \le C_8 |\mathcal{Y}_n|$ the following estimate holds*

$$e_M(\mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}}, \Phi)_{B_{1,1}} \ge C_8 n^{d-1}, \quad C_9 = C_9(d, C_7) > 0$$

for all $M \le C_7(d)|\mathcal{Y}_n|$ (for the definition of the norm in the $B_{\infty,\infty}$ see Remark 8.1).

Theorem Q ([52]). *Let $d = 1$, $1 \le q \le p \le \infty$, $1 \le \theta \le \infty$. Then for $r_1 > 0$ the following estimate holds*

$$e_M(B^{r_1}_{p,\theta})_{B_{q,1}} \asymp M^{-r_1}.$$

Theorem R ([15, 44]). *Let $d = 1$, $1 \le q \le p \le \infty$, $1 \le \theta \le \infty$. Then for $r_1 > 0$ the following estimate holds*

$$e_M(B^{r_1}_{p,\theta})_q \asymp M^{-r_1}.$$

Theorem S ([42]). *Let $d \ge 2$, $1 < q \le p < \infty$, $p \ge 2$, $1 \le \theta \le \infty$. Then for $r_1 > 0$ the following estimate holds*

$$e_M(B^{\boldsymbol{r}}_{p,\theta})_q \asymp M^{-r_1} (\log^{\nu-1} M)^{\left(r_1 + \frac{1}{2} - \frac{1}{\theta}\right)_+},$$

where $a_+ := \max\{a, 0\}$.

Further we formulate the known statements that concern the best orthogonal trigonometric approximations.

Theorem T ([52]). *Let $d = 1$, $1 \le p, q, \theta \le \infty$, $(p, q) \notin \{(1, 1), (\infty, \infty)\}$. Then for $r_1 > (1/p - 1/q)_+$ the following estimate holds*

$$e^{\perp}_M(B^{r_1}_{p,\theta})_{B_{q,1}} \asymp M^{-r_1 + \left(\frac{1}{p} + \frac{1}{q}\right)_+}. \tag{8.63}$$

Remark 8.18. *Under the conditions of Theorem T, for the quantity $e^{\perp}_M(B^{r_1}_{p,\theta})_q$ the estimate* (8.63) *holds (see, e.g., [39]).*

Theorem U ([39])**.** *Let $d \geq 2$, $1 < q \leq p < \infty$, $p \geq 2$, $1 \leq \theta \leq \infty$. Then for $r_1 > 0$ the following estimate holds*

$$e^{\perp}_M(B^{\boldsymbol{r}}_{p,\theta})_q \asymp M^{-r_1}(\log^{\nu-1} M)^{\left(r_1+\frac{1}{2}-\frac{1}{\theta}\right)_+}.$$

Theorem V ([39])**.** *Let $d \geq 2$, $1 < q < p \leq 2$. Then it holds:*
a) if either $1 \leq \theta < p$ and $r_1 \geq 1/\theta - 1/p$ or $\theta \geq p$ and $r_1 > 0$, then

$$M^{-r_1}(\log^{\nu-1} M)^{r_1+\frac{1}{2}-\frac{1}{\theta}} \ll e^{\perp}_M(B^{\boldsymbol{r}}_{p,\theta})_q \ll M^{-r_1}(\log^{\nu-1} M)^{r_1+\frac{1}{p}-\frac{1}{\theta}};$$

b) if $1 \leq \theta < p$ and $0 < r_1 < 1/\theta - 1/p$, then

$$e^{\perp}_M(B^{\boldsymbol{r}}_{p,\theta})_q \asymp M^{-r_1}.$$

Theorem W ([39])**.** *Let $d \geq 2$, $1 < p < 2$, $1 \leq \theta \leq \infty$. Then for $r_1 > 0$ the following estimate holds*

$$e^{\perp}_M(B^{\boldsymbol{r}}_{p,\theta})_p \asymp M^{-r_1}(\log^{\nu-1} m)^{\left(r_1+\frac{1}{p}-\frac{1}{\theta}\right)_+}.$$

The following statement is true.

Theorem 8.17. *Let $d \geq 2$, $1 < q \leq p \leq \infty$, $1 \leq \theta \leq \infty$. Then for $r_1 > 0$ the following estimate holds*

$$e_M(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}} \asymp M^{-r_1}(\log^{\nu-1} m)^{r_1+1-\frac{1}{\theta}}. \tag{8.64}$$

Proof. We first establish the upper estimate. Note that in view of the embedding $B^{\boldsymbol{r}}_{p,\theta} \subset B^{\boldsymbol{r}}_{q,\theta}$, $1 < q < p$, it is sufficient to consider the case $p = q$.

So, let $f \in B^{\boldsymbol{r}}_{q,\theta}$, $1 < q \leq \infty$, $1 \leq \theta \leq \infty$. As an approximation aggregate for function f we consider the polynomial

$$t_n := t_n(\boldsymbol{x}) = \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')<n} A_{\boldsymbol{s}}(f,\boldsymbol{x}),$$

where the number $n \in \mathbb{N}$ satisfies the condition $M \asymp 2^n n^{\nu-1}$.

Then defining $\boldsymbol{\gamma}'(d) := \gamma'_1 + \cdots + \gamma'_d$, by the norm definition in the space $B_{q,1}$ and the properties of convolution, we can write

$$e_M(f)_{B_{q,1}} \ll \|f - t_n\|_{B_{q,1}} = \left\| \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n} A_{\boldsymbol{s}}(f) \right\|_{B_{q,1}} =$$

$$= \sum_{\boldsymbol{s}\in\mathbb{N}^d} \left\| A_{\boldsymbol{s}} * \sum_{\substack{\boldsymbol{s}'\in\mathbb{N}^d \\ (\boldsymbol{s}',\boldsymbol{\gamma}')\geq n}} A_{\boldsymbol{s}'}(f) \right\|_q \leq \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n-\gamma'(d)} \left\| A_{\boldsymbol{s}} * \sum_{\|\boldsymbol{s}-\boldsymbol{s}'\|_\infty\leq 1} A_{\boldsymbol{s}'}(f) \right\|_q \leq$$

$$\leq \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n-\gamma'(d)} \|A_{\boldsymbol{s}}\|_1 \left\| \sum_{\|\boldsymbol{s}-\boldsymbol{s}'\|_\infty\leq 1} A_{\boldsymbol{s}'}(f) \right\|_q := J_9. \tag{8.65}$$

Further, using (8.45) we can continue estimating the quantity J_9 as follows

$$J_9 \ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n-\gamma'(d)} \left\| \sum_{\|\boldsymbol{s}-\boldsymbol{s}'\|_\infty\leq 1} A_{\boldsymbol{s}'}(f) \right\|_q \ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n-2\gamma'(d)} \|A_{\boldsymbol{s}}(f)\|_q. \tag{8.66}$$

Let us consider first the case $1 \leq \theta < \infty$ and write (8.66) in the form

$$J_9 \ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n-2\gamma'(d)} 2^{(\boldsymbol{s},\boldsymbol{r})}\|A_{\boldsymbol{s}}(f)\|_q 2^{-(\boldsymbol{s},\boldsymbol{r})} := J_{10}. \tag{8.67}$$

Using now for the expression J_2 Hölder's inequality with exponent θ (with corresponding modification of this inequality for $\theta = 1$), and Lemma A, we get

$$J_{10} \leq \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n-2\gamma'(d)} 2^{(\boldsymbol{s},\boldsymbol{r})\theta}\|A_{\boldsymbol{s}}(f)\|_q^\theta \right)^{\frac{1}{\theta}} \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n-2\gamma'(d)} 2^{-(\boldsymbol{s},\boldsymbol{r})\theta'} \right)^{\frac{1}{\theta'}} \leq$$

$$\leq \|f\|_{B^{\boldsymbol{r}}_{q,\theta}} \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n-2\gamma'(d)} 2^{-(\boldsymbol{s},\boldsymbol{r})\theta'} \right)^{\frac{1}{\theta'}} \leq \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n-2\gamma'(d)} 2^{-(\boldsymbol{s},\boldsymbol{\gamma})r_1\theta'} \right)^{\frac{1}{\theta'}} \asymp$$

$$\asymp 2^{-nr_1} n^{(\nu-1)\left(1-\frac{1}{\theta}\right)}, \quad 1/\theta + 1/\theta' = 1. \tag{8.68}$$

So, combining (8.65), (8.67) with (8.68) and taking into account that $M \asymp 2^n n^{\nu-1}$, we get the estimate

$$e_M(f)_{B_{q,1}} \ll M^{-r_1}(\log^{\nu-1} M)^{r_1+1-\frac{1}{\theta}}, \quad 1 \leq \theta < \infty.$$

Let $\theta = \infty$. For $f \in B^{\boldsymbol{r}}_{q,\infty}$ from (8.8) we have that $\|A_{\boldsymbol{s}}(f)\|_q \ll 2^{-(\boldsymbol{s},\boldsymbol{r})}$. Then in view of Lemma A and the fact that $M \asymp 2^n n^{\nu-1}$, we continue the relation (8.66) as follows

$$e_M(f)_{B_{q,1}} \ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n-2\gamma'(d)} \|A_{\boldsymbol{s}}(f)\|_q \ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n-2\gamma'(d)} 2^{-(\boldsymbol{s},\boldsymbol{r})} =$$

$$= \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\geq n-2\boldsymbol{\gamma}'(d)} 2^{-(\boldsymbol{s},\boldsymbol{\gamma})r_1} \asymp 2^{-nr_1}n^{\nu-1} \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+1}.$$

The upper estimate is proved.

Moving to the lower estimate in (8.64) we note that it suffices to consider the case $p = \infty$, $1 < q < \infty$ and $\nu = d$.

Let the number $n \in \mathbb{N}$ satisfies the relation $M \asymp 2^n n^{\nu-1}$ and $P_{\mathcal{Y}_n}$ denotes the operator of orthogonal projection on $\mathcal{T}(\mathcal{Y}_n)$. Then it is simple to show that the norm of the operator $P_{\mathcal{Y}_n}$, as an operator from $B_{q,1}$ to $B_{q,1}$ (notation $\|P_{\mathcal{Y}_n}\|_{B_{q,1}\to B_{q,1}}$), is bounded for $1 < q < \infty$. We have

$$\|P_{\mathcal{Y}_n}\|_{B_{q,1}\to B_{q,1}} = \sup_{\|f\|_{B_{q,1}}\leq 1} \left\| \sum_{|\boldsymbol{k}|\in\mathcal{Y}_n} \widehat{f}(\boldsymbol{k})e^{i(\boldsymbol{k},\boldsymbol{x})} \right\|_{B_{q,1}} \asymp \sup_{\|f\|_{B_{q,1}}\leq 1} \sum_{\boldsymbol{s}\in D_n} \|\delta_{\boldsymbol{s}}(f)\|_q \leq$$
$$\leq \sup_{\|f\|_{B_{q,1}}\leq 1} \sum_{\boldsymbol{s}\in\mathbb{N}^d} \|\delta_{\boldsymbol{s}}(f)\|_q \leq C_{10}, \quad C_{10} > 0. \tag{8.69}$$

Hence, in view of (8.69), for the trigonometric system $T = \{e^{i(\boldsymbol{k},\boldsymbol{x})}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ we can write

$$e_M(\mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}})_{B_{q,1}} = e_M(\mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}}, T)_{B_{q,1}} \geq$$
$$\geq C_{11}e_M(\mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}}, \{e^{i(\boldsymbol{k},\boldsymbol{x})}\}_{|\boldsymbol{k}|\in\mathcal{Y}_n})_{B_{q,1}}. \tag{8.70}$$

In what follows, to use the relation (8.70) for estimate of $e_M(B^{\boldsymbol{r}}_{\infty,\theta})_{B_{q,1}}$ we consider two cases.

a) Let $1 \leq \theta < \infty$. Then for the polynomial $t \in \mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}}$ we get

$$\|t\|_{B^{\boldsymbol{r}}_{\infty,\theta}} \asymp \left(\sum_{\boldsymbol{s}\in D_n} 2^{(\boldsymbol{s},\boldsymbol{r})\theta}\|A_{\boldsymbol{s}}(t)\|_\infty^\theta\right)^{1/\theta} \leq 2^{nr_1} \max_{\boldsymbol{s}\in D_n} \|A_{\boldsymbol{s}}(t)\|_\infty \left(\sum_{\boldsymbol{s}\in D_n} 1\right)^{\frac{1}{\theta}} \ll$$
$$\ll 2^{nr_1}\|t\|_{B_{\infty,\infty}} n^{\frac{d-1}{\theta}}.$$

That yields existing of a constant $C_{12}(r, d, \theta) > 0$, such that it holds

$$C_{12}(r, d, \theta)2^{-nr_1}n^{-\frac{d-1}{\theta}}\mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}} \subset B^{\boldsymbol{r}}_{\infty,\theta} \cap \mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}}.$$

b) Let $\theta = \infty$. Then for $t \in \mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}}$ we get

$$\|t\|_{B^{\boldsymbol{r}}_{\infty,\infty}} \asymp 2^{nr_1} \max_{\boldsymbol{s}\in D_n} \|A_{\boldsymbol{s}}(t)\|_\infty \ll 2^{nr_1}\|t\|_{B_{\infty,\infty}}$$

and conclude that with some constant $C_{13}(r, d) > 0$ the following embedding holds

$$C_{13}(r, d)2^{-nr_1}\mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}} \subset H^{\boldsymbol{r}}_\infty \cap \mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}}. \tag{8.71}$$

Therefore, in view of (8.70)–(8.71) and Theorem P with respect to the trigonometric system $\{e^{i(\boldsymbol{k},\boldsymbol{x})}\}_{|\boldsymbol{k}|\in\mathcal{Y}_n}$ we obtain

$$e_M(B^{\boldsymbol{r}}_{\infty,\theta})_{B_{q,1}} \geq e_M(B^{\boldsymbol{r}}_{\infty,\theta} \cap \mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}})_{B_{q,1}} \gg$$
$$\gg 2^{-nr_1} n^{-\frac{d-1}{\theta}} e_M(\mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}}, \{e^{i(\boldsymbol{k},\boldsymbol{x})}\}_{|\boldsymbol{k}|\in\mathcal{Y}_n})_{B_{1,1}} \gg 2^{-nr_1} n^{-\frac{d-1}{\theta}} n^{d-1} =$$
$$= 2^{-nr_1} n^{(d-1)\left(1-\frac{1}{\theta}\right)} \asymp M^{-r_1} (\log^{d-1} M)^{r_1+1-\frac{1}{\theta}}, \tag{8.72}$$

$1 < q < \infty$, $1 \leq \theta \leq \infty$.

To conclude the proof we note that the lower estimate in (8.64) for $q = \infty$ follows from (8.72) and the inequality $\|\cdot\|_{B_{\infty,1}} \geq \|\cdot\|_{B_{q,1}}$, $1 < q < \infty$.

Theorem 8.17 is proved. □

We now comment the obtained result. In the univariate case the orders of $e_M(B^{r_1}_{p,\theta})_{B_{q,1}}$ and $e_M(B^{r_1}_{p,\theta})_q$, $1 \leq q \leq p \leq \infty$, are formulated in Theorems Q and R. We see that the following relations hold:

$$e_M(B^{r_1}_{p,\theta})_{B_{q,1}} \asymp e_M(B^{r_1}_{p,\theta})_q \asymp M^{-r_1}. \tag{8.73}$$

We also note that in (8.73) the case $p = q = 1$ is included.

In the multivariate case ($d \geq 2$) the situation is different. So, comparing the results of Theorems 8.17 and S for corresponding values of the parameters p and q, we see that the orders of the quantities $e_M(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}}$ and $e_M(B^{\boldsymbol{r}}_{p,\theta})_q$ coincide only for $\nu = 1$. Let us mention one more important issue that was actually a motivation for investigating the best M-term trigonometric approximations of the classes $B^{\boldsymbol{r}}_{p,\theta}$ in the space $B_{q,1}$ for $d \geq 2$. We have in mind the fact that in Theorem 8.17 we obtained, in particular, estimates of the quantities $e_M(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}}$ in the cases $1 < q \leq p \leq 2$ and $p = q = \infty$, where the orders of the respective approximation characteristics of the classes $B^{\boldsymbol{r}}_{p,\theta}$ in the space L_q still remain unknown (see, e.g., Theorem S, and also [18, Open problem 7.5]).

In the following statement we get the order of the quantity $e^{\perp}_M(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}}$.

Theorem 8.18. *Let $d \geq 2$, $1 < q \leq p \leq \infty$, $q \neq \infty$, $1 \leq \theta \leq \infty$. Then for $r_1 > 0$ the following estimate holds*

$$e^{\perp}_M(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}} \asymp M^{-r_1} (\log^{\nu-1} M)^{r_1+1-\frac{1}{\theta}}. \tag{8.74}$$

Proof. The lower estimate follows from Theorem 8.17 and the relation

$$e^{\perp}_M(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}} \geq e_M(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}} \asymp M^{-r_1} (\log^{\nu-1} M)^{r_1+1-\frac{1}{\theta}}.$$

Moving to proving the upper estimate in (8.74), we note that it suffices to get it for the case $1 < q = p < \infty$.

Hence, for the function $f \in B^{\boldsymbol{r}}_{q,\theta}$, $1 < q < \infty$, as an approximation aggregate we use the polynomial

$$S_{\Theta_M}(f) := S_{\Theta_M}(f, \boldsymbol{x}) := \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')<n} \delta_{\boldsymbol{s}}(f, \boldsymbol{x}),$$

where the number $n \in \mathbb{N}$ satisfies the condition $M \asymp 2^n n^{\nu-1}$.

Therefore, we can write

$$e^{\perp}_M(f)_{B_{q,1}} \le \left\| f - \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')<n} \delta_{\boldsymbol{s}}(f) \right\|_{B_{q,1}} = \left\| \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\ge n} \delta_{\boldsymbol{s}}(f) \right\|_{B_{q,1}} =$$

$$= \sum_{\boldsymbol{s}\in\mathbb{N}^d} \left\| \delta_{\boldsymbol{s}} \left(\sum_{\substack{\boldsymbol{s}'\in\mathbb{N}^d \\ (\boldsymbol{s}',\boldsymbol{\gamma}')\ge n}} \delta_{\boldsymbol{s}'}(f) \right) \right\|_q \le \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\ge n} \|\delta_{\boldsymbol{s}}(f)\|_q := J_{11}. \tag{8.75}$$

Further let us consider two cases.

a) Let $1 \le \theta < \infty$. Then, in view of Hölder's inequality with exponent θ (with corresponding modification of this inequality for $\theta = 1$) and Lemma A, we get

$$J_{11} \le \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\ge n} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|\delta_{\boldsymbol{s}}(f)\|_q^\theta \right)^{\frac{1}{\theta}} \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\ge n} 2^{-(\boldsymbol{s},\boldsymbol{r})\theta'} \right)^{\frac{1}{\theta'}} \ll$$

$$\ll \|f\|_{B^{\boldsymbol{r}}_{q,\theta}} \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\ge n} 2^{-(\boldsymbol{s},\boldsymbol{\gamma})r_1\theta'} \right)^{\frac{1}{\theta'}} \ll 2^{-nr_1} n^{(\nu-1)\left(1-\frac{1}{\theta}\right)}.$$

b) Let $\theta = \infty$. In this case, taking into account that for $f \in B^{\boldsymbol{r}}_{q,\theta}$, $1 < q < \infty$, it holds $\|\delta_{\boldsymbol{s}}(f)\|_q \ll 2^{-(\boldsymbol{s},\boldsymbol{r})}$, $\boldsymbol{s} \in \mathbb{N}^d$, and using Lemma A, we obtain

$$J_{11} \ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\ge n} 2^{-(\boldsymbol{s},\boldsymbol{r})} \ll 2^{-nr_1} n^{\nu-1}. \tag{8.76}$$

Hence, combining (8.75)–(8.76) and noting that $M \asymp 2^n n^{\nu-1}$, we get the respective upper estimate of the quantity $e^{\perp}_M(B^{\boldsymbol{r}}_{q,\theta})_{B_{q,1}}$, and respectively

$$e^{\perp}_M(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}} \ll M^{-r_1} (\log^{\nu-1} M)^{r_1+1-\frac{1}{\theta}}, \quad 1 < q \le p \le \infty, \quad q \ne \infty.$$

Theorem 8.18 is proved. □

Let us comment the obtained result.

The result of Theorem 8.18 complements estimates of the quantity $e_m^{\perp}(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}}$, $q \in \{1, \infty\}$, which are given in Section 8.3. Besides, in Theorem 8.18 we managed to expand the allowed parameter regions in comparison to the known by this time estimates of the quantity $e_M^{\perp}(B^{\boldsymbol{r}}_{p,\theta})_q$ (see Theorems U, V). Having this in mind, comparing the results of Theorems 8.18, U and V for mutual values of the parameters p and q we conclude that orders of the considered quantities in the spaces $B_{q,1}$ and L_q differ except the case $\nu = 1$. Here we note that in the univariate case the following relations hold (see [39, 52]):

$$e_M^{\perp}(B^{r_1}_{p,\theta})_{B_{q,1}} \asymp e_M^{\perp}(B^{r_1}_{p,\theta})_q \asymp M^{-r_1}.$$

To conclude, for the Sobolev classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ in the space $B_{q,1}$ for some relations between the parameters p and q, let us formulate results similar to Theorems 8.17, 8.18 for the approximation characteristics (8.61) and (8.42).

Theorem 8.19. *Let $d \geq 2$, $1 < q \leq 2$, $q \leq p < \infty$, $\boldsymbol{\alpha} \in \mathbb{R}^d$. Then for $r_1 > 0$ the following estimates hold*

$$e_M(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{q,1}} \asymp e_M^{\perp}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_{B_{q,1}} \asymp M^{-r_1}(\log^{\nu-1} M)^{r_1+\frac{1}{2}}. \tag{8.77}$$

Theorem 8.20. *Let $d = 1$, $1 < q \leq p < \infty$, $\alpha \in \mathbb{R}$. Then for $r_1 > 0$ the following estimate holds*

$$e_M(W^{r_1}_{p,\alpha})_{B_{q,1}} \asymp e_M^{\perp}(W^{r_1}_{p,\alpha})_{B_{q,1}} \asymp M^{-r_1}. \tag{8.78}$$

Proof. The upper estimate of both of the quantities is a corollary from the results Theorems 8.6, 8.12.

The lower estimate in (8.78) follows from the relation

$$e_M(W^{r_1}_{p,\alpha})_q \asymp M^{-r_1} \tag{8.79}$$

(see [18, Thm. 7.5.1]) and inequalities (8.11).

Theorem 8.20 is proved. □

We now comment the results of Theorems 8.19, 8.20.

Comparing the estimates (8.78) and (8.79) we see that in the case $d = 1$ the respective approximation characteristics of the classes $W^{r_1}_{p,\alpha}$ in the spaces $B_{q,1}$ and L_q coincide in order. A different situation is in the multivariate case $(d \geq 2)$. For convenient comparison we formulate an analog of Theorem 8.19 in the space L_q.

Theorem X. *Let $d \geq 2$, $1 < q \leq p < \infty$, $\boldsymbol{\alpha} \in \mathbb{R}^d$. Then for $r_1 > 0$ the following estimate holds*

$$e_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_q \asymp e_m^{\perp}(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_q \asymp m^{-r_1}(\log^{\nu-1} m)^{r_1}. \tag{8.80}$$

Note that the upper estimate of both of the quantities in (8.80) is realized by approximation of functions $f \in W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ in the space L_q by their step hyperbolic Fourier sums (8.12) (see [18, Thm. 4.2.4]). The lower estimate of the quantity $e_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}})_q$ was obtained in the paper [24].

Hence, comparing (8.77) with (8.80) for $1 < q \leq 2$, $q \leq p < \infty$ we see that in the case $d \geq 2$ orders of the respective approximation characteristics of the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ coincide in the spaces $B_{q,1}$ and L_q only for $\nu = 1$.

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Andrii SHYDLICH

9 Approximation of generalized Wiener classes of functions of several variables in different metrics

9.1 Definition of functional spaces and classes

Let d be a fixed positive integer ($d \in \mathbb{N}$), let $\mathbb{R}^d$ and $\mathbb{Z}^d$ be the sets of ordered collections $k := (k_1, \ldots, k_d)$ of d real and integer numbers correspondingly. Let also $\mathbb{T}^d := [0, 2\pi)^d$ denote d-dimensional torus, and let $L_p := L_p(\mathbb{T}^d)$, $1 \le p \le \infty$, be the space of all Lebesgue-measurable on $\mathbb{R}^d$ 2π-periodic in each variable functions f with finite norm

$$\|f\|_{L_p} := \|f\|_{L_p(\mathbb{T}^d)} = \begin{cases} \left((2\pi)^{-d}\displaystyle\int_{\mathbb{T}^d} |f(x)|^p \mathrm{d}x\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}\limits_{x\in\mathbb{T}^d} |f(x)|, & p = \infty. \end{cases}$$

Set $(k, x) := k_1x_1 + k_2x_2 + \ldots + k_dx_d$, $e_k(x) := \mathrm{e}^{\mathrm{i}(k,x)}$ and for any $f \in L_1$, we denote the Fourier coefficients of f by

$$\widehat{f}(k) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(x)\overline{e}_k(x)\mathrm{d}x, \quad k \in \mathbb{Z}^d,$$

where $\overline{z}$ is the complex conjugate of z.

Further, let $\mathcal{S}^p := \mathcal{S}^p(\mathbb{T}^d)$, $0 < p \le \infty$, be the space of all functions $f \in L_1$ with the finite ℓ_p-(quasi-)norm

$$\|f\|_{\mathcal{S}^p} := \|\{\widehat{f}(k)\}_{k\in\mathbb{Z}^d}\|_{\ell_p(\mathbb{Z}^d)} = \begin{cases} \left(\displaystyle\sum_{k\in\mathbb{Z}^d} |\widehat{f}(k)|^p\right)^{1/p}, & 0 < p < \infty, \\ \sup\limits_{k\in\mathbb{Z}^d} |\widehat{f}(k)|, & p = \infty. \end{cases} \tag{9.1}$$

Approximation characteristics of the spaces $\mathcal{S}^p$ of one and several variables were actively studied in the papers of Stepanets, his students, and followers (see, for example [43,46,57], [44, Chap. 11], [1,2,30,45,56,58], etc.). It is also worth mentioning a series of papers [3,5–7,31,60] devoted mainly to direct and inverse approximation

Andrii SHYDLICH, Institute of Mathematics of the National Academy of Sciences of Ukraine, 3 Tereshchenkivska str., 01024, Kyiv, Ukraine, e-mail: shidlich@gmail.com

theorems using various linear methods in different spaces of Besicovitch-Musielak-Orlicz-Stepanets type of periodic and almost periodic functions, which can be considered as certain generalizations of the spaces $\mathcal{S}^p(\mathbb{T}^1)$. An overview of results in this direction can be also found in [32].

The spaces $\mathcal{S}^p$ are also known as Wiener spaces or Wiener-type spaces, due to their connection with the classical Wiener algebra in the case where $p = 1$, and are often denoted by $\mathcal{A}_p$. They can be seen as periodic versions of so-called Barron spaces [4, 59] and share similar properties regarding non-linear approximation. For $p = 1$ the spaces $\mathcal{S}^p$ consist of functions whose Fourier series are absolutely convergent. The questions of absolute convergence and summability of general trigonometric series and Fourier series were, in particular, studied by Wiener [61], Szász [50], Stechkin [38–41], Sunouchi [51, 52], etc. The main results in this direction, the properties of such spaces, and their generalization were described in Kahane's monograph [16]. Let us also mention a significant series of results by Móricz [21–24], etc., which establish a relationship between the absolute convergence of (multiple) Fourier series and the structural smoothness of functions, specifically by introducing enlarged Lipschitz and Zygmund classes defined by the mixed modulus of continuity.

Let $\Psi = \{\Psi_k\}_{k\in\mathbb{Z}^d}$ be a sequence of complex numbers, $\Psi_k \neq 0$, and

$$\mathcal{F}_q^\Psi := \mathcal{F}_q^\Psi(\mathbb{T}^d) := \{f \in L_1 : \ \|\{\widehat{f}(k)/\Psi_k\}_{k\in\mathbb{Z}^d}\|_{\ell_q(\mathbb{Z}^d)} \leq 1\}, \quad 0 < q \leq \infty. \tag{9.2}$$

Classes $\mathcal{F}_q^\Psi$ are also called weighted Wiener classes or generalized Wiener classes.

Next, let $\psi = \psi(t)$, $t \geq 1$, be a positive nonincreasing function. Consider the classes $\mathcal{F}_q^\Psi$ in the case when the sequence Ψ satisfies the condition

$$|\Psi_0| = \psi(1) \quad \text{and} \quad |\Psi_k| = \psi(|k|_r), \tag{9.3}$$

where $|k|_r := \|k\|_{\ell_r^d}$, $0 < r \leq \infty$, is the ℓ_r-(quasi-)norm defined for $k \in \mathbb{Z}^d$ similarly to (9.1). In this case, we denote $\mathcal{F}_q^\Psi = \mathcal{F}_q^\Psi(\mathbb{T}^d) = \mathcal{F}_{q,r}^\psi(\mathbb{T}^d) =: \mathcal{F}_{q,r}^\psi$.

The main goal of the paper is to find the exact order estimates for linear and nonlinear approximations of classes $\mathcal{F}_{q,r}^\psi$ in various metrics depending on the rate of decay to zero of functions ψ. Similar estimates play important role for estimates of the corresponding approximations of functional classes such as Sobolev classes, Besov classes, etc. (see, for example, [10, 18, 25, 53]), and can also serve as upper bounds for (nonlinear) sampling errors in L_p (see [15, 19, 20], etc.).

If $\psi(t) = t^{-s}$, $s \in \mathbb{N}$ and $r = \infty$, then $\mathcal{F}_{q,\infty}^\psi =: \mathcal{F}_{q,\infty}^s$ is a set of functions whose sth partial derivatives have absolutely convergent Fourier series. It is also called periodic (isotropic) Wiener class. If $q = 2$, $\mathcal{F}_{q,\infty}^s$ is equivalent (modulo constants) to the unit ball of the Sobolev class W_2^s. Approximative characteristics of the classes $\mathcal{F}^{\psi}_{q,r}$ for different $r \in (0,\infty]$ and for the various functions ψ were investigated by many authors (see, e.g., [10,17,18,29,33,35–37,53]). In particular, DeVore and Temlyakov [10] found the exact order estimates for the best m-term trigonometric approximations of the classes $\mathcal{F}^{s}_{q,\infty}$, $s>0$, in the spaces L_p. Temlyakov [53] obtained the exact order estimates for approximations of these classes by m-term greedy polynomials in L_p. In the case, where $\psi(t)$ is a positive function that decreases to zero no faster than some power function, the best m-term one-sided trigonometric approximations and approximations by m-term one-sided Greedy-liked polynomials of the classes $\mathcal{F}^{\psi}_{q,\infty}$ were studied in [18]. Linear approximation in Wiener type spaces was studied in [17].

If the sequence Ψ satisfies the conditions similar to (9.3), where ψ is a power function: $\psi(t)=t^{-s}$, and instead of the functional $|k|_r$ we consider the functional $|k|_{mix} := \prod_{i=1}^{d}(1+|k_i|)^r$, then the classes $\mathcal{F}^{\Psi}_{q}$ are denoted by $\mathrm{S}^{\psi}_{q}\mathcal{A}$ and called multivariate weighted Wiener classes with mixed smoothness. Approximative characteristics of such classes were studied in [15, 19, 20, 25], etc.

Note that nonlinear approximation of the generalized Wiener classes was studied in [43], [44, Chap. 11], [9, 25], etc. In particular, Stepanets [43], [44, Chap. 11] and V. K. Nguyen and V. D. Nguyen [25] found the exact values of the best m-term trigonometric approximations of the classes $\mathcal{F}^{\Psi}_{q}$ in the spaces $\mathcal{S}^p$ for all $0<p,q\leq\infty$.

9.2 Approximative characteristics

Let $\mathscr{X}$ be one of the spaces L_p, $1\leq p\leq\infty$, or $\mathcal{S}^p$, $0<p\leq\infty$, $m\in\mathbb{N}$, let γ_m be a collection of m different vectors of $\mathbb{Z}^d$, and let f be any function from $\mathscr{X}$. The quantity

$$E_{\gamma_m}(f)_{\mathscr{X}} = \inf_{c_k\in\mathbb{C}} \left\| f - \sum_{k\in\gamma_m} c_k\, e_k \right\|_{\mathscr{X}} \tag{9.4}$$

is called the best approximation of the function f by m-term polynomials corresponding to the collection γ_m in the space $\mathscr{X}$.

Next, let $S_{\gamma_m}(f) = \sum_{k\in\gamma_m}\widehat{f}(k)e_k$ be the Fourier sum corresponding to the collection γ_m, and

$$\mathscr{E}_{\gamma_m}(f)_{\mathscr{X}} = \| f - S_{\gamma_m}(f)\|_{\mathscr{X}} \tag{9.5}$$

be the approximation of the function f by the Fourier sum $S_{\gamma_m}(f)$ in $\mathscr{X}$.

If $\mathfrak{N}$ is a subset of the space $\mathscr{X}$, then $E_{\gamma_m}(\mathfrak{N})_{\mathscr{X}}$ and $\mathscr{E}_{\gamma_m}(\mathfrak{N})_{\mathscr{X}}$ denote the exact upper bounds of the quantities (9.4) and (9.5) over the set $\mathfrak{N}$, i.e.,

$$E_{\gamma_m}(\mathfrak{N})_{\mathscr{X}} = \sup_{f\in\mathfrak{N}} E_{\gamma_m}(f)_p \quad \text{and} \quad \mathscr{E}_{\gamma_m}(\mathfrak{N})_{\mathscr{X}} = \sup_{f\in\mathfrak{N}} \mathscr{E}_{\gamma_m}(f)_{\mathscr{X}}. \tag{9.6}$$

Denote by Γ_m a set of all collections of m different vectors of $\mathbb{Z}^d$. The quantities

$$\mathscr{D}_m(\mathfrak{N})_{\mathscr{X}} = \inf_{\gamma_m \in \Gamma_m} E_{\gamma_m}(\mathfrak{N})_{\mathscr{X}} \quad \text{and} \quad \mathscr{D}_m^{\perp}(\mathfrak{N})_{\mathscr{X}} = \inf_{\gamma_m \in \Gamma_m} \mathscr{E}_{\gamma_m}(\mathfrak{N})_{\mathscr{X}} \tag{9.7}$$

are called the basis width and projection width (or Fourier-width) of order m of the set $\mathfrak{N}$ in $\mathscr{X}$.

Further, for $f \in \mathscr{X}$, let $\{k_l\}_{l=1}^{\infty} = \{k_l(f)\}_{l=1}^{\infty}$ denote a rearrangement of vectors of $\mathbb{Z}^d$ such that

$$|\widehat{f}(k_1)| \geq |\widehat{f}(k_2)| \geq \dots . \tag{9.8}$$

In general case, this rearrangement is not unique. In such case, we take any rearrangement satisfying (9.8).

We define Σ_m to be the class of all complex trigonometric polynomials of the form $T = \sum_{k \in \gamma_m} c_k e_k$, where γ_m is a collection from the set Γ_m.

In addition to (9.4)–(9.7), we consider the quantities

$$\|f - G_m(f)\|_{\mathscr{X}} = \left\| f(\cdot) - \sum_{l=1}^{m} \widehat{f}(k_l) e_{k_l} \right\|_{\mathscr{X}}, \tag{9.9}$$

$$\sigma_m^{\perp}(f)_{\mathscr{X}} = \inf_{\gamma_m \in \Gamma_m} \left\| f - \sum_{k \in \gamma_m} \widehat{f}(k) e_k \right\|_{\mathscr{X}} = \inf_{\gamma_m \in \Gamma_m} \mathscr{E}_{\gamma_m}(f)_{\mathscr{X}}, \tag{9.10}$$

and

$$\sigma_m(f)_{\mathscr{X}} = \inf_{T \in \Sigma_m} \|f - T\|_{\mathscr{X}} = \inf_{\gamma_m \in \Gamma_m} E_{\gamma_m}(f)_{\mathscr{X}}. \tag{9.11}$$

The quantities (9.11) and (9.10) are respectively called the best m-term trigonometric and the best m-term orthogonal trigonometric approximations of the function f in the space $\mathscr{X}$. The quantity (9.9) is called the approximation of the function f by m-term greedy polynomials in the space $\mathscr{X}$.

For a set $\mathfrak{N} \subset \mathscr{X}$, we put

$$\sigma_m^{\perp}(\mathfrak{N})_{\mathscr{X}} = \sup_{f \in \mathfrak{N}} \sigma_m^{\perp}(f)_{\mathscr{X}} \quad \text{and} \quad \sigma_m(\mathfrak{N})_{\mathscr{X}} = \sup_{f \in \mathfrak{N}} \sigma_m(f)_{\mathscr{X}}.$$

In general case, the quantities (9.9) depend on the choice of the rearrangement satisfying (9.8). So, for the unique definition, we put

$$G_m(\mathfrak{N})_{\mathscr{X}} = \sup_{f \in \mathfrak{N}} \inf_{\{k_l(f)\}_{l=1}^{\infty}} \left\| f(\cdot) - \sum_{l=1}^{m} \widehat{f}(k_l(f)) e_{k_l(f)} \right\|_{\mathscr{X}}. \tag{9.12}$$

In (9.12), for any function $f \in \mathfrak{N}$, we consider the infimum on all rearrangements, satisfying (9.8), but it should be noted that results, formulated in this paper, are also true for any other rearrangements, satisfying (9.8).

Research of the quantities in (9.9)–(9.11) goes back to the paper of Stechkin [41]. Order estimates of these quantities on different classes of functions of one and several variables were obtained by many authors. In particular, the bibliography of papers with the similar results can be found in [8, 11, 28, 54, 55].

It follows from (9.4)–(9.12) that

$$\sigma_m(f)_{L_p} \le \sigma_m^{\perp}(f)_{L_p} \le \|f - G_m(f)\|_{L_p} \quad \forall f \in L_p, \tag{9.13}$$

$$\sigma_m(f)_{\mathcal{S}^p} = \sigma_m^{\perp}(f)_{\mathcal{S}^p} = \|f - G_m(f)\|_{\mathcal{S}^p} \quad \forall f \in \mathcal{S}^p, \tag{9.14}$$

and for any $\gamma_m \subset \mathbb{Z}^d$ and $\mathfrak{N} \subset \mathscr{X}$

$$\sigma_m(\mathfrak{N})_{\mathscr{X}} \le \mathscr{D}_m(\mathfrak{N})_{\mathscr{X}} \le E_{\gamma_m}(\mathfrak{N})_{\mathscr{X}}, \quad \sigma_m^{\perp}(\mathfrak{N})_{\mathscr{X}} \le \mathscr{D}_m^{\perp}(\mathfrak{N})_{\mathscr{X}} \le \mathscr{E}_{\gamma_m}(\mathfrak{N})_{\mathscr{X}}.$$

9.3 Approximative characteristics of the classes $\mathcal{F}_q^{\Psi}$ in the spaces $\mathcal{S}^p$

9.3.1 Exact values of best approximations and basis widths

Let $\Psi = \{\Psi_k\}_{k\in\mathbb{Z}^d}$ be a sequence of complex numbers, $\Psi_k \ne 0$, such that there exists a non-increasing rearrangement $\bar{\Psi} = \{\bar{\Psi}_j\}_{j=1}^{\infty}$ of the number system $\{|\Psi_k|\}_{k\in\mathbb{Z}^d}$. It is clear that in this case the system $\{|\Psi_k|\}_{k\in\mathbb{Z}^d}$ is bounded, i.e.

$$|\Psi_k| \le K \quad \forall k \in \mathbb{Z}^d.$$

Hereinafter, K, c, $K_0, \ldots$ are positive constants that do not depend on the corresponding variable (k, t, etc.).

A sufficient condition guaranteeing the existence of the rearrangement $\bar{\Psi} = \{\bar{\Psi}_j\}_{j=1}^{\infty}$ is the condition

$$\lim_{|k|\to\infty} |\Psi_k| = 0,$$

however such rearrangement also exists, for example, when $\{|\Psi_k|\}_{k\in\mathbb{Z}^d}$ is a constant.

In the case where $\mathfrak{N} = \mathcal{F}_q^{\Psi}$ and $\mathscr{X} = \mathcal{S}^p$, the values of the characteristics (9.6) and (9.7) were found by Stepanets [44, Chapter XI], [45] for all $0 < p, q < \infty$. In order to formulate this result, for any collection γ_m of m different vectors of $\mathbb{Z}^d$, by $\Psi_{\gamma_m} = \{\Psi_{\gamma_m}(k)\}_{k\in\mathbb{Z}^d}$ denote a system of numbers such that

$$\Psi_{\gamma_m}(k) = \begin{cases} 0, & k \in \gamma_m, \\ \Psi_k, & k \overline{\in} \gamma_m, \end{cases} \tag{9.15}$$

and by $\bar{\Psi}_{\gamma_m} = \{\bar{\Psi}_{\gamma_m}(j)\}_{j=1}^{\infty}$ denote a non-increasing rearrangement of the system $\{|\Psi_{\gamma_m}(k)|\}_{k\in\mathbb{Z}^d}$.

Theorem Y ([44, Chapter XI], [45]). *Let $0 < p, q < \infty$, $m \in \mathbb{N}$ and $\Psi = \{\Psi_k\}_{k\in\mathbb{Z}^d}$ be a sequence of complex numbers, $\Psi_k \neq 0$, such that there exists a non-increasing rearrangement $\bar{\Psi} = \{\bar{\Psi}_j\}_{j=1}^{\infty}$ of the number system $\{|\Psi_k|\}_{k\in\mathbb{Z}^d}$.*

(i) In the case $0 < p \le q < \infty$,

$$E_{\gamma_m}(\mathcal{F}_q^{\Psi})_{\mathcal{S}^p} = \mathscr{E}_{\gamma_m}(\mathcal{F}_q^{\Psi})_{\mathcal{S}^p} = \bar{\Psi}_{\gamma_m}(1) \quad \forall \gamma_m \in \Gamma_m, \tag{9.16}$$

and

$$\mathscr{D}_m(\mathcal{F}_q^{\Psi})_{\mathcal{S}^p} = \mathscr{D}_m^{\perp}(\mathcal{F}_q^{\Psi})_{\mathcal{S}^p} = \bar{\Psi}_{m+1}. \tag{9.17}$$

(ii) In the case when $0 < q < p < \infty$ and $\sum_{k\in\mathbb{Z}^d} |\Psi_k|^{\frac{pq}{q-p}} < \infty$,

$$E_{\gamma_m}(\mathcal{F}_q^{\Psi})_{\mathcal{S}^p} = \mathscr{E}_{\gamma_m}(\mathcal{F}_q^{\Psi})_{\mathcal{S}^p} = \left(\sum_{k=1}^{\infty} \bar{\Psi}_{\gamma_m}^{\frac{pq}{q-p}}(k)\right)^{\frac{q-p}{pq}} \qquad \forall \gamma_m \in \Gamma_m,$$

and

$$\mathscr{D}_m(\mathcal{F}_q^{\Psi})_{\mathcal{S}^p} = \mathscr{D}_m^{\perp}(\mathcal{F}_q^{\Psi})_{\mathcal{S}^p} = \left(\sum_{k=m+1}^{\infty} \bar{\Psi}_k^{\frac{pq}{q-p}}\right)^{\frac{q-p}{pq}}. \tag{9.18}$$

Moreover, for any collection $\gamma_m^* = \{k_1, \dots, k_m\}$ from the set Γ_m such that

$$\gamma_m^* = \{k_j \in \mathbb{Z}^d : \ |\Psi_{k_j}| = \bar{\Psi}_j, \ j = 1, 2, \dots, m\}, \tag{9.19}$$

the following relation holds in both cases:

$$\mathscr{D}_m(\mathcal{F}_q^{\Psi})_{\mathcal{S}^p} = \mathscr{D}_m^{\perp}(\mathcal{F}_q^{\Psi})_{\mathcal{S}^p} = E_{\gamma_m^*}(\mathcal{F}_q^{\Psi})_{\mathcal{S}^p} = \mathscr{E}_{\gamma_m^*}(\mathcal{F}_q^{\Psi})_{\mathcal{S}^p}. \tag{9.20}$$

Next, we formulate a theorem that complements this statement for cases of infinite values of parameters p and q.

Theorem 9.1. *Let $m \in \mathbb{N}$ and $\Psi = \{\Psi_k\}_{k\in\mathbb{Z}^d}$ be a sequence of complex numbers, $\Psi_k \neq 0$, such that there exists a non-increasing rearrangement $\bar{\Psi} = \{\bar{\Psi}_j\}_{j=1}^{\infty}$ of the system $\{|\Psi_k|\}_{k\in\mathbb{Z}^d}$.*

(i) If $0 < q < p = \infty$ or $p = q = \infty$, then relations (9.16) *and* (9.17) *hold.*

(ii) If $0 < p < q = \infty$ and the series $\sum_{k\in\mathbb{Z}^d} |\Psi_k|^p$ converges, then

$$E_{\gamma_m}(\mathcal{F}_q^{\Psi})_{\mathcal{S}^p} = \mathscr{E}_{\gamma_m}(\mathcal{F}_q^{\Psi})_{\mathcal{S}^p} = \left(\sum_{k=1}^{\infty} \bar{\Psi}_{\gamma_m}^p(k)\right)^{1/p} \qquad \forall \gamma_m \in \Gamma_m, \tag{9.21}$$

and

$$\mathscr{D}_m(\mathcal{F}_q^{\Psi})_{\mathcal{S}^p} = \mathscr{D}_m^{\perp}(\mathcal{F}_q^{\Psi})_{\mathcal{S}^p} = \left(\sum_{k=m+1}^{\infty} \bar{\Psi}_k^p\right)^{1/p}. \tag{9.22}$$

Moreover, for any collection $\gamma_m^* \in \Gamma_m$ satisfying (9.19)*, relation* (9.20) *holds.*

The proof of this statement, as well as the proofs of all other statements in this article, will be given in Section 9.5.

9.3.2 Exact values of best n-term approximation

Exact values of the quantities $\sigma_m(\mathcal{F}_q^\Psi)_{\mathcal{S}^p}$ were obtained in [43], [44, Chapter XI] ((i), (ii)) and [25] ((iii)-(v)). They follow from the following statement.

Theorem Z ([43], [44, Chapter XI], [25])**.** *Let $0 < p, q \le \infty$, $m \in \mathbb{N}$ and $\Psi = \{\Psi_k\}_{k\in\mathbb{Z}^d}$ be a sequence of complex numbers such that there exists a non-increasing rearrangement $\bar{\Psi} = \{\bar{\Psi}_j\}_{j=1}^\infty$ of the number system $\{|\Psi_k|\}_{k\in\mathbb{Z}^d}$.*

(i) If $0 < q \le p < \infty$, then

$$\sigma_m(\mathcal{F}_q^\Psi)_{\mathcal{S}^p} = \sup_{l>m} \frac{(l-m)^{1/p}}{\left(\sum\limits_{j=1}^{l} \bar{\Psi}_j^{-q}\right)^{1/q}}. \tag{9.23}$$

(ii) If $0 < p < q < \infty$ and the series $\sum_{k\in\mathbb{Z}^d} |\Psi_k|^{\frac{pq}{q-p}}$ converges, then

$$\sigma_m(\mathcal{F}_q^\Psi)_{\mathcal{S}^p} = \left((l_m - m)^{\frac{q}{q-p}} \left(\sum_{j=1}^{l_m} \bar{\Psi}_j^{-q}\right)^{\frac{p}{p-q}} + \sum_{j=l_m+1}^{\infty} \bar{\Psi}_j^{\frac{pq}{q-p}} \right)^{\frac{q-p}{pq}}, \tag{9.24}$$

where the number l_m is defined by

$$\bar{\Psi}_{l_m}^{-q} \le \frac{1}{l_m - m} \sum_{j=1}^{l_m} \Psi_j^{-q} < \bar{\Psi}_{l_m+1}^{-q}. \tag{9.25}$$

(iii) If $0 < q < p = \infty$, then

$$\sigma_m(\mathcal{F}_q^\Psi)_{\mathcal{S}^p} = \left(\sum_{j=1}^{m+1} \bar{\Psi}_j^{-q}\right)^{-1/q}. \tag{9.26}$$

(iv) If $0 < p < q = \infty$ and the series $\sum_{k\in\mathbb{Z}^d} |\Psi_k|^p$ converges, then

$$\sigma_m(\mathcal{F}_q^\Psi)_{\mathcal{S}^p} = \left(\sum_{j=m+1}^{\infty} \bar{\Psi}_j^{p}\right)^{1/p}. \tag{9.27}$$

(v) If $p = q = \infty$, then

$$\sigma_m(\mathcal{F}_q^\Psi)_{\mathcal{S}^p} = \bar{\Psi}_{m+1}. \tag{9.28}$$

From the formulation of Theorems Y, Z, and 9.1 we can see that the values of $\mathscr{D}_m(\mathcal{F}^\Psi_q)_{\mathcal{S}^p}$, $\mathscr{D}^\perp_m(\mathcal{F}^\Psi_q)_{\mathcal{S}^p}$ and $\sigma_m(\mathcal{F}^\Psi_q)_{\mathcal{S}^p}$ significantly depend on the behavior of the sequence Ψ. Therefore, in order to obtain their estimates (as $m \to \infty$) for a specific sequence Ψ, it is necessary to additionally investigate the behavior of the functionals given in the right-hand sides of the corresponding relations.

In this paper, by studying such functionals and using Theorems Y, Z, and 9.1, we find the asymptotic behavior of the above quantities for classes $\mathcal{F}^\Psi_q$ in the case when the sequence Ψ satisfies condition (9.3), where $\psi = \psi(t)$, $t \ge 1$, is a positive non-increasing function, i.e., for classes $\mathcal{F}^\psi_{q,r}$.

Note that in the terms of similar functionals, solutions of other problems of approximation theory are formulated (see, e.g., [27] (Chapter 6), [49], [12], [13], [47]). Therefore, the study of such functionals is interesting in itself.

Let us provide a few auxiliary facts and estimates regarding the structure of sequences of the form (9.3). It is clear that for any the sequence Ψ satisfying condition (9.3) with non-increasing positive function ψ, there exists a non-increasing rearrangement $\bar{\Psi} = \{\bar{\Psi}_j\}_{j=1}^\infty$ of the number system $\{|\Psi_k|\}_{k\in\mathbb{Z}^d}$ and it has the step-wise form:

$$\bar{\Psi}_j = \psi(s), \quad j \in (V_{s-1}, V_s], \;\; s = 0, 1, 2, \dots, \tag{9.29}$$

where

$$V_s = V_s(d) := \#\{k \in \mathbb{Z}^d : \; \|k\|_{\ell^d_r} \le s\}, \quad s = 0, 1, \dots, \quad V_{-1} := 0. \tag{9.30}$$

Therefore, to study functionals that depend on such sequences Ψ, it is necessary to have convenient estimates for the numbers V_s.

Let $B^d_r = \{x \in \mathbb{R}^d : \; \|x\|_{\ell^d_r} \le 1\}$ be the unit ℓ_r-ball of the space $\mathbb{R}^d$. It is known (see, e.g., [62]) that the volume of the ball B^d_r is calculated by the formula

$$M_{r,d} := \mathrm{vol}(B^d_r) = \frac{(2\Gamma(1+1/r))^d}{\Gamma(1+d/r)}, \quad r \in (0, \infty], \tag{9.31}$$

and in particular, $M_{1,d} = \frac{2^d}{d!}$, $M_{2,d} = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$ and $M_{\infty,d} = 2^d$.

For a real number a, we denote $(a)_+ = \max\{0, a\}$. For positive sequences $a(s)$ and $b(s)$ (or functions $a(s)$ and $b(s)$, $s \ge 1$), the expression "$a(s) \asymp b(s)$" means that there are constants $0 < K_1 < K_2$ such that for any $s \in \mathbb{N}$ (or $s \ge 1$), $a(s) \le K_2 b(s)$ (in this case, we write "$a(s) \ll b(s)$") and $a(s) \ge K_1 b(s)$ (in this case, we write "$a(s) \gg b(s)$").

The following lemma follows from known general results on asymptotic behavior of the number of lattice points in scaled convex bodies (see, e.g., [14, Chapter 14]).

Lemma 9.1. *Let* $r \in (0, \infty]$ *and* $d \in \mathbb{N}$. *Then for any* $s = 0, 1, 2, \ldots$,

$$M_{r,d}\big((s - c_{r,d})_+\big)^d \leq V_s \leq M_{r,d}(s + c_{r,d})^d, \tag{9.32}$$

where $c_{r,d} := \frac{d^{1/r}}{2}$ *and*

$$\nu_s := V_s - V_{s-1} \asymp s^{d-1}. \tag{9.33}$$

Proof. Relation (9.33) follows from (9.32). Therefore, it suffices to prove (9.32). For each $k \in \mathbb{Z}^d$, let $Q_k := k + [-\frac{1}{2}, \frac{1}{2}]^d$ be the unit cube centered at k. The family $\{Q_k\}_{k \in \mathbb{Z}^d}$ forms a tiling of $\mathbb{R}^d$, and $\mathrm{vol}(Q_k) = 1$ for all k. By definition, the values V_s can be expressed as the volume of the union of these cubes:

$$V_s = \mathrm{vol}\left(\bigcup_{\|k\|_{\ell_r^d} \leq s} Q_k \right).$$

Note the value $c_{r,d}$ is the maximum distance from the center to any point in the unit cube $[-\frac{1}{2}, \frac{1}{2}]^d$, i.e., $c_{r,d} = \sup_{y \in [-1/2, 1/2]^d} \|y\|_{\ell_r^d}$.

Upper bound. Suppose $\|k\|_{\ell_r^d} \leq s$. For any point $x \in Q_k$, we have $x - k \in [-\frac{1}{2}, \frac{1}{2}]^d$ and by the triangle inequality

$$\|x\|_{\ell_r^d} \leq \|k\|_{\ell_r^d} + \|x - k\|_{\ell_r^d} \leq s + c_{r,d}.$$

This implies that the union of cubes is contained within the scaled ball of the radius $s + c_{r,d}$, i.e.,

$$\bigcup_{\|k\|_{\ell_r^d} \leq s} Q_k \subset (s + c_{r,d}) B_r^d.$$

Taking the volume of both sides of this relation, we obtain

$$V_s \leq \mathrm{vol}((s + c_{r,d}) B_r^d) = M_{r,d}(s + c_{r,d})^d.$$

Lower bound. It is sufficient to consider the case when $s > c_{r,d}$. Let $x \in (s - c_{r,d}) B_r^d$. Since the cubes Q_k tile $\mathbb{R}^d$, x must belong to Q_k for some $k \in \mathbb{Z}^d$. For this k, we have

$$\|k\|_{\ell_r^d} \leq \|x\|_{\ell_r^d} + \|k - x\|_{\ell_r^d} \leq (s - c_{r,d}) + c_{r,d} = s.$$

This shows that every point in the smaller ball is covered by a cube whose center is in the set counted by V_s, i.e.,

$$(s - c_{r,d}) B_r^d \subset \bigcup_{\|k\|_{\ell_r^d} \leq s} Q_k.$$

This similarly yields

$$M_{r,d}(s - c_{r,d})^d \leq V_s.$$

□

Remark 9.1. *Denoting for a given positive integer s by n_s a number such that*

$$V_{n_s-1} < s \le V_{n_s}, \tag{9.34}$$

we see that

$$\left((s/M_{r,d})^{1/d} - c_{r,d}\right)_+ \le n_s < (s/M_{r,d})^{1/d} + c_{r,d} + 1. \tag{9.35}$$

9.3.3 Order estimates of best n-term approximations and basis widths of the classes $\mathcal{F}^{\psi}_{q,r}$ in the spaces $\mathcal{S}^p$

Estimates of the approximation characteristics of classes $\mathcal{F}^{\psi}_{q,r}$ and methods for obtaining them depend significantly on the asymptotic behavior of functions ψ. Depending on this, we further distinguish the following different subsets of ψ with common properties and formulate the corresponding results.

First, denote by B the set of all positive non-increasing functions $\psi(t)$, $t \ge 1$, which satisfy the so-called Δ_2-condition, i.e., for all $t \ge 1$

$$1 < \frac{\psi(t)}{\psi(2t)} \le K_3. \tag{9.36}$$

Natural representatives of the set B are functions of the form: $\psi(t) \equiv c$, $\psi(t) = \ln^{\varepsilon}(t + \mathrm{e})$ for $\varepsilon < 0$, $\psi(t) = t^{-r} \ln^{\varepsilon}(t + \mathrm{e})$ for $r > 0$ and $\varepsilon \in \mathbb{R}$, etc.

Theorem 9.2. *Assume that $0 < p, q, r \le \infty$, $\psi \in B$ and in the case $p < q$, moreover, for all t, larger than a certain number t_0, $\psi(t)$ is convex and satisfies the condition*

$$\frac{t|\psi'(t)|}{\psi(t)} \ge \beta, \quad \psi'(t) := \psi'(t+), \tag{9.37}$$

with a certain $\beta > d(1/p - 1/q)$. Then

$$\mathscr{D}_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \asymp \begin{cases} \psi\left(m^{1/d}\right), & 0 < q \le p \le \infty, \\ \psi\left(m^{1/d}\right) m^{1/p-1/q}, & 0 < p < q \le \infty. \end{cases} \tag{9.38}$$

and

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \asymp \psi\left(m^{1/d}\right) m^{1/p-1/q}, \quad 0 < p, q \le \infty. \tag{9.39}$$

Remark 9.2. *Note that condition* (9.37) *guarantees that $\sum_{k\in\mathbb{Z}^d} |\Psi_k|^{\frac{pq}{q-p}} < \infty$.*

Proof. Indeed, in this case, for all $\tau \ge t_0$,

$$\frac{|\psi'(\tau)|}{\psi(\tau)} \ge \frac{\beta}{\tau}. \tag{9.40}$$

Integrating each part of this relation in the range from t_0 to t, $t{>}t_0$, we obtain the estimate $\psi(t){\ll}t^{-\beta}$, where $\beta > \frac{d(q-p)}{pq}$.

Therefore, taking into account (9.29) and (9.33), we conclude that

$$\sum_{k\in\mathbb{Z}^d} |\Psi_k|^{\frac{pq}{q-p}} = \sum_{n=1}^{\infty} \nu_n \psi^{\frac{pq}{q-p}}(n) \ll \sum_{n=1}^{\infty} n^{d-1}\psi^{\frac{pq}{q-p}}(n) \ll \sum_{n=1}^{\infty} n^{d-1} n^{-\frac{pq\beta}{q-p}} < \infty. \tag{9.41}$$

□

To obtain similar estimates in the case where the functions ψ tend to zero faster than any power function, we give the following definitions.

Consider the set $\mathfrak{M}$ of all positive convex functions $\psi(t)$, $t \geq 1$ such that

$$\lim_{t\to\infty} \psi(t) = 0.$$

Denote by $\mathfrak{M}'_\infty$ the set of all functions $\psi \in \mathfrak{M}$, satisfying conditions

$$\alpha(\psi,t) := \frac{\psi(t)}{t|\psi'(t)|} \downarrow 0, \quad \psi'(t) := \psi'(t+), \tag{9.42}$$

and $\frac{\psi(t)}{|\psi'(t)|} \uparrow \infty$. Denote by $\mathfrak{M}^c_\infty$ the set of all functions $\psi \in \mathfrak{M}$ satisfying (9.42) and

$$K_4 \leq \frac{\psi(t)}{|\psi'(t)|} \leq K_5 \quad \forall t \geq 1. \tag{9.43}$$

Finally, denote by $\mathfrak{M}''_\infty$ the set of all $\psi \in \mathfrak{M}$ satisfying condition $\frac{\psi(t)}{|\psi'(t)|} \downarrow 0$.

Natural representatives of the sets $\mathfrak{M}'_\infty$, $\mathfrak{M}^c_\infty$ and $\mathfrak{M}''_\infty$ are functions of the form $\exp(-at^s)$, $a > 0$, in cases where $s \in (0,1)$, $s = 1$ and $s > 1$, respectively.

For any positive integers d and m and any $0 < r \leq \infty$, we denote

$$\widetilde{n}_m := \widetilde{n}_m(r,d) = (m/M_{r,d})^{1/d}, \tag{9.44}$$

where $M_{r,d}$ is the number defined by the relation (9.31).

Theorem 9.3. *Assume that* $0 < p,q,r \leq \infty$ *and* $\psi \in \mathfrak{M}'_\infty \cup \mathfrak{M}^c_\infty$. *Then*

$$\mathscr{D}_m(\mathcal{F}^\psi_{q,r})_{\mathcal{S}^p} \asymp \begin{cases} \psi(\widetilde{n}_m), & 0 < q \leq p \leq \infty, \\ \psi(\widetilde{n}_m)(m\alpha(\psi,\widetilde{n}_m))^{1/p-1/q}, & 0 < p < q \leq \infty, \end{cases} \tag{9.45}$$

where $\widetilde{n}_m$ *is defined by* (9.44), *and*

$$\sigma_m(\mathcal{F}^\psi_{q,r})_{\mathcal{S}^p} \asymp \psi(\widetilde{n}_m)(m\alpha(\psi,\widetilde{n}_m))^{1/p-1/q}, \quad 0 < p,q \leq \infty. \tag{9.46}$$

Since for any $\psi \in \mathfrak{M}$ satisfying (9.42), inequality (9.40) holds for any $\beta > 0$ and sufficiently large τ, then it can be similarly proven that for any ψ from the sets $\mathfrak{M}'_\infty$, $\mathfrak{M}^c_\infty$ or $\mathfrak{M}''_\infty$, the series $\sum_{k\in\mathbb{Z}^d} |\Psi_k|^s$ with any $s > 0$ are convergent.

In the case where $d = 1$, the classes $\mathcal{F}^\psi_{q,r}(\mathbb{T}^1) =: \mathcal{F}^\psi_q$ do not depend on r, the constant $M_{r,1} = 2$. Therefore, for any function $\psi \in \mathfrak{M}'_\infty \cup \mathfrak{M}^c_\infty$,

$$\mathscr{D}_m(\mathcal{F}^\psi_q)_{\mathcal{S}^p(\mathbb{T}^1)} \asymp \begin{cases} \psi\left(\dfrac{m}{2}\right), & 0 < q \le p \le \infty, \\ \psi\left(\dfrac{m}{2}\right)\left(m\alpha\left(\psi, \dfrac{m}{2}\right)\right)^{1/p-1/q}, & 0 < p < q \le \infty, \end{cases} \tag{9.45'}$$

and

$$\sigma_m(\mathcal{F}^\psi_q)_{\mathcal{S}^p(\mathbb{T}^1)} \asymp \psi\left(\frac{m}{2}\right)\left(m\alpha\left(\psi, \frac{m}{2}\right)\right)^{1/p-1/q}, \quad 0 < p, q \le \infty. \tag{9.46'}$$

Consider the case when ψ belongs to the set $\mathfrak{M}''_\infty$. First of all, note that the following statement follows from relations (9.17), (9.28) and (9.29):

Proposition 9.1. *Assume that* $0 < r \le \infty$, $m \in \mathbb{N}$ *and* $\psi = \psi(t)$, $t \ge 1$, *is a positive non-increasing function. Then for any* $0 < q \le p \le \infty$ *the following relation holds:*

$$\mathscr{D}_m(\mathcal{F}^\psi_{q,r})_{\mathcal{S}^p} = \sigma_m(\mathcal{F}^\psi_{\infty,r})_{\mathcal{S}^\infty} = \psi(n_{m+1}), \tag{9.47}$$

where the number n_{m+1} *is defined by* (9.34) *for* $s = m + 1$.

Let, as above, the numbers $V_s = V_s(d)$, $s = 0, 1, \ldots$, be defined by (9.30).

Theorem 9.4. *Assume that* $0 < r \le \infty$, $m \in [V_{s-1}, V_s)$, $s \in \mathbb{N}$ *and* $\psi \in \mathfrak{M}''_\infty$.

(i) Let $0 < p < q \le \infty$ *and the function* ψ *also satisfies the condition*

$$\lim_{t\to\infty} \frac{t^\beta \psi(t+1)}{\psi(t)} = 0 \tag{9.48}$$

with $\beta = \frac{d-1}{p}$. *Then*

$$\sigma_m(\mathcal{F}^\psi_{q,r})_{\mathcal{S}^p} \asymp \psi(s)\frac{(V_s - m)^{1/p}}{m^{\frac{d-1}{qd}}}. \tag{9.49}$$

(ii) Let $0 < q \le p < \infty$ *and* ψ *satisfies condition* (9.48) *with* $\beta = \frac{d-1}{q}$. *Then*

$$\sigma_m(\mathcal{F}^\psi_{q,r})_{\mathcal{S}^p} \asymp \frac{\psi(s)}{(m+1-V_{s-1})^{1/q-1/p}}, \tag{9.50}$$

provided that $m = V_{s-1}$ *or that the following additional condition is satisfied:*

$$p(V_s - m) \ge q(V_s - V_{s-1}), \tag{9.51}$$

and relation (9.49) *holds, provided that the condition* (9.51) *is not satisfied.*

(iii) If $0 < q < p = \infty$ *and* ψ *satisfies* (9.48) *with* $\beta = \frac{d-1}{q}$*, then* (9.50) *holds.*

(iv) If $0 < p < q \le \infty$ *and* ψ *satisfies* (9.48) *with* $\beta = (d-1)(1/p - 1/q)$*, then*

$$\mathscr{D}_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \asymp \psi(s)(V_s - m)^{1/p-1/q}. \tag{9.52}$$

Remark 9.3. *For any* $\psi \in \mathfrak{M}''_\infty$*, condition* (9.48) *is equivalent to the following:*

$$\lim_{t\to\infty} \left(\frac{|\psi'(t)|}{\psi(t)} - \beta \ln t \right) = +\infty. \tag{9.53}$$

Therefore, for $d = 1$*, condition* (9.48) *holds for any* $\psi \in \mathfrak{M}''_\infty$*.*

Proof. Indeed, if $\psi \in \mathfrak{M}''_\infty$ and (9.53) holds, then for any $M > 0$ and sufficiently large t, we have

$$\ln \frac{\psi(t)}{\psi(t+1)} = \int_t^{t+1} \frac{|\psi'(\tau)|}{\psi(\tau)} \mathrm{d}\tau > \frac{|\psi'(t)|}{\psi(t)} > \beta \ln t + M.$$

Therefore, relation (9.48) is satisfied.

On the other hand side, if $\psi \in \mathfrak{M}''_\infty$ and (9.48) is satisfied, then for any $M > 0$ and sufficiently large t, we have

$$M \le \ln \frac{\psi(t)}{\psi(t+1)} - \beta \ln(t+1) = \int_t^{t+1} \frac{|\psi'(\tau)|}{\psi(\tau)} \mathrm{d}\tau - \beta \ln(t+1) \le \frac{|\psi'(t+1)|}{\psi(t+1)} - \beta \ln(t+1),$$

and relation (9.53) holds. □

Remark 9.4. *From* (9.49) *and* (9.32)*, it follows that under the conditions when relation* (9.49) *holds, we have*

$$\frac{\psi(s)}{m^{\frac{d-1}{qd}}} \ll \sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \ll \psi(s) m^{\frac{d-1}{d}(1/p-1/q)}.$$

Note that in the case when $r = \infty$, for any $s \in \mathbb{N}$ we have $V_s = (2s+1)^d$. Therefore, if $m \in [V_{s-1}, V_s)$, the number s is defined by the equality $s = [\frac{(m+1)^{1/d}}{2}]$. Here and below, $[x]$ denotes the greatest integer in x.

If $d = 1$, the classes $\mathcal{F}^{\psi}_{q,r} =: \mathcal{F}^{\psi}_{q}$ does not depend on r, and for any $m \in [V_{s-1}, V_s)$ we have $m = V_{s-1} = V_s - 1$ and $s = [\frac{m+1}{2}]$. Therefore, for any $\psi \in \mathfrak{M}''_\infty$,

$$\sigma_m(\mathcal{F}^{\psi}_{q})_{\mathcal{S}^p(\mathbb{T}^1)} \asymp \mathscr{D}_m(\mathcal{F}^{\psi}_{q})_{\mathcal{S}^p(\mathbb{T}^1)} \asymp \psi\left(\left[\frac{m+1}{2}\right]\right), \quad 0 < p,\, q \le \infty.$$

If $d > 1$, the obtained estimates depend significantly on the placement of the number m on the half-segment $[V_{s-1}, V_s)$. Considering some specific subsequences $m(s)$ in Theorem 9.4, we obtain the following corollary.

Corollary 9.1. *Assume that $d > 1$, $m \in [V_{s-1}, V_s)$, $s \in \mathbb{N}$, the parameters p, q, r and ψ satisfy condition of Theorem 9.4.*

(i) Let $m = m(s) = V_s - c_s$, $1 \le c_s \le c$. Then for $0 < p, q \le \infty$

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \asymp \frac{\psi(s)}{m^{\frac{d-1}{qd}}}, \tag{9.54}$$

and for $0 < p < q \le \infty$

$$\mathscr{D}_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \asymp \psi(s). \tag{9.55}$$

(ii) Let the sequence $m = m(s)$ be such that

$$V_s - m(s) \asymp V_s - V_{s-1}. \tag{9.56}$$

Then for any $0 < p < q \le \infty$

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \asymp \psi(s) m^{\frac{d-1}{d}(1/p-1/q)} \tag{9.57}$$

and

$$\mathscr{D}_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \asymp \psi(s) m^{\frac{d-1}{d}(1/p-1/q)}.$$

If $0 < q \le p < \infty$, then relation (9.57) holds, in particular, when $m(s) - V_{s-1} \to \infty$ as $s \to \infty$. If $m = m(s) = V_{s-1} + c_s$, $0 \le c_s \le c$, then for $0 < q < p < \infty$

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \asymp \psi(s). \tag{9.58}$$

In the case $0 < p = q < \infty$, if $m = m(s) = V_{s-1} + c_s$, $1 \le c_s \le c$, then relation (9.57) holds, and if $m = m(s) = V_{s-1}$, then relation (9.58) holds.

Remark 9.5. *Comparing the obtained estimates, we conclude the following:*

(i) If the function ψ belongs to the set B and satisfies condition (9.37) with a certain $\beta > d(1/p - 1/q)$ or to the set $\mathfrak{M}'_\infty \cup \mathfrak{M}^c_\infty$, then for $0 < p \le q \le \infty$

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \asymp \mathscr{D}_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p}. \tag{9.59}$$

(ii) If $\psi \in B \cup \mathfrak{M}'_\infty$ and $0 < q < p \le \infty$, then

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} = o(\mathscr{D}_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p}), \quad m \to \infty. \tag{9.60}$$

(iii) If $\psi \in \mathfrak{M}^c_\infty$ and $0 < q < p \le \infty$, then (9.60) holds for $d > 1$, and

$$\sigma_m(\mathcal{F}^{\psi}_{q})_{\mathcal{S}^p(\mathbb{T}^1)} \asymp \mathscr{D}_m(\mathcal{F}^{\psi}_{q})_{\mathcal{S}^p(\mathbb{T}^1)}. \tag{9.61}$$

(iv) If $\psi \in \mathfrak{M}''_\infty$, then for any $0 < p,\, q \le \infty$ relation (9.61) holds. If $d > 1$, then relation (9.59) holds, in particular, when the sequence $m = m(s)$ satisfies (9.56) and $0 < p < q \le \infty$, when $p = q = \infty$, or when $m = m(s) = V_{s-1} + c_s$, $0 \le c_s \le c$, and $0 < q < p < \infty$.

Note that for finite values $0 < p, q < \infty$, the statements of Theorems 9.2, 9.3 and 9.4 were proven in [33, 35–37]. However, given the relative inaccessibility of these sources for English-speaking readers, we present these statements in this work with proof, slightly changing their presentation and structure.

In the case when $\psi(t) = 1$, relation (9.67) also follows from Lemma 4.4 [20]. For $d = 1$ and $0 < p,\, q < \infty$, statements similar to Theorems 9.2, 9.3 and 9.4 were proven in [34]. Exact order estimates of integral analogues of the quantities $\sigma_m(\mathcal{F}^\psi_{q,r})_{\mathcal{S}^p}$ were found in [48].

9.4 Approximation characteristics of the classes $\mathcal{F}^\psi_{q,r}$ in the spaces L_p

In the case where $2 \le p < \infty$, based on the Hausdorff-Young theorem (see, for example, [54, §0.1]), for any $f \in L_p$, the following inequality holds:

$$\|f\|_{L_p} \le \|f\|_{\mathcal{S}^{p'}}. \tag{9.62}$$

Here and below, for any $1 < p < \infty$, we set $p' := \frac{p}{p-1}$ and $p' := \infty$ when $p = 1$.

If $1 \le p < 2$, then for any $f \in L_p(\mathbb{T}^d)$

$$\|f\|_{L_p} \le \|f\|_{L_2} = \|f\|_{\mathcal{S}^2}. \tag{9.63}$$

Thus, from the estimates for the approximate quantities in the spaces $\mathcal{S}^p$ obtained in Subsection 9.3.3, estimates from above of the similar quantities in the spaces L_p also follow. Here, we consider some of the cases in which the corresponding estimates from below are obtained.

As mentioned above, in the case where ψ is a power function, i.e., $\psi(t) = t^{-s}$, $s > 0$, for all $1 \le p \le \infty$ the exact order estimates of the quantities $\sigma_m(\mathcal{F}^\psi_{q,\infty})_{L_p}$ and $G_m(\mathcal{F}^\psi_{q,\infty})_{L_p}$ were obtained in [10] and [53], correspondingly. In particular, from Theorems 6.1 [10] and 3.1 [53], it follows that for all $s > d(1 - 1/q)_+$,

$$\sigma_m(\mathcal{F}^s_{q,\infty})_{L_p} \asymp m^{-\frac{s}{d} - \frac{1}{q} + \frac{1}{2}}, \quad 1 \le p \le \infty, \tag{9.64}$$

and

$$G_m(\mathcal{F}^s_{q,\infty})_{L_p} \asymp \begin{cases} m^{-\frac{s}{d} - \frac{1}{q} + \frac{1}{2}}, & 1 \le p < 2, \\ m^{-\frac{s}{d} - \frac{1}{q} + 1 - \frac{1}{p}}, & 2 \le p < \infty. \end{cases} \tag{9.65}$$

From the following Theorem 9.5, in particular, it follows that for $\sigma_m(\mathcal{F}^{\psi}_{q,\infty})_{L_p}$ and $G_m(\mathcal{F}^{\psi}_{q,\infty})_{L_p}$, the estimates of forms (9.64) and (9.65) are satisfied for a wider set of the functions ψ.

Theorem 9.5. *Assume that* $0 < r, q \le \infty$, $1 \le p < \infty$, $\psi \in B$ *and in the case* $\frac{p}{p-1} < q$, *moreover, for all* t, *larger than a certain number* t_0, $\psi(t)$ *is convex and satisfies the condition* (9.37) *with* $\beta > d(1/2 - 1/q)$ *when* $1 < p \le 2$ *and* $\beta > d(1 - 1/p - 1/q)$ *when* $2 \le p < \infty$. *Then*

$$G_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \asymp \sigma^{\perp}_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \asymp \begin{cases} \psi\left(m^{1/d}\right)m^{1/2-1/q}, & 1 \le p \le 2, \\ \psi\left(m^{1/d}\right)m^{1-1/p-1/q}, & 2 \le p < \infty \end{cases} \tag{9.66}$$

for all $1 \le p \le 2$

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \asymp \psi\left(m^{1/d}\right)m^{1/2-1/q}, \tag{9.67}$$

and for all $2 < p < \infty$

$$\psi\left(m^{1/d}\right)m^{1/2-1/q} \ll \sigma_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \ll \psi\left(m^{1/d}\right) m^{1-1/p-1/q}.$$

In the case $2 < p \le \infty$, the following theorem is true.

Theorem 9.6. *Assume that* $1 \le r \le \infty$, $2 < p \le \infty$, $0 < q < \infty$, *the function* ψ *belongs to the set* B *and for all* t, *larger than a certain number* t_0, $\psi(t)$ *is convex and satisfies condition* (9.37) *with* $\beta > d(1 - 1/q)_+$. *Then relation* (9.67) *holds.*

The conditions in Theorems 9.5 and 9.6 guarantee the embedding $\mathcal{F}^{\psi}_{q,r} \subset L_p$.

Putting $r = \infty$ and $\psi(t) = t^{-s}$, from Theorems 9.5 and 9.6 we obtain the following corollary:

Corollary 9.2. *Assume that* $1 \le p < \infty$, $0 < q < \infty$, s *is a positive number, which in the case* $\frac{p}{p-1} < q$, *satisfies the inequality* $s > \beta$, *where* β *is defined in Theorem 9.5. Then relation relation* (9.65) *holds for* $1 \le p < \infty$, *and relation* (9.64) *holds for* $1 \le p \le 2$. *If* $s > d(1 - 1/q)_+$, *then relation* (9.64) *holds for* $1 \le p \le \infty$.

This statement complements the above-mentioned results of [10] and [53] in the following sense:

(i) From Corollary 9.2 it follows that in the case $1 < q \le \frac{p}{(p-1)}$, relation (9.64) (for $1 \le p \le 2$) and relation (9.65) (for $1 \le p < \infty$) also hold for all $s > 0$.

(ii) If $1 < p \le 2$ and $q > \frac{p}{p-1}$, then relations (9.64) and (9.65) also hold for all s such that $d(1/2 - 1/q) < s \le d(1 - 1/q)$.

(iii) If $2 < p < \infty$ and $q > \frac{p}{p-1}$, then relation (9.65) also holds for all s such that $d(1 - 1/p - 1/q) < s \le d(1 - 1/q)$.

(iv) In the case $2 < p \le \infty$, conditions on s in Corollary 9.2 (for validity of relation (9.64)) are the same as in Theorem 6.1 [10].

Note also that if $0 < q \le \frac{p}{p-1}$, then the conditions of Theorem 9.5 are satisfied, for example, for the functions $\psi(t) = t^{-s}\ln^{\varepsilon}(t+\mathrm{e})$, where $s > 0$, $\varepsilon \in \mathbb{R}$, $\psi(t) = \ln^{\varepsilon}(t+\mathrm{e})$, $\varepsilon < 0$, and $\psi(t) \equiv c$. If $1 < \frac{p}{p-1} < q$ and $1 < p \le 2$, then the conditions of Theorem 9.5 are satisfied for $\psi(t) = t^{-s}\ln^{\varepsilon}(t+\mathrm{e})$, where $\varepsilon \in \mathbb{R}$ and $s > d(\frac{1}{2} - \frac{1}{q})$. If $1 < \frac{p}{p-1} < q$ and $2 < p < \infty$, then the conditions of Theorem 9.5 are satisfied for $\psi(t) = t^{-s}\ln^{\varepsilon}(t+\mathrm{e})$, where $\varepsilon \in \mathbb{R}$ and $s > d(1 - 1/p - \frac{1}{q})$. The conditions of Theorem 9.6 are satisfied for $\psi(t) = t^{-s}\ln^{\varepsilon}(t+\mathrm{e})$, where $\varepsilon \in \mathbb{R}$ and $s > d(1-1/q)_+$.

Consider the case when the function ψ decreases faster that any power function.

Theorem 9.7. *Assume that* $m \in \mathbb{N}$, $0 < r$, $q \le \infty$, $1 \le p < \infty$, $n \in [V_{m-1}, V_m)$ *and the function* ψ *belongs to the set* $\mathfrak{M}''_{\infty}$ *and satisfies* (9.48) *with* $\beta > \max\{\frac{d-1}{p'}, \frac{d-1}{q}\}$.

(i) If $m = m(s) = V_s - c_s$, $1 \le c_s \le c$, *then for any* $0 < q \le \infty$

$$\sigma_m^{\perp}(\mathcal{F}^{\psi}_{q,r})_{L_p} \asymp G_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \asymp \frac{\psi(s)}{m^{(d-1)/(qd)}}.$$

(ii) If $m = m(s) = V_{s-1} + c_s$, $0 \le c_s \le c$, *and* $0 < q < p' < \infty$ *or if* $m = m(s) = V_{s-1}$ *and* $0 < p' = q < \infty$, *then*

$$\sigma_m^{\perp}(\mathcal{F}^{\psi}_{q,r})_{L_p} \asymp G_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \asymp \psi(s).$$

As noted above, if $d = 1$, the classes $\mathcal{F}^{\psi}_{q,r} = \mathcal{F}^{\psi}_{q}$ does not depend on r, and $s = [\frac{m+1}{2}]$ for any $m \in [V_{s-1}, V_s)$. Therefore, the following corollary follows from Theorem 9.7.

Corollary 9.3. *For any* $1 \le p < \infty$, $0 < q \le \infty$ *and any function* $\psi \in \mathfrak{M}''_{\infty}$

$$\sigma_m^{\perp}(\mathcal{F}^{\psi}_{q})_{L_p(\mathbb{T}^1)} \asymp G_m(\mathcal{F}^{\psi}_{q})_{L_p(\mathbb{T}^1)} \asymp \psi\left(\left[\frac{m+1}{2}\right]\right).$$

The similar estimate can be proven in the case when $d = 1$ and $\psi \in \mathfrak{M}^{c}_{\infty}$.

Theorem 9.8. *For any* $1 \le p < \infty$, $0 < q \le \infty$ *and any function* $\psi \in \mathfrak{M}^{c}_{\infty}$

$$\sigma_m^{\perp}(\mathcal{F}^{\psi}_{q})_{L_p(\mathbb{T}^1)} \asymp G_m(\mathcal{F}^{\psi}_{q})_{L_p(\mathbb{T}^1)} \asymp \mathscr{D}_m^{\perp}(\mathcal{F}^{\psi}_{q,r})_{L_p} \asymp \psi\left(\frac{m}{2}\right) \asymp \psi\left(\left[\frac{m+1}{2}\right]\right).$$

The following statement also holds for the quantities $\mathscr{D}_m^{\perp}(\mathcal{F}^{\psi}_{q,r})_{L_p}$.

Theorem 9.9. *Assume that* $m \in \mathbb{N}$, $0 < r, q \le \infty$, $1 \le p < \infty$ *and* $m \in [V_{s-1}, V_s)$.

(i) If $0 < q \le p'$, then for any positive non-increasing function ψ, we have

$$\mathscr{D}_m^{\perp}(\mathcal{F}_{q,r}^{\psi})_{L_p} = \psi(s). \tag{9.68}$$

(ii) If $1 < p' < q \le \infty$ and $\psi \in \mathfrak{M}''_{\infty}$ and condition (9.48) is satisfied with $\beta = \frac{(d-1)(q-p')}{p'q}$, then for the sequence $m = m(s) = V_s - c_s$, $1 \le c_s \le c$, the following estimate holds:

$$\mathscr{D}_m^{\perp}(\mathcal{F}_{q,r}^{\psi})_{L_p} \asymp \psi(s). \tag{9.69}$$

Corollary 9.4. *Let $1 \le p < \infty$. For any positive non-increasing function ψ and $0 < q \le \frac{p}{p-1}$,*

$$\mathscr{D}_m^{\perp}(\mathcal{F}_{q}^{\psi})_{L_p(\mathbb{T}^1)} = \psi\left(\left[\frac{m+1}{2}\right]\right). \tag{9.68$'$}$$

If $1 < p' < q \le \infty$, then for any $\psi \in \mathfrak{M}''_{\infty}$,

$$\mathscr{D}_m^{\perp}(\mathcal{F}_{q}^{\psi})_{L_p(\mathbb{T}^1)} \asymp \psi\left(\left[\frac{m+1}{2}\right]\right). \tag{9.69$'$}$$

In the case when $\psi \in B$, estimates for $\mathscr{D}_m^{\perp}(\mathcal{F}_{q,r}^{\psi})_{L_p}$ are given by the following statement.

Theorem 9.10. *Assume that conditions of Theorem 9.5 are satisfied. Then*

$$\mathscr{D}_m^{\perp}(\mathcal{F}_{q,r}^{\psi})_{L_p} \asymp \begin{cases} \psi\left(m^{1/d}\right), & 0 < q \le \dfrac{p}{p-1}, \quad 1 \le p < \infty, \\ \psi\left(m^{1/d}\right) m^{1/2-1/q}, & \dfrac{p}{p-1} < q \le \infty, \quad 1 < p < 2, \\ \psi\left(m^{1/d}\right) m^{1-1/p-1/q}, & \frac{p}{p-1} < q \le \infty, \quad 2 \le p < \infty. \end{cases} \tag{9.70}$$

Remark 9.6. *Analyzing the results of Section 9.4, we conclude the following:*

(i) If $d = 1$ and $\psi \in \mathfrak{M}''_{\infty} \cup \mathfrak{M}^{c}_{\infty}$, then for any $1 \le p < \infty$ and $0 < q \le \infty$

$$\sigma_m^{\perp}(\mathcal{F}_{q}^{\psi})_{L_p(\mathbb{T}^1)} \asymp G_n(\mathcal{F}_{q}^{\psi})_{L_p(\mathbb{T}^1)} \asymp \mathscr{D}_m^{\perp}(\mathcal{F}_{q}^{\psi})_{L_p(\mathbb{T}^1)}.$$

(ii) If $d > 1$ and condition of Theorems 9.7 and (9.9) are satisfied, then the similar estimate

$$\sigma_m^{\perp}(\mathcal{F}_{q,r}^{\psi})_{L_p} \asymp G_m(\mathcal{F}_{q,r}^{\psi})_{L_p} \asymp \mathscr{D}_m^{\perp}(\mathcal{F}_{q,r}^{\psi})_{L_p} \tag{9.71}$$

holds when $m = m(s) = V_{s-1} + c_s$, $0 \le c_s \le c$, and $0 < q < p' < \infty$, or when $m = m(s) = V_{s-1}$ and $0 < p' = q < \infty$. If $m = m(s) = V_s - c_s$, $1 \le c_s \le c$, then for any $0 < q \le \infty$

$$\sigma_m^{\perp}(\mathcal{F}_{q,r}^{\psi})_{L_p} \asymp G_n(\mathcal{F}_{q,r}^{\psi})_{L_p} = o\Big(\mathscr{D}_m^{\perp}(\mathcal{F}_{q,r}^{\psi})_{L_p}\Big). \tag{9.72}$$

(iii) In the case when conditions of Theorem 9.5 are satisfied, for all $p' \le q \le \infty$ *relation* (9.71) *holds, and for all* $0 < q \le p'$, $1 \le p < \infty$ *relation* (9.72) *holds.*

Note that for finite values $0 < q < \infty$, the statements of Theorems 9.5-9.10 were were proven in [33, 35–37]. In the case when $\psi(t) = 1$, $0 < q \le 1$ and $2 \le p < \infty$, the upper estimate for the quantity $\sigma_m(\mathcal{F}^{\psi}_{q,r})_{L_p}$ can be also obtained from Theorem 4.4 [19].

9.5 Proof of theorems

9.5.1 Proof of Theorem 9.1

Proof of Theorem 9.1. Consider the case when $0 < q < p = \infty$ or $p = q = \infty$. Due to (9.4), (9.5), (9.1), (9.2) and (9.15), for any $f \in \mathcal{F}^{\Psi}_q$ and $\gamma_m \in \Gamma_m$, we have

$$E_{\gamma_m}(f)_{\mathcal{S}^p} = \mathscr{E}_{\gamma_m}(f)_{\mathcal{S}^p} = \| f - S_{\gamma_m}(f)\|_{\mathcal{S}^p} = \|\{\Psi_{\gamma_m}(k)\widehat{f}(k)/\Psi_k\}_{k\in\mathbb{Z}^d}\|_{l_p(\mathbb{Z}^d)}. \quad (9.73)$$

Therefore, for any $f \in \mathcal{F}^{\Psi}_q$ and $\gamma_m \in \Gamma_m$, the following upper estimate holds:

$$E_{\gamma_m}(f)_{\mathcal{S}^\infty} = \sup_{k\in\mathbb{Z}^d}\left|\frac{\Psi_{\gamma_m}(k)\widehat{f}(k)}{\Psi_k}\right| \le \bar{\Psi}_{\gamma_m}(1)\sup_{k\in\mathbb{Z}^d}\left|\frac{\widehat{f}(k)}{\Psi_k}\right| \le$$
$$\le \bar{\Psi}_{\gamma_m}(1)\|\{\widehat{f}(k)/\Psi_k\}_{k\in\mathbb{Z}^d}\|_{l_q(\mathbb{Z}^d)} \le \bar{\Psi}_{\gamma_m}(1). \quad (9.74)$$

Let $k_0 \in \mathbb{Z}^d$ be a vector such that $|\Psi_{k_0}| = \bar{\Psi}_{\gamma_m}(1)$. The function $f_0(x) := \Psi_{k_0}e_{k_0}(x)$ belongs to the set $\mathcal{F}^{\Psi}_q$ and

$$E_{\gamma_m}(f_0)_{\mathcal{S}^p} = \mathscr{E}_{\gamma_m}(f_0)_{\mathcal{S}^p} = |\Psi_{k_0}| = \bar{\Psi}_{\gamma_m}(1). \quad (9.75)$$

Combining (9.74) and (9.75), we see that in this case relation (9.16) holds.

Now, assume that $0 < p < q = \infty$ and the series $\sum_{k\in\mathbb{Z}^d}|\Psi_k|^p$ converges. By virtue of (9.73), for any $f \in \mathcal{F}^{\Psi}_\infty$ and $\gamma_m \in \Gamma_m$, we have

$$E_{\gamma_m}(f)^p_{\mathcal{S}^p} = \sum_{k\in\mathbb{Z}^d}\left|\frac{\Psi_{\gamma_m}(k)\widehat{f}(k)}{\Psi_k}\right|^p \le \sup_{k\in\mathbb{Z}^d}\left|\frac{\widehat{f}(k)}{\Psi_k}\right|^p \sum_{k\in\mathbb{Z}^d}|\Psi_{\gamma_m}(k)|^p \le \sum_{j=1}^{\infty}\bar{\Psi}^p_{\gamma_m}(j). \quad (9.76)$$

Since

$$\sum_{k=1}^{\infty}\bar{\Psi}^p_{\gamma_m}(k) \le \sum_{k=1}^{\infty}\bar{\Psi}^p_k = \sum_{k\in\mathbb{Z}^d}|\Psi_k|^p < \infty,$$

for any $\varepsilon > 0$ there exists a number N_ε such that $\sum_{j=N_\varepsilon+1}^{\infty}\bar{\Psi}^p_{\gamma_m}(j) < \varepsilon$. Consider any collection $\gamma^*_{N_\varepsilon} = \{k_1,\ldots,k_{N_\varepsilon}\}$ from the set $\Gamma_{N_\varepsilon}\setminus\gamma_m$ such that

$$\gamma^*_{N_\varepsilon} = \{k_j \in \mathbb{Z}^d :\ |\Psi_{k_j}| = \bar{\Psi}_{\gamma_m}(j),\ j = 1,2,\ldots,m\}$$

Then the function $f_\varepsilon(x) := \sum_{k\in\gamma^*_{N_\varepsilon}} \Psi_k e_k(x)$ belongs to the set $\mathcal{F}^\Psi_\infty$ and

$$E_{\gamma_m}(f_\varepsilon)^p_{\mathcal{S}^p} = \sum_{k\in\gamma^*_{N_\varepsilon}} |\Psi_k|^p = \sum_{j=1}^{N_\varepsilon} \bar{\Psi}^p_{\gamma_m}(j) > \sum_{j=1}^{\infty} \bar{\Psi}^p_{\gamma_m}(j) - \varepsilon. \tag{9.77}$$

Combining (9.76) and (9.77) and taking into account the arbitrariness of ε, we see that (9.21) is indeed true.

Finally, considering the infima of all quantities (9.16) and (9.21) over all $\gamma_m \in \Gamma_m$, we see that in the corresponding cases, relations (9.18) and (9.22) are true, and for any collection $\gamma^*_m \in \Gamma_m$ satisfying (9.19), relation (9.20) holds. □

9.5.2 Auxiliary estimates

Before proving the theorems on estimates of best n-term approximations and basis widths in the spaces $\mathcal{S}^p$, we give several auxiliary statements.

Firstly, we show how the belonging of a function to the sets B, $\mathfrak{M}'_\infty$, $\mathfrak{M}^c_\infty$ or $\mathfrak{M}''_\infty$ affects its values at different points.

As follows from (9.36), for any $\psi \in B$ and $c > 1$, we have

$$\psi(t) \asymp \psi(ct).$$

In the case, when $\psi \in \mathfrak{M}'_\infty \cup \mathfrak{M}^c_\infty$, we can obtain the following statement.

Proposition 9.2. *Let $d \geq 1$ and $c > 1$. Then for any $\psi \in \mathfrak{M}'_\infty \cup \mathfrak{M}^c_\infty$*

$$\psi(t^{1/d}) \asymp \psi((t+c)^{1/d}) \quad \textit{and} \quad \frac{\psi(t^{1/d})}{|\psi'(t^{1/d})|} \asymp \frac{\psi(ct^{1/d})}{|\psi'(ct^{1/d})|}. \tag{9.78}$$

Proof. If ψ belongs to $\mathfrak{M}'_\infty$ or $\mathfrak{M}^c_\infty$, then $\frac{|\psi'(t)|}{\psi(t)} \leq K_6$ for any $t \geq 1$. Then

$$\ln \frac{\psi(t^{1/d})}{\psi((t+c)^{1/d})} = \int_{t^{1/d}}^{(t+c)^{1/d}} \frac{|\psi'(\tau)|}{\psi(\tau)} d\tau \leq K_6 \left((t+c)^{\frac{1}{d}} - t^{\frac{1}{d}}\right) \leq K_6 c.$$

Taking into account this relation and monotonicity of ψ, we see that the first relation in (9.78) is true.

For $\psi \in \mathfrak{M}^c_\infty$ the validity of the second relation in (9.78) is obvious. For $\psi \in \mathfrak{M}'_\infty$ it follows from monotonicity of the functions $\alpha(\psi, t) = \frac{\psi(t)}{t|\psi'(t)|}$ and $\frac{\psi(t)}{|\psi'(t)|}$:

$$1 \geq \frac{\psi(t^{1/d})/|\psi'(t^{1/d})|}{\psi(ct^{1/d})/|\psi'(ct^{1/d})|} = \frac{1}{c} \cdot \frac{\alpha(\psi, t^{1/d})}{\alpha(\psi, ct^{1/d})} \geq \frac{1}{c}.$$

□

Finally, in the case when $\psi \in \mathfrak{M}''_\infty$, for any positive c we have

$$\ln \frac{\psi(t)}{\psi(t+c)} = \int_t^{t+c} \frac{|\psi'(\tau)|}{\psi(\tau)} d\tau \geq \frac{|\psi'(t)|}{\psi(t)} \uparrow \infty.$$

Next, we formulate several known facts for functions from the sets $\mathfrak{M}'_\infty$, $\mathfrak{M}^c_\infty$ and $\mathfrak{M}''_\infty$. For this purpose, following Stepanets [44, Chapter 3] (see also [42]), for any $\psi \in \mathfrak{M}$, consider the functions $\eta(t) = \eta(\psi, t)$ and $\mu(t) = \mu(\psi, t)$ such that

$$\eta(t) = \psi^{-1}\left(\frac{\psi(t)}{2}\right) \quad \text{and} \quad \mu(t) = \frac{t}{\eta(t) - t}, \quad t \geq 1,$$

where ψ^{-1} is the inverse function of ψ, as well as the sets

$$\mathfrak{M}^+_\infty = \{\psi \in \mathfrak{M} : \ \mu(\psi, t) \uparrow \infty\} \quad \text{and} \quad F = \{\psi \in \mathfrak{M} : \ \eta'(\psi, t) \leq K\}.$$

Remark 9.7. *It follows from Theorems 12.1 and 13.1 [44, Chapter 3] (see also [42, Theorems 1 and 2]) that all sets $\mathfrak{M}'_\infty$, $\mathfrak{M}^c_\infty$ and $\mathfrak{M}''_\infty$ belong to the set $\mathfrak{M}^+_\infty \subset F$. By virtue of Remarks 13.1 and 13.2 from [44, Chapter 3] (see also [42, Remarks 1 and 2]), for any function $\psi \in F$ (in particular, for any $\psi \in \mathfrak{M}'_\infty \cup \mathfrak{M}^c_\infty \cup \mathfrak{M}''_\infty$) and for all $t \geq 1$, the following relations hold:*

$$K_7(\eta(\psi, t) - t) \leq \psi(t)/|\psi'(t)| = t\alpha(\psi, t) \leq K_8(\eta(\psi, t) - t) \tag{9.79}$$

and

$$2(\eta(\psi, t) - t) \leq \eta(\psi, \eta(\psi, t)) - \eta(\psi, t) \leq K_9(\eta(\psi, t) - t). \tag{9.80}$$

Now we prove statements that give estimates for some integrals and sums containing functions from the sets B, $\mathfrak{M}'_\infty$, $\mathfrak{M}^c_\infty$ or $\mathfrak{M}''_\infty$

Proposition 9.3. *Let $d \geq 1$ and $0 < s < \infty$. Then for any $\psi \in B$*

$$\int_1^l \frac{t^{d-1}\mathrm{d}t}{\psi^s(t)} \asymp \sum_{k=1}^{l} \frac{k^{d-1}}{\psi^s(k)} \asymp \frac{l^d}{\psi^s(l)}. \tag{9.81}$$

If, in addition, for all t greater than a certain number t_0, $\psi(t)$ is convex and satisfies condition (9.37) *with $\beta > d/s$, then*

$$\int_l^\infty t^{d-1}\psi^s(t)\mathrm{d}t \asymp \sum_{k=l+1}^{\infty} k^{d-1}\psi^s(k) \asymp l^d\psi^s(l). \tag{9.82}$$

Proof. Since the function $\frac{t^{d-1}}{\psi^s(t)}$ is monotonically increasing, we have

$$I_l := \int_1^l \frac{t^{d-1}\mathrm{d}t}{\psi^s(t)} \leq \sum_{n=1}^{l} \frac{n^{d-1}}{\psi^s(n)} \leq I_{l+1}, \tag{9.83}$$

and for any $\psi \in B$, the following necessary estimate is true:

$$\frac{l^d}{\psi^s(l)} \ll \frac{(l/2)^d}{\psi^s(l/2)} \ll \sum_{l/2 \le n \le l} \frac{n^{d-1}}{\psi^s(n)} \ll \sum_{k=1}^{l} \frac{n^{d-1}}{\psi^s(n)} \ll \frac{l^d}{\psi^s(l)}.$$

Now, assume that for all t greater than a certain number t_0, the function ψ is also convex and satisfies condition (9.37) with $\beta > \frac{d}{s}$. Then the function $t^d\psi^s(t)$ decreases to zero for $t > t_0$. Therefore, for $l > t_0$, $l \in \mathbb{N}$,

$$\int_l^\infty t^{d-1}\psi^s(t)\mathrm{d}t =: J_l \ge \sum_{n=l+1}^{\infty} n^{d-1}\psi^s(n) \ge J_{l+1}. \tag{9.84}$$

Since $\psi \in B$, then

$$J_l \ge \int_l^{2l} t^{d-1}\psi^s(t)\mathrm{d}t \gg l^d\psi^s(l). \tag{9.85}$$

Applying (9.37) and integrating by parts, we obtain

$$J_l \le \frac{1}{\beta}\int_l^\infty t^d\psi^{s-1}(t)|\psi'(t)|dt = \frac{l^d\psi^s(l)}{s\beta} + \frac{d}{s\beta}J_l.$$

Then in view of (9.37), we see that

$$J_l \le \frac{1}{s\beta - d} l^d\psi^s(l) \ll l^d\psi^s(l) \tag{9.86}$$

Combining (9.85), (9.86) and (9.84), we obtain (9.82). □

Proposition 9.4. *Let $d \ge 1$ and $0 < s < \infty$. Then for any $\psi \in \mathfrak{M}'_\infty \cup \mathfrak{M}^c_\infty$*

$$\int_1^l \frac{t^{d-1}\mathrm{d}t}{\psi^s(t)} \asymp \sum_{n=1}^{l} \frac{n^{d-1}}{\psi^s(n)} \asymp \frac{l^d\alpha(\psi, l)}{\psi^s(l)} \tag{9.87}$$

and

$$\int_l^\infty t^{d-1}\psi^s(t)\mathrm{d}t \asymp \sum_{n=l+1}^{\infty} n^{d-1}\psi^s(n) \asymp l^d\psi^s(l)\alpha(\psi, l). \tag{9.88}$$

Proof. First let us prove relation (9.87). Consider the function

$$f(t) = f(\psi, t) := \frac{\psi^s(t)}{t^{d-1}}, \quad t \ge 1. \tag{9.89}$$

Its derivative has the form

$$f'(t) = \left(\frac{\psi^s(t)}{t^{d-1}}\right)' = -\frac{\psi^s(t)}{t^{d-1}}\left(s\frac{|\psi'(t)|}{\psi(t)} + \frac{d-1}{t}\right), \quad t \ge 1.$$

Therefore, for any $\psi \in \mathfrak{M}'_\infty \cup \mathfrak{M}^c_\infty$, we have

$$\frac{1}{t\alpha(f,t)} := \frac{|f'(t)|}{f(t)} = \frac{|\psi'(t)|}{\psi(t)}(s+(d-1)\alpha(\psi,t)) \asymp \frac{|\psi'(t)|}{\psi(t)} \asymp \frac{1}{t\alpha(\psi,t)} \tag{9.90}$$

and $\alpha(f,t) = \frac{f(t)}{t|f'(t)|} \downarrow 0$. Using (9.89) and (9.83), integrating by parts, we obtain

$$I_l = \int_1^l \frac{\mathrm{d}t}{f(t)} = \frac{l}{f(l)} - \frac{1}{f(1)} - \int_1^l \frac{\mathrm{d}t}{\alpha(f,t)f(t)} \geq \frac{l}{f(l)} - \frac{1}{f(1)} - \frac{I_l}{\alpha(f,l)},$$

and

$$I_l \geq \frac{\alpha(f;l)}{1+\alpha(f;l)}\left(\frac{l}{f(l)} - \frac{1}{f(1)}\right) \gg \frac{l\alpha(f,l)}{f(l)}. \tag{9.91}$$

Further, if $\psi \in \mathfrak{M}'_\infty$, then by (9.90), we have $\frac{f(t)}{|f'(t)|} \uparrow \infty$ and $f(t) \geq \frac{f'^2(t)}{f''(t)}$, $f''(t) := f''(t+)$, for almost all $t \geq 1$. Hence,

$$I_l \leq \int_1^l \frac{f''(t)\mathrm{d}t}{f'^2(t)} = \frac{1}{|f'(l)|} - \frac{1}{|f'(1)|} \leq \frac{1}{|f'(l)|} = \frac{l\alpha(f,l)}{f(l)}. \tag{9.92}$$

Due to (9.91) and (9.92), taking into account (9.89) and (9.90), we see for any $\psi \in \mathfrak{M}'_\infty$

$$I_l = \int_1^l \frac{t^{d-1}\mathrm{d}t}{\psi^s(t)} = \int_1^l \frac{\mathrm{d}t}{f(t)} \asymp \frac{l\alpha(f,l)}{f(l)} \asymp \frac{l^d\alpha(\psi,l)}{\psi^s(l)}. \tag{9.93}$$

If $\psi \in \mathfrak{M}^c_\infty$, by (9.90) and (9.43), we have $K_{10} \leq \frac{f(t)}{|f'(t)|} \leq K_{11}$, $t \geq 1$, and

$$I_l \leq K_{11}\int_1^l \frac{|f'(t)|\mathrm{d}t}{f^2(t)} = K_{11}\left(\frac{1}{f(l)} - \frac{1}{f(1)}\right) \asymp \frac{1}{f(l)} \asymp \frac{l\alpha(f,l)}{f(l)}. \tag{9.94}$$

Based on (9.91), (9.94) (9.89), and (9.90), we see for $\psi \in \mathfrak{M}^c_\infty$, relation (9.93) also holds.

Combining (9.93) and (9.83), we get the second estimate in (9.87).

Now let us prove relation (9.88). For this purpose, we first show that the integral J_l defined in (9.84) satisfies relation

$$J_l \asymp l^d\psi^s(l)\alpha(\psi,l). \tag{9.95}$$

Since $\alpha(\psi,t) \downarrow 0$ for $\psi \in \mathfrak{M}'_\infty \cup \mathfrak{M}^c_\infty$, integrating by parts, we obtain

$$J_l = -\frac{l^d\psi^s(l)}{d} + \frac{s}{d}\int_l^\infty \frac{t^{d-1}\psi^s(t)}{\alpha(\psi,t)}\mathrm{d}t \geq -\frac{l^d\psi^s(l)}{d} + \frac{s}{d\alpha(\psi,l)}J_l$$

and for sufficiently large l (such that $\alpha(\psi,l) < \frac{s}{d}$),

$$J_l \leq \frac{\alpha(\psi,l)}{s-d\alpha(\psi,l)}l^d\psi^s(l) \ll l^d\psi^s(l)\alpha(\psi,l). \tag{9.96}$$

On the other hand-side, by virtue of monotonicity of ψ and (9.79), we have

$$J_l \geq \int_l^{\eta(\psi,t)} t^{d-1}\psi^s(t)\mathrm{d}t \geq l^{d-1}\psi^s(\eta(\psi,l))(\eta(\psi,l)-l) \gg l^d\psi^s(l)\alpha(\psi,l).$$

Therefore, relation (9.95) is true indeed.

Further, the derivative of the function $h(t) := t^{d-1}\psi^s(t)$ has the form

$$h'(t) = t^{d-2}\psi^s(t)\left(d-1-s\frac{t|\psi'(t)|}{\psi(t)}\right) \quad \forall t > 1.$$

Hence, based on (9.42), we see that the function $h(t)$ decreases for sufficiently large t (such that $(d-1)\alpha(\psi,t) < s$), and therefore, relation (9.84) holds, which, based on (9.95) and (9.78), gives the necessary second estimate in (9.88). □

Proposition 9.5. *Let $d \geq 1$ and $0 < s < \infty$. Then for any $\psi \in \mathfrak{M}''_\infty$*

$$\sum_{n=1}^{l} \frac{n^{d-1}}{\psi^s(n)} \asymp \frac{l^{d-1}}{\psi^s(l)} \tag{9.97}$$

and

$$\sum_{n=l+1}^{\infty} n^{d-1}\psi^s(n) \asymp l^{d-1}\psi^s(l+1). \tag{9.98}$$

Proof. For any $\psi \in \mathfrak{M}''_\infty$, we have $\psi(t) \leq K_{12}|\psi'(t)|$, $t \geq 1$. Therefore,

$$I_l = \int_1^l \frac{t^{d-1}\mathrm{d}t}{\psi^s(t)} \leq K_{12}l^{d-1}\int_1^l \frac{|\psi'(t)|\mathrm{d}t}{\psi^{s+1}(t)} = K_{12}l^{d-1}\left(\frac{1}{s\psi^s(l)} - \frac{1}{s\psi^s(1)}\right) \ll \frac{l^{d-1}}{\psi^s(l)}.$$

and estimate (9.97) follows from the following relation:

$$\frac{l^{d-1}}{\psi^s(l)} \leq \sum_{k=1}^{l} \frac{k^{d-1}}{\psi^s(k)} \leq \frac{l^{d-1}}{\psi^s(l)} + \int_1^l \frac{t^{d-1}dt}{\psi^s(t)} \ll \frac{l^{d-1}}{\psi^s(l)}.$$

To prove the above estimate (9.96), we actually used only the fact that ψ belongs to the set $\mathfrak{M}$ and satisfies the condition (9.42). Hence, estimate (9.96) also holds for any $\psi \in \mathfrak{M}''_\infty$. However, since in this case $\frac{\psi(t)}{|\psi'(t)|} = t\alpha(\psi,t) \downarrow 0$, then

$$J_l = \int_l^\infty t^{d-1}\psi^s(t)\mathrm{d}t \ll l^d\psi^s(l)\alpha(\psi,l) \ll l^{d-1}\psi^s(l).$$

It was shown that the function $h(t) = t^{d-1}\psi^s(t)$ decreases for all t such that $\alpha(\psi,t) < \frac{s}{d-1}$. Therefore,

$$\sum_{k=l+1}^{\infty} k^{d-1}\psi^s(k) \ll (l+1)^{d-1}\psi^s(l+1) + J_{l+1} \ll l^{d-1}\psi^s(l+1),$$

and since

$$\sum_{k=l+1}^{\infty} k^{d-1}\psi^s(k) \geq l^{d-1}\psi^s(l+1),$$

relation (9.98) is also true. □

9.5.3 Proof of Theorem 9.2

Proof of Theorem 9.2. First, find the estimates for the quantities $\sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p}$.

Case $0 < q \leq p < \infty$. By virtue of (9.23), (9.29) and (9.33), we have

$$\sigma^p_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} = \sup_{l>m}(l-m)\left(\sum_{n=1}^{n_l-1}\frac{\nu_n}{\psi^q(n)} + \frac{l-V_{n_l-1}}{\psi^q(n_l)}\right)^{-p/q}, \qquad (9.99)$$

where the number n_l is defined by (9.34) for $s = l$.

By virtue of (9.35) and (9.36), we have

$$n_l \asymp l^{1/d} \quad \text{and} \quad \psi(n_l) \asymp \psi\left(l^{1/d}\right). \qquad (9.100)$$

In view of (9.81), we conclude that

$$\sigma^p_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \asymp \sup_{l>m}\frac{l-m}{\left(\frac{n_l^d}{\psi^q(n_l)}\right)^{p/q}} \asymp \sup_{l>m}\frac{\psi^p\left(l^{1/d}\right)(l-m)}{l^{\frac{p}{q}}} \ll \psi^p\left(m^{1/d}\right)\sup_{l>m}\frac{l-m}{l^{\frac{p}{q}}}. \qquad (9.101)$$

For $t > 0$, $m \in \mathbb{N}$ and $s \in (1,\infty)$, the function $h(t) = h(t,s) = \frac{t-m}{t^s}$ attains its maximal value at the point $t_* = \frac{sm}{s-1}$, and

$$h(t_*, s) = \left(1 - \frac{1}{s}\right)^s m^{1-s}.$$

If $s = 1$, then the function $h(t) = h(t;1)$ is non-decreasing and tends to 1 as t increases. Therefore,

$$\sup_{t>0} h(t;1) = \sup_{t>0}\frac{t-m}{t} = \lim_{t\to+\infty}\frac{t-m}{t} = 1. \qquad (9.102)$$

Combining (9.101)–(9.102), we obtain necessary upper estimate:

$$\sigma^p_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \ll \psi^p\left(m^{1/d}\right)m^{1-p/q}.$$

Taking into account (9.101) and (9.36), we also obtain the lower estimate:

$$\sigma^p_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \gg \frac{\psi^p\left((2m)^{1/d}\right)(2m-m)}{(2m)^{p/q}} \asymp \psi^p\left(m^{1/d}\right)m^{1-p/q}.$$

Case $0 < p < q < \infty$. Let $\Psi = \{\Psi_k\}_{k\in\mathbb{Z}^d}$ be a sequence of complex numbers such that there exists a non-increasing rearrangement $\bar{\Psi} = \{\bar{\Psi}_j\}_{j=1}^{\infty}$ of the number system $\{|\Psi_k|\}_{k\in\mathbb{Z}^d}$. For a fixed $m \in \mathbb{N}$ and $l \in \mathbb{N}$, $l > m$, consider the functional

$$Q_m(\Psi, l) := \frac{l-m}{\sum\limits_{j=1}^{l} \bar{\Psi}_j^{-q}}.$$

We have

$$Q_m(\Psi, l+1) = Q_m(\Psi, l) + \frac{\bar{\Psi}_{l+1}^{-q}}{\sum\limits_{j=1}^{l+1} \bar{\Psi}_j^{-q}}(\bar{\Psi}_{l+1}^{q} - Q_m(\Psi, l))$$

and

$$\bar{\Psi}_{l+1}^{q} = Q_m(\Psi, l+1) + \frac{\sum\limits_{j=1}^{l} \bar{\Psi}_j^{-q}}{\sum\limits_{j=1}^{l+1} \bar{\Psi}_j^{-q}}(\bar{\Psi}_{l+1}^{q} - Q_m(\Psi, l))$$

Therefore, in view of monotonicity of $\bar{\Psi}$ and relation (9.25), we conclude that

$$Q_m(\Psi, l) > Q_m(\Psi, l+1) > \Psi^q(l+1) \quad \forall l \geq l_m$$

and

$$Q_m(\Psi, l) \leq Q_m(\Psi, l+1) \leq \Psi^q(l+1) \quad \forall l \in [m, l_m).$$

This yields that

$$\sup_{l>m} Q_n(\Psi, l) = Q_m(\Psi, l_m). \tag{9.103}$$

According to (9.25), we get $\Psi(l_m + 1) > \Psi(l_m)$. Hence, if the system $\{|\Psi_k|\}_{k\in\mathbb{Z}^d}$ has the stepwise form (9.29), then

$$l_m = V_{n_{l_m}} = \sum_{i=0}^{n_{l_m}} \nu_i, \tag{9.104}$$

where n_{l_m} is defined in (9.34) for $s = l_m$. Thus, we have

$$\sigma_m(\mathcal{F}_{q,r}^{\psi})_{\mathcal{S}^p} = \left((l_m - m)^{\frac{q}{q-p}} \left(\sum_{n=1}^{n_{l_m}} \frac{\nu_n}{\psi^q(n)} \right)^{\frac{p}{p-q}} + \sum_{n=n_{l_m}+1}^{\infty} \nu_n \psi^{\frac{pq}{q-p}}(n) \right)^{\frac{q-p}{pq}}, \tag{9.105}$$

where

$$\psi^{-q}(n_{l_m}) \le \frac{1}{l_m - m} \sum_{j=1}^{n_{l_m}} \frac{\nu_n}{\psi^q(n)} < \psi^{-q}(n_{l_m}+1). \tag{9.106}$$

It follows from (9.103) that

$$\sup_{l>m}(l-m)\left(\sum_{n=1}^{n_l-1} \frac{\nu_n}{\psi^q(n)} + \frac{l - V_{n_l-1}}{\psi^q(n_l)}\right)^{-1} = (l_m - m)\left(\sum_{n=1}^{n_{l_m}} \frac{\nu_n}{\psi^q(n)}\right)^{-1}. \tag{9.107}$$

Then similarly to the case $0 < p \le q < \infty$, we show that

$$(l_m - m)\left(\sum_{n=1}^{n_{l_m}} \frac{\nu_n}{\psi^q(n)}\right)^{-1} \asymp \psi^q\left(m^{1/d}\right). \tag{9.108}$$

Taking into account (9.106), (9.108) and (9.36), we see that

$$\psi(n_{l_m}) \asymp \psi\left(m^{1/d}\right). \tag{9.109}$$

Since $\psi \in B$, then in view of (9.33), (9.37) and (9.82), we conclude that

$$\sum_{n=l+1}^{\infty} \nu_n \psi^{\frac{pq}{q-p}}(n) \asymp \sum_{n=l+1}^{\infty} n^{d-1}\psi^{\frac{pq}{q-p}}(n) \asymp l^d \psi^{\frac{pq}{q-p}}(l). \tag{9.110}$$

Let us also show that

$$n_{l_m} \asymp m^{1/d}. \tag{9.111}$$

Indeed, by virtue of (9.104) and (9.100), we see that $\widetilde{m} := K_{13} m^{1/d} \le K_{13} l_m^{1/d} \le n_{l_m}$. On the other hand, integrating each part of (9.40) in the range from $\widetilde{m}$ to n_{l_m}, $\widetilde{m} > t_0$, we obtain $\frac{\psi(\widetilde{m})}{\psi(n_{l_m})} \ge (\frac{n_{l_m}}{\widetilde{m}})^\beta$. Therefore, in view of (9.109) and (9.36), we see that $\widetilde{m} \gg n_{l_m}$ and relation (9.111) is true.

Combining (9.105), (9.108), (9.81), (9.109), (9.110) and (9.111) we obtain (9.39):

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \asymp \left(\frac{m\psi^{\frac{q^2}{q-p}}\left(m^{1/d}\right)}{\psi^q\left(m^{1/d}\right)} + m\psi^{\frac{pq}{q-p}}\left(m^{1/d}\right)\right)^{\frac{q-p}{pq}} \asymp \psi\left(m^{1/d}\right) m^{1/p-1/q}.$$

Case $0 < q < p = \infty$. By virtue of (9.26), (9.29) and (9.33), we have

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} = \left(\sum_{n=1}^{n_{m+1}-1} \frac{\nu_n}{\psi^q(n)} + \frac{m+1-V_{n_{m+1}-1}}{\psi^q(n_{m+1})}\right)^{-\frac{1}{q}}.$$

Taking into account (9.33), (9.81) and (9.100), we see that

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \asymp \left(\sum_{n=1}^{n_{m+1}} \frac{\nu_n}{\psi^q(n)}\right)^{-1/q} \asymp \psi\left(m^{1/d}\right) m^{-1/q}.$$

Case $0 < p < q = \infty$. Similarly to the proof of (9.41) and (9.110), we can show that condition (9.37) with $\beta = d(1/p - \frac{1}{q}) = \frac{d}{p}$ guarantees the convergence of the series

$$\sum_{k\in\mathbb{Z}^d} |\Psi_k|^p = \sum_{j=1}^{\infty} \bar{\Psi}_j^p = \sum_{n=1}^{\infty} \nu_n \psi^p(n)$$

and

$$\sum_{n=l+1}^{\infty} \nu_n \psi^p(n) \asymp l^d \psi^p(l). \tag{9.112}$$

Then using (9.27), (9.29), (9.33), (9.112) and (9.100), we obtain (9.39):

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} = \left((V_{n_{m+1}} - m - 1)\psi^p(n_{m+1}) + \sum_{n=n_{m+1}+1}^{\infty} \nu_n \psi^p(n)\right)^{1/p} \asymp \psi(m^{1/d}) m^{1/p}.$$

Case $p = q = \infty$. In this case, estimate (9.39) follows from (9.28), (9.29) and (9.100)

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} = \bar{\Psi}_{m+1} = \psi(n_{m+1}) \asymp \psi\left(m^{1/d}\right).$$

Finally, consider the quantities $\mathscr{D}_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p}$. In the case $0 < q \le p \le \infty$, estimate (9.38) follows from (9.17), (9.29) and (9.100):

$$\mathscr{D}_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} = \psi(n_{m+1}) \asymp \psi(m^{1/d}).$$

For $0 < p < q < \infty$, estimate (9.38) follows from (9.18), (9.29), (9.110) and (9.100):

$$\mathscr{D}_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \asymp \left(\sum_{n=n_{m+1}+1}^{\infty} \nu_n \psi^{\frac{pq}{q-p}}(n)\right)^{\frac{q-p}{pq}} \asymp \psi(n_{m+1}) n_{m+1}^{\frac{q-p}{dpq}} \asymp \psi\left(m^{1/d}\right) m^{1/p-1/q},$$

and for $0 < p < q = \infty$, it similarly follows from (9.22), (9.29), (9.112) and (9.100). □

9.5.4 Proof of Theorem 9.3

Proof of Theorem 9.3. First, consider the quantities $\mathscr{D}_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p}$. In the case $0 < q \le p \le \infty$, estimate (9.45) follows from the corresponding results of Theorems A and 9.1, relations (9.34), (9.35) and (9.44), taking into account Proposition 9.2:

$$\mathscr{D}_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} = \psi(n_{m+1}) \asymp \psi(\widetilde{n}_m). \tag{9.113}$$

In the case $0 < p < q \le \infty$, estimate (9.45) similarly follows from the results of Theorems A and 9.1, relations (9.33), (9.88), (9.35) and (9.44), taking into account Proposition 9.2:

$$\mathscr{D}_m(\mathcal{F}^{\psi}_{q,r})_{S^p} \asymp \left(\sum_{n=n_{m+1}+1}^{\infty} \nu_n \psi^{\frac{pq}{q-p}}(n) \right)^{\frac{q-p}{pq}} \asymp \left(\sum_{n=n_{m+1}+1}^{\infty} n^{d-1} \psi^{\frac{pq}{q-p}}(n) \right)^{\frac{q-p}{pq}} \asymp$$
$$\asymp \left(n^d_{m+1} \psi^{\frac{pq}{q-p}}(n_{m+1}) \alpha(\psi, n_{m+1}) \right)^{\frac{q-p}{pq}} \asymp \psi(\widetilde{n}_m)(m\alpha(\psi, \widetilde{n}_m))^{1/p-1/q}. \tag{9.114}$$

Now, find the estimates for the quantities $\sigma_m(\mathcal{F}^{\psi}_{q,r})_{S^p}$. Note that in the *cases* $p = q = \infty$ *and* $0 < p < q = \infty$, estimate (9.46) is obtained similarly to the estimates (9.113) and (9.114), using the corresponding results of Theorem B.

In the *case* $0 < q < p = \infty$, it follows from relations (9.26), (9.34), (9.35), (9.44) by using Propositions 9.4 and 9.2:

$$\sigma_m(\mathcal{F}^{\Psi}_{q})_{S^p} \asymp \left(\sum_{n=1}^{n_{m+1}} \frac{\nu_n}{\psi^q(n)} \right)^{-1/q} \asymp \left(\sum_{n=1}^{n_{m+1}} \frac{n^{d-1}}{\psi^q(n)} \right)^{-1/q}$$
$$\asymp \left(\frac{n^d_{m+1} \alpha(\psi, n_{m+1})}{\psi^q(n_{m+1})} \right)^{-1/q} \asymp \psi(\widetilde{n}_m)(m\alpha(\psi, \widetilde{n}_m))^{-1/q}.$$

Case $0 < q \le p < \infty$. Let $M_{r,d}$ be the number defined by (9.31) and c be a fixed real number. For $m \in \mathbb{N}$ and any $l \ge m$, consider the function

$$W_m(l, c) := \frac{l - m}{\left(\int\limits_1^{k(l)} \frac{t^{d-1} \mathrm{d}t}{\psi^q(t)} \right)^{p/q}}, \quad \text{where} \quad k(l) = k(l, c) = (l/M_{r,d})^{1/d} + c.$$

This function is continuously differentiable, its derivative has the form

$$W'_m(l, c) = \left(\int_1^{k(l)} \frac{t^{d-1} \mathrm{d}t}{\psi^q(t)} - \frac{p(l - m)}{q\psi^q(k(l))} \cdot \frac{k^{d-1}(l)}{dM^{1/d}_{r,d} l^{(d-1)/d}} \right) \left(\int_1^{k(l)} \frac{t^{d-1} \mathrm{d}t}{\psi^q(t)} \right)^{p/q-1}, \tag{9.115}$$

where

$$\lim_{l \to \infty} \frac{k^{d-1}(l)}{l^{(d-1)/d}} = \lim_{l \to \infty} \left(\frac{(l/M_{r,d})^{1/d} + c}{l^{1/d}} \right)^{d-1} = M^{\frac{1-d}{d}}_{r,d}. \tag{9.116}$$

There exists a unique point $l_m > m$ such that $W'_m(l_m, c) = 0$. The function $W_m(l, c)$ increase for $m \le l < l_m$ and decrease for $l > l_m$. Moreover, it follows from (9.115), (9.116) and (9.93) that

$$(l_m - m)\left(\int_1^{k(l_m)} \frac{t^{d-1}\mathrm{d}t}{\psi^q(t)}\right)^{-1} = \frac{dqM_{r,d}^{1/d} l^{(d-1)/d}}{k^{d-1}(l)}\psi^q(k(l_m)) \asymp \psi^q(k(l_m)). \quad (9.117)$$

Due to (9.117) and (9.87), for any $\psi \in \mathfrak{M}'_\infty \cup \mathfrak{M}^c_\infty$, we obtain

$$\sup_{l\geq m} W_m(l,c) = W_m(l_m,c) = \frac{l_m - n}{\int\limits_1^{k(l_m)} \frac{t^{d-1}\mathrm{d}t}{\psi^r(t)}} \cdot \left(\int_1^{k(l_m)} \frac{t^{d-1}\mathrm{d}t}{\psi^q(t)}\right)^{1-p/q} \asymp$$

$$\asymp \psi^q(k(l_m))\left(\frac{k^d(l_m)\alpha(\psi,k(l_m))}{\psi^q(k(l_m))}\right)^{1-p/q} = \frac{\psi^p(k(l_m))}{(k^d(l_m)\alpha(\psi,k(l_m)))^{p/q-1}}. \quad (9.118)$$

Since $l_m \geq m$ and $k(l_m) \geq (m/M_{r,d})^{1/d} - |c| = \widetilde{n}_m - |c|$, for any $\psi \in \mathfrak{M}'_\infty \cup \mathfrak{M}^c_\infty$, taking into account Proposition 9.2, we have

$$\frac{\psi^p(k(l_m))}{(k^d(l_m)\alpha(\psi,k(l_m)))^{p/q-1}} \leq \frac{\psi^p(\widetilde{n}_m - |c|)}{\left((\widetilde{n}_m - |c|)^{d-1}\frac{\psi(k(l_m))}{|\psi'(k(l_m))|}\right)^{p/q-1}} \ll$$

$$\ll \frac{\psi^p(\widetilde{n}_m)}{\left(\widetilde{n}_m^{d-1}\frac{\psi(\widetilde{n}_m)}{|\psi'(\widetilde{n}_m)|}\right)^{p/q-1}} = \frac{\psi^p(\widetilde{n}_m)}{(m\alpha(\psi,\widetilde{n}_m))^{p/q-1}}.$$

Combining the last estimate and (9.118), we conclude that for any $\psi \in \mathfrak{M}'_\infty \cup \mathfrak{M}^c_\infty$

$$\sup_{l\geq m} W_m(l,c) \ll \frac{\psi^p(\widetilde{n}_m)}{(m\alpha(\psi,\widetilde{n}_m))^{p/q-1}}. \quad (9.119)$$

By virtue of (9.99), (9.35) and (9.119), we obtain the upper estimate in (9.46):

$$\sigma_m^p(\mathcal{F}^\psi_{q,r})_{S^p} \leq \sup_{l>m} \frac{l-m}{\left(\sum\limits_{n=1}^{n_l-1} \frac{\nu_n}{\psi^q(n)}\right)^{p/q}} \ll \sup_{l>m} \frac{l-m}{\left(\sum\limits_{n=1}^{n_l-1} \frac{n^{d-1}}{\psi^q(n)}\right)^{p/q}} \ll$$

$$\ll \sup_{l>m} \frac{l-m}{\left(\int\limits_1^{n_l-1} \frac{t^{d-1}\mathrm{d}t}{\psi^q(t)}\right)^{p/q}} \ll \sup_{l>m} W_m(l, c_{r,d}+1) \ll \frac{\psi^p(\widetilde{n}_m)}{(m\alpha(\psi,\widetilde{n}_m))^{p/q-1}}.$$

Let us also find the lower estimate. Due to (9.99), (9.35), (9.87) and (9.78), we have

$$\sigma_m^p(\mathcal{F}^\psi_{q,r})_{S^p} \gg \sup_{l>m} \frac{l-m}{\left(\sum\limits_{n=1}^{n_l} \frac{n^{d-1}}{\psi^q(n)}\right)^{p/q}} \gg \sup_{l>m, l\in\mathbb{N}} \frac{\psi^p(n_l)(l-m)}{(n_l^d\alpha(\psi,n_l))^{p/q}} \gg$$

$$\gg \sup_{l>m, l\in\mathbb{N}} \frac{\psi^p\left((l/M_{r,d})^{1/d}\right)(l-m)}{\left(l\alpha\left(\psi,(l/M_{r,d})^{1/d}\right)\right)^{p/q}} =: \sup_{l>m, l\in\mathbb{N}} R_m(l). \tag{9.120}$$

Consider the function

$$g(t) = g(\psi;t) := \psi\left((t/M_{r,d})^{1/d}\right), \quad t \geq 1. \tag{9.121}$$

It is easy to see that $g \in \mathfrak{M}$ and

$$\alpha(g,t) = \frac{g(t)}{t|g'(t)|} = dM_{r,d}^{1/d}\alpha\left(\psi,(t/M_{r,d})^{1/d}\right) \downarrow 0$$

for any $\psi \in \mathfrak{M}'_\infty \cup \mathfrak{M}^c_\infty$. Therefore, similar to ψ, the function $g \in \mathfrak{M}^+_\infty \subset F$. Hence, by virtue of (9.79) and (9.80), for the quantity $\eta(t) = \eta(g,t)$, we have

$$\eta(\eta(t)) - \eta(t) \asymp \eta(t) - t \asymp \frac{g(t)}{|g'(t)|} \asymp t\alpha\left(\psi,(t/M_{r,d})^{1/d}\right). \tag{9.122}$$

Set $l_0 = [\eta(m)] + 1$. Then by Proposition 9.2, Remark 9.7, relations (9.121), (9.122) and (9.44),

$$\begin{aligned} R_m(l_0) &\gg \frac{\psi^p\left((\eta(g,m)/M_{r,d})^{1/d}\right)(\eta(g,m)-m)}{\left(\eta(g,m)\alpha\left(\psi,(\eta(g,m)/M_{r,d})^{1/d}\right)\right)^{p/q}} \gg \\ &\gg \frac{g^p(m)}{(\eta(g,m)-m))^{p/q-1}} \gg \frac{\psi^p(\widetilde{n}_m)}{(m\alpha(\psi,\widetilde{n}_m)\,)^{p/q-1}}. \end{aligned} \tag{9.123}$$

Combining relations (9.120) and (9.123), we obtain the necessary lower bound:

$$\sigma^p_m(\mathcal{F}^\psi_{q,r})_{\mathcal{S}^p} \gg \frac{\psi^p(\widetilde{n}_m)}{(m\alpha(\psi,\widetilde{n}_m)\,)^{p/q-1}}$$

which complies the proof of relation (9.46) in the case $0 < q \leq p < \infty$.

Case $0 < p < q < \infty$. Let us use relation (9.105)–(9.107), which holds for any positive non-increasing function ψ. Similarly to the previous cases, we prove that

$$\sup_{l>m} \frac{l-m}{\sum\limits_{n=1}^{n_l-1}\frac{\nu_n}{\psi^q(n)} + \frac{l-V_{n_l-1}}{\psi^q(n_l)}} = \frac{l_m - m}{\sum\limits_{n=1}^{n_{l_m}}\frac{\nu_n}{\psi^q(n)}} \asymp \psi^q(\widetilde{n}_m) \asymp \psi^q(n_{l_m}), \tag{9.124}$$

where $\widetilde{n}_m$ and n_{l_m} are defined by (9.44) and (9.106). Further, let us show that

$$l_m \asymp \widetilde{n}_m. \tag{9.125}$$

Since $l_m > m$, by (9.35) we have $n_{l_m} > n_m \geq \widetilde{n}_m - c_2$. If $n_{l_m} \leq \widetilde{n}_m$, then relation (9.125) holds. Assume that $n_{l_m} > \widetilde{n}_m$. Since $\alpha(\psi,t) \downarrow 0$, then $\frac{1}{t} \leq K_{14}\frac{|\psi'(t)|}{\psi(t)}$ for all $t \geq 1$. Integrating both parts of this inequality in the range from $\widetilde{n}_m$ to n_{l_m}, we obtain

$$\ln \frac{n_{l_m}}{\widetilde{n}_m} \leq K_{14} \ln \frac{\psi(\widetilde{n}_m)}{\psi(n_{l_m})} \quad \text{and} \quad n_{l_m} \ll \widetilde{n}_m.$$

Thus, relation (9.125) is true indeed.

By virtue of (9.32), (9.87), (9.125) and (9.78),

$$\sum_{n=1}^{n_{l_m}} \frac{\nu_n}{\psi^q(n)} \asymp \sum_{n=1}^{n_{l_m}} \frac{n^{d-1}}{\psi^q(n)} \asymp \frac{n_{l_m}^d \alpha(\psi, n_{l_m})}{\psi^q(n_{l_m})} \asymp \frac{m\alpha(\psi, \widetilde{n}_m)}{\psi^q(\widetilde{n}_m)}. \tag{9.126}$$

Finally, due to (9.32), (9.88), (9.125) and (9.78), we get

$$\sum_{n=n_{l_m}+1}^{\infty} \nu_n \psi^{\frac{pq}{q-p}}(n) \asymp \sum_{n=n_{l_m}+1}^{\infty} n^{d-1} \psi^{\frac{pq}{q-p}}(n) \asymp$$
$$\asymp n_{l_m}^d \psi^{\frac{pq}{q-p}}(n_{l_m}) \alpha(\psi, n_{l_m}) \asymp m\psi^{\frac{pq}{q-p}}(\widetilde{n}_m) \alpha(\psi, \widetilde{n}_m). \tag{9.127}$$

Combining relations (9.105), (9.124), (9.126) and (9.127), we obtain (9.46):

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \asymp \left(\psi^{\frac{q^2}{q-p}}(\widetilde{n}_m) \frac{m\alpha(\psi, \widetilde{n}_m)}{\psi^q(\widetilde{n}_m)} + m\psi^{\frac{pq}{q-p}}(\widetilde{n}_m) \alpha(\psi, \widetilde{n}_m) \right)^{\frac{q-p}{pq}} \asymp$$
$$\asymp \psi(\widetilde{n}_m)(m\alpha(\psi, \widetilde{n}_m))^{1/p-1/q}.$$

□

9.5.5 Proof of Theorem 9.4

Proof of Theorem 9.4. First, we find the estimates for the quantities $\sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p}$.

(i) *Case* $0 < p < q < \infty$. Let us use the representation (9.105), in which n_{l_m} is defined in (9.34) for $s = l_m$, and l_m satisfies relations (9.106) and (9.104). By virtue of (9.33) and (9.98),

$$\sum_{n=n_{l_m}+1}^{\infty} \nu_n \psi^{\frac{pq}{q-p}}(n) \asymp \sum_{n=n_{l_m}+1}^{\infty} n^{d-1} \psi^{\frac{pq}{q-p}}(n) \asymp n_{l_m}^{d-1} \psi^{\frac{pq}{q-p}}(n_{l_m}+1). \tag{9.128}$$

Since $l_m \geq m+1$ and $m \in [V_{s-1}, V_s)$, it follows from (9.104) and (9.34) that $l_m = V_{n_{l_m}} \geq V_{n_{m+1}}$.

By virtue of (9.97), (9.32) and (9.48), for sufficiently large s we have

$$\sum_{j=1}^{s} \frac{\nu_j}{\psi^q(j)} \asymp \sum_{j=1}^{s} \frac{j^{d-1}}{\psi^q(j)} \asymp \frac{s^{d-1}}{\psi^q(s)} < \psi^{-q}(s+1). \tag{9.129}$$

Therefore, for such s and $m \in [V_{s-1}, V_s)$, the following relation holds

$$\frac{V_s - m}{\sum_{j=1}^{s} \frac{\nu_j}{\psi^q(j)}} \asymp \frac{V_s - m}{\sum_{j=1}^{s} \frac{j^{d-1}}{\psi^q(j)}} \asymp \frac{(V_s - m)\psi^q(s)}{s^{d-1}} \gg \psi^q(s+1),$$

which implies that $l_m \leq V_s$, and therefore, $l_m = V_s$ and $n_{l_m} = s$. Combining this equality, relations (9.105), (9.128) and (9.129), and taking into account relations (9.35) and (9.48), we see that relation (9.49) is true indeed:

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{S^p} \asymp \left(\left(\frac{(V_s - m)\psi^q(s)}{s^{d-1}} \right)^{\frac{q}{q-p}} \frac{s^{d-1}}{\psi^q(s)} + s^{d-1}\psi^{\frac{pq}{q-p}}(s+1) \right)^{\frac{q-p}{pq}} \asymp$$

$$\asymp \frac{(V_s - m)^p \psi(s)}{s^{\frac{d-1}{q}}} \left(1 + \left(\frac{s^{1/2d-1p}\psi(s+1)}{\psi(s)(V_s - m)^p} \right)^{\frac{pq}{q-p}} \right)^{\frac{q-p}{pq}} \asymp \psi(s) \frac{(V_s - m)^p}{m^{\frac{d-1}{dq}}}.$$

(ii) *Case* $0 < p < q = \infty$. Due to (9.27) and $m \in [V_{s-1}, V_s)$, we have

$$\sigma^p_m(\mathcal{F}^{\psi}_{q,r})_{S^p} = (V_s - m)\psi^p(s) + \sum_{n=s+1}^{\infty} \nu_n \psi^p(n).$$

Taking into account (9.33) and (9.98), we obtain

$$\sigma^p_m(\mathcal{F}^{\psi}_{q,r})_{S^p} \asymp (V_s - m)\psi^p(s) + s^{d-1}\psi^p(s+1),$$

and by virtue of (9.48), we see that for $0 < p < q = \infty$, relation (9.49) also holds.

Case $0 < q \leq p < \infty$. By virtue of (9.99) and (9.97), we have

$$\sigma^p_m(\mathcal{F}^{\psi}_{q,r})_{S^p} = \sup_{l>m} \frac{l - m}{\left(\sum_{n=1}^{n_l - 1} \frac{n^{d-1}}{\psi^q(n)} + \frac{l - V_{n_l - 1}}{\psi^q(n_l)} \right)^{p/q}} \asymp \sup_{l>m,\, l\in\mathbb{N}} \frac{l - m}{\left(\frac{(n_l - 1)^{d-1}}{\psi^q(n_l - 1)} + \frac{l - V_{n_l - 1}}{\psi^q(n_l)} \right)^{p/q}}.$$

In this case, condition (9.48) is also satisfied for $\beta = \frac{d-1}{p}$. Therefore,

$$\sigma^p_m(\mathcal{F}^{\psi}_{q,r})_{S^p} \asymp \sup_{l\in\mathbb{N},\, l>m} \frac{\psi^p(n_l)(l - m)}{(l - V_{n_l - 1})^{p/q}} = \sup_{j>m} \left(\psi^p(n_j) \max_{l\in I_j} \frac{l - m}{(l - V_{n_j - 1})^{p/q}} \right), \tag{9.130}$$

where I_{m+1} denotes the sets of positive integers from the half-interval $(m, V_s]$ and I_j, $j = m+2, m+3, \ldots$, denotes the sets of positive integers l from the half-intervals $(V_{n_j - 1}, V_{n_j}]$.

By virtue of (9.32), (9.34) and (9.48), for $j > m$ we have

$$\psi^p(n_j)\max_{l\in I_j}\frac{l-m}{l-V_{n_j-1}}=\frac{\psi^p(n_j)(V_{n_j}-m)}{V_{n_j}-V_{n_j-1}}\asymp\frac{\psi^p(n_j)(V_{n_j}-m)}{n_j^{d-1}}\gg\psi^p(n_j+1)\max_{l\in I_{j+1}}\frac{l-m}{l-V_{n_j}}.$$

Thus, if $p = q$, then the supremum in (9.130) is attained at the point $j = m+1$, and in this case, relation (9.49) holds:

$$\sigma_m(\mathcal{F}^{\psi}_{p,r})_{\mathcal{S}^p}\asymp\psi(n_{m+1})\frac{(V_{n_{m+1}}-m)^{1/p}}{n_{m+1}^{\frac{d-1}{p}}}\asymp\psi(n_{m+1})\frac{(V_{n_{m+1}}-m)^{1/p}}{m^{\frac{d-1}{dp}}}\asymp\psi(s)\frac{(V_s-m)^{1/p}}{m^{\frac{d-1}{dp}}}.$$

Now, let $0 < q < p < \infty$. For fixed m and $v > m$, the function

$$f_m(l,v)=\frac{l-m}{(l-v)^{p/q}},\quad l>v, \tag{9.131}$$

attains its maximal value at the point $l_v^*=\frac{pm-qv}{p-q}$,

$$f_m(l_v^*,v)=\frac{q(p-q)^{p/q-1}}{p^{p/q}}\frac{1}{(m-v)^{p/q-1}}\asymp\frac{1}{(m-v)^{p/q-1}}, \tag{9.132}$$

$f_m(l,v)$ increases for $l\in(v,l_v^*)$ and decreases for $l>l_v^*$.

If $m > V_{s-1}$, according to (9.32), (9.48) and (9.132), we have

$$\psi^p(s)\max_{l\in I_{m+1}}\frac{l-m}{(l-V_{s-1})^{p/q}}\gg\frac{\psi^p(s)}{(V_s-V_{s-1})^{p/q}}\asymp\frac{\psi^p(s)}{s^{\frac{(d-1)p}{q}}}\gg\psi^p(s+1)\gg$$

$$\gg\frac{\psi^p(s+1)}{(m-V_s)^{p/q-1}}=\psi^p(s+1)\sup_{l>V_s}f_m(l,V_s)\geq\psi^p(n_j)\max_{l\in I_j}\frac{l-m}{(l-V_{n_j-1})^{p/q}}$$

for any $j > m+1$. Thus, the supremum in (9.130) is attained at $j = m+1$, and

$$\sigma_m^p(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p}\asymp\psi^p(s)\max_{l\in I_{m+1}}\frac{(l-m)}{(l-V_{s-1})^{p/q}}\asymp\psi^p(s)\max_{l\in(m,V_s]}f(l,v_m),$$

where $f(l,v)$ is defined by (9.131) and $v_m = V_{s-1}$. Further, if condition (9.51) holds, the point $l_{v_m}^*=\frac{pm-qV_{s-1}}{p-q}$ belongs to $(m,V_s]$, and taking into account (9.132), we obtain (9.50):

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p}\asymp\psi(s)f^{1/p}(l_{r_m}^*,v_m)\asymp\frac{\psi(s)}{(m+1-V_{s-1})^{1/q-1/p}}.$$

If condition (9.51) is not satisfied, then $f(l,v_m)$ increases on $(m,V_s]$. Therefore, taking into account (9.32), we obtain (9.49):

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p}\asymp\psi(s)f^{1/p}(V_s,r_m)\asymp\frac{\psi(s)(V_s-m)^{1/p}}{(V_s-V_{s-1})^{1/q}}\asymp\frac{\psi(s)(V_s-m)^{1/p}}{m^{\frac{d-1}{dq}}}.$$

Similarly, if $m = V_{s-1}$, for any $j > m$ we have

$$\psi^p(s) \max_{l\in I_{m+1}} \frac{l-m}{(l-V_{s-1})^{p/q}} = \psi^p(s) \max_{l\in I_{m+1}} (l-V_{s-1})^{-p/q+1} = \psi^p(s) \gg$$
$$\gg \frac{\psi^p(s+1)}{(m+1-V_s)^{\frac{p}{q}-1}} = \psi^p(s+1) \sup_{l>V_s} f_m(l, V_s) \geq \psi^p(n_j) \max_{l\in I_j} \frac{l-m}{(l-V_{n_j-1})^{p/q}}.$$

Therefore, in this case relation (9.50) holds:

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \asymp \psi(s) = \frac{\psi(s)}{(m+1-V_{s-1})^{1/q-1/p}}.$$

(iii) *Case* $0 < q < p = \infty$. Due to (9.26), (9.29) and (9.129), we have

$$\sigma^q_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^\infty} =$$
$$= \left(\sum_{n=1}^{s-1} \frac{\nu_n}{\psi^q(n)} + \frac{m+1-V_{s-1}}{\psi^q(s)}\right)^{-1/q} \asymp \left(\frac{(s-1)^{d-1}}{\psi^q(s-1)} + \frac{m+1-V_{s-1}}{\psi^q(s)}\right)^{-1/q} \asymp$$
$$\asymp \frac{\psi(s)}{(m+1-V_{s-1})^{1/q}} \left(1 + \frac{s^{d-1}\psi^q(s)}{\psi^q(s-1)} \frac{1}{m+1-V_{s-1}}\right)^{-1/q}.$$

Since ψ satisfies (9.48) with $\beta = \frac{d-1}{q}$, we conclude that relation (9.50) also holds.

(iv) Find estimates for $\mathscr{D}_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p}$ in the *case* $0 < p < q \leq \infty$. Due to (9.18) and (9.29),

$$\mathscr{D}_m^{\frac{q-p}{pq}}(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} = (V_s - m)\psi^{\frac{pq}{q-p}}(s) + \sum_{n=s+1}^{\infty} \nu_n \psi^{\frac{pq}{q-p}}(n).$$

Taking into account (9.33) and (9.98), we obtain

$$\mathscr{D}_m^{\frac{q-p}{pq}}(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^p} \asymp (V_s - m)\psi^{\frac{pq}{q-p}}(s) + (s+1)^{d-1}\psi^{\frac{pq}{q-p}}(s+1).$$

By virtue of (9.48), we see that for $0 < p < q < \infty$ relation (9.52) is valid. □

9.5.6 Proof of Theorem 9.5

Proof of Theorem 9.5. Upper estimates. In the case $1 \leq p \leq 2$, due to (9.13), (9.63) and (9.14), we have

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \ll \sigma^{\perp}_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \ll G_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \ll G_m(\mathcal{F}^{\psi}_{q,r})_{L_2} \ll \sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^2}. \quad (9.133)$$

Thus, to obtain the necessary upper bound, it suffices to use (9.39) for $\mathcal{S}^p = \mathcal{S}^2$.

In the case $2 \le p < \infty$, using relations (9.13), (9.62) and (9.14), we obtain

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \ll \sigma^{\perp}_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \ll G_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \ll G_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^{p'}} \asymp \sigma_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^{p'}}, \quad (9.134)$$

and the upper bound similarly follows from relation (9.39).

Lower estimate. Let $\mathscr{T}_n$, $n \in \mathbb{N}$, denote the set of all polynomials of the form

$$\tau_n = \sum_{|k|_\infty \le n} \widehat{\tau}_n(k) e_k,$$

and let $\mathcal{A}_q(\mathscr{T}_n)$, $0 < q \le \infty$, denote the subset of all polynomials $\tau_n \in \mathscr{T}_n$ such that $\|\tau_n\|_{\mathcal{S}^q} \le 1$. From Theorem 5.2 of [10], it follows that for any $0 < q \le \infty$, $1 \le p \le \infty$, $n = 1, 2, \dots$ and $m = \frac{(2n+1)^d - 1}{2}$,

$$\sigma_m(\mathcal{A}_q(\mathscr{T}_n))_{L_p} \ge K m^{1/2-1/q}.$$

For a fixed $n \in \mathbb{N}$, consider the set $\psi(dn)\mathcal{A}_q(\mathscr{T}_n) = \{\tau \in \mathscr{T}_n : \|\tau\|_{\mathcal{S}^q} \le \psi(dn)\}$. Due to monotonicity ψ, for any $\tau \in \psi(dn)\mathcal{A}_q(\mathscr{T}_n)$ and $0 < r \le \infty$, we have

$$\sum_{k\in\mathbb{Z}^d} \left| \frac{\widehat{\tau}(k)}{\psi(\|k\|_r)} \right|^q \le \sum_{|k|_\infty \le n} \left| \frac{\widehat{\tau}(k)}{\psi(d|k|_\infty)} \right|^q \le \sum_{|k|_\infty \le n} \left| \frac{\widehat{\tau}(k)}{\psi(dn)} \right|^q \le 1$$

Therefore, $\psi(dn)\mathcal{A}_q(\mathscr{T}_n)$ is contained in the set $\mathcal{F}^{\psi}_{q,r}$. In view of definition of the set B, for all $n = 1, 2, \dots$ and $m = \frac{(2n+1)^d - 1}{2}$, we obtain

$$\sigma_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \ge \sigma_m(\psi(dn)\mathcal{A}_q(\mathscr{T}_n))_{L_p} \gg \psi(dn) m^{1/2-1/q} \gg \psi\left(m^{1/d}\right) m^{1/2-1/q}.$$

Taking into account the relation (9.13), monotonicity of the quantity σ_m and inclusion $\psi \in B$, we see that for all $1 \le p \le \infty$,

$$G_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \gg \sigma^{\perp}_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \gg \sigma_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \gg \psi\left(m^{1/d}\right) m^{1/2-1/q}.$$

In the case $2 < p < \infty$, for the quantities $\sigma^{\perp}_m(\mathcal{F}^{\psi}_{q,r})_{L_p}$ and $G_m(\mathcal{F}^{\psi}_{q,r})_{L_p}$, this estimate can be improved. For this purpose, consider the function

$$h_1 = \sum_{|k|_r \le n_m} \widehat{h}_1(k) e_k = \mathfrak{h}_1 \sum_{|k|_r \le n_m} e_k,$$

where

$$\mathfrak{h}_1^{-q}(m) := \sum_{|j|_r \le n_m} \psi^{-q}(|j|_r), \quad n_m := n_m(r,d) = \left[\left(\frac{2m}{M_{r,d}} \right)^{\frac{1}{d}} \right]$$

and $M_{r,d}$ is a constant defined in (9.31). It is obviously that $f_1 \in \mathcal{F}^{\psi}_{q,r}$ and by (9.81) and (9.36)

$$\mathfrak{h}_1^{-q}(m) \asymp \sum_{j=1}^{n_m} \frac{j^{d-1}}{\psi^q(j)} \asymp \frac{n_m^d}{\psi^q(n_m)} \asymp \frac{m}{\psi^q\left(m^{\frac{1}{d}}\right)}.$$

For any collection $\gamma_n \subset \mathbb{Z}^d$, using Nikol'skii's inequality [26] and (9.32), we obtain

$$\left\| h_1 - \sum_{k\in\gamma_m} \widehat{h}_1(k)e_k \right\|_{L_p} \gg \mathfrak{h}_1(m)m^{-\frac{1}{p}} \left\| \sum_{|k|_1\le n_m,\, k\notin\gamma_m} e_k \right\|_{L_\infty} \asymp \psi(m^{1/d})m^{1-1/p-1/q}.$$

Therefore, for all $2 \le p < \infty$, the following estimates are true:

$$G_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \gg \sigma_m^{\perp}(\mathcal{F}^{\psi}_{q,r})_{L_p} \gg \sigma_m^{\perp}(h_1)_{L_p} \gg \psi\left(m^{1/d}\right) m^{1-1/p-1/q}.$$

□

9.5.7 Proof of Theorem 9.6

Proof of Theorem 9.6. Proof of this theorem is similar to the proof of the upper estimates in Theorem 6.1 [10]. It uses the following lemma from [10].

Lemma 9.2 ([10]). *For any* $0 < q \le \infty$, $n = 1, 2, \ldots$, *and* $1 \le m \le (2n+1)^d$,

$$\sigma_m(\mathcal{A}_q(\mathscr{T}_n))_{L_\infty} \le Cm^{1/2-1/q}L(n^d/m), \quad 0 < q \le 1, \tag{9.135}$$

where $L(x) = (1 + (\ln x)_+)^{1/2}$ *and*

$$\sigma_m(\mathcal{A}_q(\mathscr{T}_n))_{L_\infty} \le Cn^{d-d/q}m^{-1/2}L(n^d/m), \quad 1 < q \le \infty, \tag{9.136}$$

with C *depending only on* q *and* d.

For any $f \in \mathcal{F}^{\psi}_{q,\infty}$, we use the decomposition

$$f = \sum_{j=0}^{\infty} f_j, \quad \text{where} \quad f_j := \sum_{2^{j-1}\le|k|_\infty<2^j} \widehat{f}(k)e_k, \quad j \ge 1, \quad f_0 := \widehat{f}(0).$$

We note that

$$f_j/\psi(2^{j-1}) \in \mathcal{A}_q(\mathscr{T}_{2^j}), \quad j = 1, 2, \ldots \tag{9.137}$$

For any $N = 1, 2, \ldots$, we approximate f as follows. Let N_0 be the largest integer j such that $m_j := [(j-N)^{-2}2^{Nd}] \ge 1$, i.e. $N_0 = [2^{\frac{Nd}{2}} + N]$. If $j \le N$, we set $P_j := f_j$. If $N < j \le N_0$, then by virtue of (9.135), (9.136) and (9.137), there is a polynomial $P_j \in \Sigma_{m_j}$ such that

$$\|f_j - P_j\|_{L_p} \ll m_j^{1/2-1/q} L(2^{jd}/m_j)\psi(2^{j-1}), \quad 0 < q \le 1. \tag{9.138}$$

and

$$\|f_j - P_j\|_{L_p} \ll 2^{j(d-d/q)} m_j^{-1/2} L(2^{jd}/m_j)\psi(2^{j-1}), \quad 1 < q < \infty. \tag{9.139}$$

Set $P = \sum_{j=0}^{N_0} P_k$. Since

$$(2 \cdot 2^N + 1)^d + \sum_{j=N+1}^{N_0} (j-N)^{-2} 2^{Nd} \le a 2^{Nd},$$

where a depends only on d, then P is a linear combination of at most $a2^{Nd}$ exponentials e_k. Hence, P is in $\Sigma_{a2^{Nd}}$. We also have

$$\|f - P\|_{L_p} \le \sum_{j=N+1}^{N_0} \|f_j - P_j\|_{L_p} + \sum_{j=N_0+1}^{\infty} \|f_j\|_{L_p} =: S_1 + S_2. \tag{9.140}$$

For all $x \ge 1$, we have $[x] \ge x/2$. Therefore, for sufficiently large N and $N < j \le N_0$, from the definition of $L(x)$, we have

$$L(2^{jd}/m_j) \le (1 + \ln(2^{d(j-N)+1}(j-N)^2))^{1/2} \ll (j-N)^{1/2}. \tag{9.141}$$

First, consider the case $0 < q \le 1$. Reasoning similar to the proof of upper estimate in (9.82), it is easy to show that if the function ψ belongs to the set B and for all t, larger than a certain number t_0, ψ is convex and it satisfies condition (9.37) with a fixed $\beta \ge 0$, then for any $\alpha \in \mathbb{R}$ and sufficiently large $t > N$, the function $h_{\alpha,\beta}(t) := 2^{\beta t}(t-N)^\alpha \psi(2^{t-1})$ decreases to zero, as well as

$$\sum_{j=N+1}^{\infty} 2^{\beta j}(j-N)^\alpha \psi(2^{j-1}) \ll 2^{\beta N}\psi(2^N). \tag{9.142}$$

In this case, $\beta = 0$. By virtue of (9.138), (9.141) and (9.142), we obtain the estimate of the first sum S_1 in (9.140):

$$S_1 \ll \sum_{j=N+1}^{\infty} (j-N)^{2(1/q-1/2)} 2^{-Nd(1/q-1/2)} (j-N)^{\frac{1}{2}} \psi(2^{j-1}) \ll$$
$$\ll 2^{-N\left(\frac{d}{q}-\frac{d}{2}\right)} \sum_{j=N+1}^{\infty} (j-N)^{2/q-1/2}\psi(2^{j-1}) \ll 2^{-N\left(\frac{d}{q}-\frac{d}{2}\right)}\psi(2^N). \tag{9.143}$$

To estimate S_2, we note that from (9.137)

$$S_2 \le \sum_{j=N_0+1}^{\infty} \|f_j\|_{L_\infty} \le \sum_{j=N_0+1}^{\infty} \left(\sum_{2^{j-1} \le \|k\|_\infty < 2^j} |\widehat{f}(k)| \right) \le$$
$$\le \sum_{j=N_0+1}^{\infty} \|f_j\|_{S^q} \ll \sum_{j=N_0+1}^{\infty} \psi(2^{j-1}) \ll \psi(2^{N_0}).$$

Further, note that if for all t, larger than a certain number t_0, ψ is convex and satisfies (9.37), then for any $\alpha > 0$, we have $\psi(2^{N(\alpha+1)}) \ll \psi(2^N)2^{-N\alpha}$.

From the definition of N_0, we have $N_0 \ge N + 2^{\frac{Nd}{2}} - 1$. It follows that if N is sufficiently large (depending only on d and q), then $N_0 \ge N(1 + \frac{d}{q} - \frac{d}{2})$. Hence,

$$S_2 \ll \psi\left(2^{N\left(1+\frac{d}{q}-\frac{d}{2}\right)}\right) \ll 2^{-N\left(\frac{d}{q}-\frac{d}{2}\right)}\psi(2^N).$$

Using this and (9.143) in (9.140), we find that

$$\sigma_{a2^{Nd}}(f)_{L_p} \le \|f - P\|_{L_p} \ll 2^{-N\left(\frac{d}{q}-\frac{d}{2}\right)}\psi(2^N). \tag{9.144}$$

In the case $1 < q < \infty$, condition (9.37) is satisfied with $\beta = d - \frac{d}{q}$. By virtue of (9.139), (9.141) and (9.142), we have

$$S_1 \ll 2^{-\frac{N}{2}} \sum_{j=N+1}^{\infty} 2^{j\left(d-\frac{d}{q}\right)}(j-N)^{\frac{3}{2}}\psi(2^{j-1}) \ll \psi(2^N)2^{-N\left(\frac{d}{q}-\frac{d}{2}\right)}. \tag{9.145}$$

To estimate S_2, we use Hölder's inequality, (9.137), (9.142) and the inequalities $N_0 \ge N + 2^{\frac{Nd}{2}} - 1$ and $h_{\alpha,\beta}(N_0) \le h_{\alpha,\beta}(N+1)$ with $\alpha = 1$ and $\beta = d - \frac{d}{q}$,

$$S_2 \le \sum_{j=N_0+1}^{\infty} \|f_j\|_{L_\infty} \le \sum_{j=N_0+1}^{\infty} \psi(2^{j-1}) \left(\sum_{2^{j-1} \le \|k\|_\infty < 2^j} \left| \frac{\widehat{f}(k)}{\psi(2^{j-1})} \right| \right) \le$$
$$\le \sum_{j=N_0+1}^{\infty} \psi(2^{j-1})2^{(j-1)(d-\frac{d}{q})} \ll \psi(2^{N_0})2^{N_0(d-\frac{d}{q})} \ll \psi(2^N)2^{N(\frac{d}{2}-\frac{d}{q})}.$$

Using this and (9.145) in (9.140), we see that in this case, relation (9.144) is also true. Therefore, the upper estimate in (9.37) follows from the monotonicity of σ_m and inclusion $\psi \in B$. □

9.5.8 Proof of Theorem 9.7

Proof of Theorem 9.7. Upper estimates. Let $m = m(s) = V_s - c_s$, $1 \le c_s \le c$. Then due to relations (9.133) and (9.134) and estimate (9.54), we see that in this case for any $0 < q \le \infty$ and $1 \le p < \infty$

$$\sigma_m^{\perp}(\mathcal{F}_{q,r}^{\psi})_{L_p} \ll G_m(\mathcal{F}_{q,r}^{\psi})_{L_p} \ll \frac{\psi(s)}{m^{\frac{d-1}{qd}}}. \tag{9.146}$$

If $m = m(s) = V_{s-1} + c_s$, $0 \le c_s \le c$, and $0 < q < p' < \infty$ or if $m = m(s) = V_{s-1}$ and $0 < p' = q < \infty$, then the upper estimate similarly follows from (9.133), (9.134) and (9.58)

$$\sigma_m^{\perp}(\mathcal{F}_{q,r}^{\psi})_{L_p} \ll G_m(\mathcal{F}_{q,r}^{\psi})_{L_p} \ll \psi(s). \tag{9.147}$$

Lower estimates. Let $\{k_1^*, k_2^*, \ldots\}$ be a rearrangement of vectors from $\mathbb{Z}^d$ such that

$$|\Psi_{k_j^*}| = \bar{\Psi}_j, \quad j = 1, 2, \ldots, \tag{9.148}$$

where, as above, $\bar{\Psi} = \{\bar{\Psi}_j\}_{j=1}^{\infty}$ is a non-increasing rearrangement of the system $\{|\Psi_k|\}_{k\in\mathbb{Z}^d}$.

Consider the function

$$h_2 = \sum_{j=1}^{m+1} \widehat{h}_3(k_j^*) e_{k_j} = \mathfrak{h}_2(m) \sum_{j=1}^{m+1} e_{k_j}, \quad \text{where} \quad \mathfrak{h}_2^{-q}(m) = \sum_{i=1}^{m+1} |\Psi_{k_i^*}|^{-q}.$$

It is easy to see that $h_2 \in \mathcal{F}_{q,r}^{\psi}$ and due to (9.148) and (9.29), for $m \in [V_{s-1}, V_s)$ we have

$$\mathfrak{h}_2^{-q}(m) = \sum_{j=1}^{m+1} \bar{\Psi}_j^{-q} \asymp \sum_{k=1}^{s-1} \frac{V_k - V_{k-1}}{\psi^q(k)} + \frac{m+1-V_{s-1}}{\psi^q(s)}, \tag{9.149}$$

where in view of (9.33) and (9.97),

$$\sum_{k=1}^{s-1} \frac{V_k - V_{k-1}}{\psi^q(k)} \asymp \sum_{k=1}^{s-1} \frac{k^{d-1}}{\psi^q(k)} \asymp \frac{(s-1)^{d-1}}{\psi^q(s-1)}. \tag{9.150}$$

Combining (9.149) and (9.150), taking into account (9.48), we conclude that

$$\mathfrak{h}_2^{-q}(m) \asymp \frac{m+1-V_{s-1}}{\psi^q(s)} \left(1 + \frac{(s-1)^{d-1}\psi^q(s)}{\psi^q(s-1)(m+1-V_{s-1})}\right) \asymp \frac{m+1-V_{s-1}}{\psi^q(s)}.$$

Then for any collection $\gamma_m \in \Gamma_m$, we have

$$\left\| h_2 - \sum_{k\in\gamma_m} \widehat{h}_2(k) e_k \right\|_{L_p} = \mathfrak{h}_2(m) \left\| \sum_{j\in[1,m+1]\setminus\gamma_m} e_{k_j^*} \right\|_{L_p} \asymp \frac{\psi(s)}{(m+1-V_{s-1})^{1/q}}.$$

Therefore, for all $1 \le p < \infty$, the following estimate holds:

$$G_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \gg \sigma^{\perp}_n(\mathcal{F}^{\psi}_{q,r})_{L_p} \gg \sigma^{\perp}_n(h_2)_{L_p} \gg \frac{\psi(s)}{(m+1-V_{s-1})^{1/q}}. \tag{9.151}$$

If $m = m(s) = V_{s-1} + c_s$, $0 \le c_s \le c$, then the quantities on the right-hand sides of relations (9.151) and (9.147) are equivalent. If $m = m(s) = V_s - c_s$, $1 \le c_s \le c$, then the equivalence of quantities on the right-hand sides of relations (9.151) and (9.146) follows from (9.33), (9.34) and (9.35). □

9.5.9 Proof of Theorem 9.8

Proof of Theorem 9.8. Theorem 9.8 can be proven similarly to Theorem 9.7. *Upper estimates* for $\sigma^{\perp}_m(\mathcal{F}^{\psi}_q)_{L_p(\mathbb{T}^1)}$ and $G_n(\mathcal{F}^{\psi}_q)_{L_p(\mathbb{T}^1)}$ follows from (9.133) and (9.134), $(9.46')$, taking into account (9.43).

Similarly to (9.133) and (9.134), we have

$$\mathscr{D}^{\perp}_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \le \mathscr{D}^{\perp}_m(\mathcal{F}^{\psi}_{q,r})_{L_2} = \mathscr{D}_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^2}, \quad 1 \le p \le 2, \tag{9.152}$$

and

$$\mathscr{D}^{\perp}_m(\mathcal{F}^{\psi}_{q,r})_{L_p} \le \mathscr{D}^{\perp}_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^{p'}} = \mathscr{D}_m(\mathcal{F}^{\psi}_{q,r})_{\mathcal{S}^{p'}}, \quad 2 \le p < \infty. \tag{9.153}$$

Therefore, to obtain upper estimates for $\mathscr{D}^{\perp}_m(\mathcal{F}^{\psi}_q)_{L_p(\mathbb{T}^1)}$, it sufficient to use estimates (9.152), (9.153), (9.38) and $(9.45')$, taking into account (9.43).

To obtain *lower estimates*, consider a rearrangement $\{k^*_1, k^*_2, \ldots\}$ of the set $\mathbb{Z}$ such that

$$|\Psi_{k^*_j}| = \bar{\Psi}_j, \quad j = 1, 2, \ldots, \tag{9.154}$$

where $\bar{\Psi} = \{\bar{\Psi}_j\}_{j=1}^{\infty}$ is a non-increasing rearrangement of the system $\{|\Psi_k|\}_{k\in\mathbb{Z}}$.

For a given $m \in \mathbb{N}$, consider the function

$$h_3 = \sum_{j=1}^{m+1} \widehat{h}_3(k^*_j) e_{k_j} = \mathfrak{h}_3(m) \sum_{j=1}^{m+1} e_{k_j}, \quad \text{where} \quad \mathfrak{h}_3^{-q}(m) = \sum_{j=1}^{m+1} |\Psi_{k^*_i}|^{-q}.$$

It is clear that $h_3 \in \mathcal{F}^{\psi}_q$, and due to (9.154), (9.29), (9.87) and (9.43), we see that

$$\mathfrak{h}_3^{-q}(m) \asymp \sum_{j=1}^{m+1} \bar{\Psi}_j^{-q} \asymp \sum_{k=1}^{[\frac{m+1}{2}]} \psi^{-q}(k) \asymp \frac{\left[\dfrac{m+1}{2}\right] \alpha\left(\psi, \left[\dfrac{m+1}{2}\right]\right)}{\psi^s\left(\left[\dfrac{m+1}{2}\right]\right)} \asymp$$
$$\asymp \psi^{-q}\left(\left[\frac{m+1}{2}\right]\right).$$

Therefore, for any collection $\gamma_m \in \Gamma_m$, we have

$$\left\| h_3 - \sum_{k\in\gamma_m} \widehat{h}_3(k)e_k \right\|_{L_p(\mathbb{T}^1)} = \mathfrak{h}_3(m) \left\| \sum_{\substack{j=1 \\ k_j^*\notin\gamma_m}}^{n+1} e_{k_j^*} \right\|_{L_p(\mathbb{T}^1)} \asymp \psi\left(\left[\frac{m+1}{2}\right]\right)$$

and the necessary lower estimate is valid:

$$\sigma_m^{\perp}(\mathcal{F}_q^{\psi})_{L_p(\mathbb{T}^1)} \gg \sigma_m^{\perp}(h_3)_{L_p(\mathbb{T}^1)} \gg \psi\left(\left[\frac{m+1}{2}\right]\right) \asymp \psi(\frac{m}{2}).$$

□

9.5.10 Proof of Theorem 9.9

Proof of Theorem 9.9. To obtain *upper estimates* in (9.68) and (9.69), it is sufficient to use relations (9.152), (9.153) (9.47) and (9.55). To obtain *lower estimates*, for any collection $\gamma_m \in \Gamma_m$, we choose any vector $k_0 = k_0(\gamma_m) \in \mathbb{Z}^d \setminus \gamma_m$ such that

$$|\psi(|k_0|_r)| = \bar{\Psi}_{\gamma_m}(1) = \sup_{k\in\mathbb{Z}^d\setminus\gamma_m} |\Psi_{\gamma_m}(k)| = \sup_{k\in\mathbb{Z}^d\setminus\gamma_m} |\psi(|k|_r)|,$$

where, as in Section 9.3, the system $\Psi_{\gamma_m} = \{\Psi_{\gamma_m}(k)\}_{k\in\mathbb{Z}^d}$ is defined by (9.15) and $\bar{\Psi}_{\gamma_m} = \{\bar{\Psi}_{\gamma_m}(j)\}_{j=1}^{\infty}$ is a non-increasing rearrangement of $\{|\Psi_{\gamma_m}(k)|\}_{k\in\mathbb{Z}^d}$.

Then the function $h_4 := \psi(|k_0|_r)e_{k_0}$ belongs to the set $\mathcal{F}_{q,r}^{\psi}$ and

$$\mathscr{E}_{\gamma_m}(h_4)_{L_p} = \left\|\psi(|k_0|_r)e_{k_0}\right\|_{L_p} = \psi(|k_0|_r) = \bar{\Psi}_{\gamma_m}(1).$$

Taking into account (9.29), (9.34), (9.35) and (9.36), we obtain the necessary lower estimate:

$$\mathscr{D}_m^{\perp}(\mathcal{F}_{q,r}^{\psi})_{L_p} \geq \inf_{\gamma_m\in\Gamma_m} \mathscr{E}_{\gamma_m}(h_4)_{L_p} = \inf_{\gamma_m\in\Gamma_m} \bar{\Psi}_{\gamma_m}(1) = \bar{\Psi}_{m+1} = \psi(s).$$

□

9.5.11 Proof of Theorem 9.10

Proof of Theorem 9.10. Upper estimates in (9.70) follow from relations (9.152), (9.153) and (9.38).

In the case when $0 < q \leq \frac{p}{p-1}$, the *lower estimate* follows from (9.68), taking into account relations (9.34), (9.35) and the definition of the set B. In the case when $\frac{p}{p-1} < q \leq \infty$, the *lower estimate* in (9.70) follows from (9.66). □

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Sergii YANCHENKO

10 Approximation characteristics of Nikol'skii–Besov classes of non-periodic functions of one and many variables

In this chapter, we present and somewhat systematize the results obtained by the author in [29–31,33], as well as by Wang Heping and Sun Yongsheng [10,28]. These results relate to the search for exact-order estimates of some approximation characteristics of Nikol'skii–Besov classes of functions with a dominant mixed smoothness (derivative) $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$. These classes of functions were introduced in the works of S. M. Nikol'skii [13] and T. I. Amanov [1]. Namely, we investigated the approximation of the classes $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$ in the metric of the Lebesgue space $L_q(\mathbb{R}^d)$ using entire functions of exponential type. Section 10.2 presents exact-order estimates of the approximation of the classes of functions $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$ by entire functions of exponential type with the supports of their Fourier transform in the step hyperbolic cross. In this section formulates the well-known results obtained by Wang Heping and Sun Yongsheng, and the main attention is paid to the results obtained by the author. In Section 10.3, we investigate the approximation of functions from these classes by entire functions of exponential type with a spectrum of a special form. Due to the greater flexibility in the choice of the approximating aggregate, in some cases, we manage to obtain better estimates of the approximation compared to the approximation in the step hyperbolic cross.

In the classical form, the definitions of these spaces and, accordingly, classes of functions (unit spheres) $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$, were denoted in terms of conditions for the mixed difference of the $\boldsymbol{k}$-th order of the function with the vector step $\boldsymbol{h}$. As it turned out later, in the study of approximation properties, the definition of the norm of functions plays a key role indirectly through the so-called "decomposition" representation of the elements of these spaces, which was obtained in the work of S. M. Nikol'skii and P. I. Lizorkin [12] both for classes of non-periodic functions and in the case of classes of periodic functions. In addition to their independent interest from the point of view of function theory, studies of such spaces find applications in the theory of partial differential equations, in solving boundary value problems of regular elliptic differential equations, in computational mathematics, in image reproduction, etc.

Sergii YANCHENKO, Institute of Mathematics of the National Academy of Sciences of Ukraine, 3 Tereschenkivska str., 01024, Kyiv, Ukraine, e-mail: yan.sergiy@gmail.com

10.1 Definition of Nikol'skii–Besov classes of functions with dominant mixed smoothness

Here, the definition of the Nikol'skii–Besov spaces of functions with dominant mixed smoothness $S^r_{p,\theta}B(\mathbb{R}^d)$ is presented through so-called decomposition representation of the norm of elements from these spaces. This representation is essentially used in proving the obtained results and is based on the application of the Fourier transform, which can be defined in terms of generalized functions (see, e.g., [3, Ch. 11], [14, Ch. 1, § 5]).

First, we give the necessary notation and definitions.

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space with the elements $\boldsymbol{x}=(x_1,\dots,x_d)$, $x_i \in \mathbb{R}$, and $(\boldsymbol{x},\boldsymbol{y}) = x_1y_1 + \ldots + x_dy_d$. Denote by $L_q(\mathbb{R}^d)$, $1 \leqslant q \leqslant \infty$, the space of all functions $f(\boldsymbol{x}) = f(x_1,\dots,x_d)$ measurable on $\mathbb{R}^d$ with the finite norm

$$\|f\|_{L_q(\mathbb{R}^d)} := \left(\int\limits_{\mathbb{R}^d} |f(\boldsymbol{x})|^q d\boldsymbol{x}\right)^{1/q}, \quad 1 \leqslant q < \infty,$$

$$\|f\|_{L_\infty(\mathbb{R}^d)} := \operatorname*{ess\,sup}_{\boldsymbol{x}\in\mathbb{R}^d} |f(\boldsymbol{x})|.$$

Let $S = S(\mathbb{R}^d)$ be the Schwarz space of test complex-valued functions φ infinitely differentiable on $\mathbb{R}^d$ and decreasing at infinity together with their derivatives faster than any power of the function $(x_1^2 + \ldots + x_d^2)^{-\frac{1}{2}}$, considered in the appropriate topology. Let S' denote the space of linear continuous functionals on S. The elements of the space S' are generalized functions. If $f \in S'$, then $\langle f, \varphi\rangle$ denote the value of a functional f on the test function $\varphi \in S$.

The Fourier transform $\mathfrak{F}\varphi\colon S \to S$ is defined by the formula

$$(\mathfrak{F}\varphi)(\boldsymbol{\lambda}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \varphi(\boldsymbol{t}) e^{-i(\boldsymbol{\lambda},\boldsymbol{t})} d\boldsymbol{t} \equiv \widetilde{\varphi}(\boldsymbol{\lambda}),$$

where $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_d) \in \mathbb{R}^d$, $\boldsymbol{t} = (t_1,\dots,t_d) \in \mathbb{R}^d$ and $(\boldsymbol{\lambda},\boldsymbol{t}) = \sum_{i=1}^d \lambda_i t_i$ is the scalar product of the vectors $\boldsymbol{\lambda}$ and $\boldsymbol{t}$ in $\mathbb{R}^d$.

The inverse Fourier transform is defined as follows:

$$(\mathfrak{F}^{-1}\varphi)(\boldsymbol{t}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \varphi(\boldsymbol{\lambda}) e^{i(\boldsymbol{\lambda},\boldsymbol{t})} d\boldsymbol{\lambda} \equiv \widehat{\varphi}(\boldsymbol{t}).$$

The Fourier transform of generalized functions (for which we preserve the same notation) is defined by the formula

$$\langle \mathfrak{F}f, \varphi\rangle = \langle f, \mathfrak{F}\varphi\rangle, \quad \langle \widetilde{f}, \varphi\rangle = \langle f, \widetilde{\varphi}\rangle,$$

where $f \in S'$ and $\varphi \in S$.

The inverse transform of generalized functions is also denoted by $\mathfrak{F}^{-1} f$ and defined, by analogy with the direct Fourier transformation, by the following rule:

$$\langle \mathfrak{F}^{-1} f, \varphi \rangle = \langle f, \mathfrak{F}^{-1} \varphi \rangle, \quad \langle \widehat{f}, \varphi \rangle = \langle f, \widehat{\varphi} \rangle.$$

For any function φ continuous on $\mathbb{R}^d$, the closure of the set of all points $\boldsymbol{x} \in \mathbb{R}^d$ such that $\varphi(\boldsymbol{x}) \neq 0$ is called the support of the function φ and denoted by $\operatorname{supp} \varphi$.

The generalized function f vanishes in an open set G when $\langle f, \varphi \rangle = 0$ for all $\varphi \in S$ and $\operatorname{supp} \varphi \subset G$. The union of all neighborhoods where f is equal to zero is an open set and called the null set of the generalized function f. It is denoted by G_f. The complement of the largest open set G_f to $\mathbb{R}^d$ is called the support of the generalized function f, i.e., $\operatorname{supp} f$ equal to $\bar{G}_f$ is a closed set.

According to the formula

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^d} f(\boldsymbol{x}) \varphi(\boldsymbol{x}) d\boldsymbol{x}, \quad \varphi \in S,$$

each function $f \in L_p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$, defines a linear continuous functional on S and, therefore, is an element of S' in this sense. Hence, the Fourier transform of a function $f \in L_p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$, can be regarded as the Fourier transform of the generalized function $\langle f, \varphi \rangle$.

For $\boldsymbol{s} \in \mathbb{Z}_+^d$, we also consider the sets

$$Q^*_{2^{\boldsymbol{s}}} = \{\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d) \colon \ \eta(s_j) 2^{s_j - 1} \leqslant |\lambda_j| < 2^{s_j}, \ \lambda_j \in \mathbb{R}, \ j = \overline{1, d}\},$$

where $\eta(0) = 0$ and $\eta(t) = 1$, $t > 0$.

Let $A \subset \mathbb{R}^d$ be a measurable set. By χ_A we denote the characteristic function of the set A. For $f \in L_p(\mathbb{R}^d)$, $1 < p < \infty$, the function

$$\delta^*_{\boldsymbol{s}}(f, \boldsymbol{x}) = \mathfrak{F}^{-1}(\chi_{Q^*_{2^{\boldsymbol{s}}}} \cdot \mathfrak{F} f)$$

is correctly defined and belongs to $L_p(\mathbb{R}^d)$.

Then the spaces $S^{\boldsymbol{r}}_{p,\theta} B(\mathbb{R}^d)$, $1 < p < \infty$, $1 \leqslant \theta \leqslant \infty$, $\boldsymbol{r} > \boldsymbol{0}$, can be defined as follows [12]:

$$S^{\boldsymbol{r}}_{p,\theta} B(\mathbb{R}^d) := \{f \in L_p(\mathbb{R}^d) \colon \ \|f\|_{S^{\boldsymbol{r}}_{p,\theta} B(\mathbb{R}^d)} < \infty\},$$

where

$$\|f\|_{S^{\boldsymbol{r}}_{p,\theta} B(\mathbb{R}^d)} \asymp \left(\sum_{\boldsymbol{s} \geqslant \boldsymbol{0}} 2^{(\boldsymbol{s}, \boldsymbol{r})\theta} \|\delta^*_{\boldsymbol{s}}(f, \cdot)\|^{\theta}_{L_p(\mathbb{R}^d)} \right)^{1/\theta} \tag{10.1}$$

for $1 \leqslant \theta < \infty$ and

$$\|f\|_{S^{\boldsymbol{r}}_{p,\infty} B(\mathbb{R}^d)} \equiv \|f\|_{S^{\boldsymbol{r}}_{p} H(\mathbb{R}^d)} \asymp \sup_{\boldsymbol{s} \geqslant \boldsymbol{0}} 2^{(\boldsymbol{s}, \boldsymbol{r})} \|\delta^*_{\boldsymbol{s}}(f, \cdot)\|_{L_p(\mathbb{R}^d)}. \tag{10.2}$$

Here and below, for positive quantities a and b, the notation $a \asymp b$ means that there exist positive constants C_1 and C_2 that do not depend on an essential parameter in the values a and b (e.g., C_1 and C_2 in the expressions (10.1) and (10.2) do not depend on the function f) such that $C_1 a \leqslant b$ (in this case, we write $a \ll b$) and $C_2 a \geqslant b$ (in this case, we write $a \gg b$). All of the constants C_i, $i = 1, 2, \ldots$, appearing in the chapter, could depend only on the parameters from the definition of the class, metrics where we measure the approximation error, and from the dimension of the space $\mathbb{R}^d$.

In the case $1 \leqslant p \leqslant \infty$, by modifying $\delta_{\boldsymbol{s}}^*(f, \boldsymbol{x})$, for example, according to the de la Vallée Poussin type, the norm of functions from the spaces $S_{p,\theta}^{\boldsymbol{r}} B(\mathbb{R}^d)$ can be defined in a different form.

Further, let

$$K_m(t) = \int_{\mathbb{R}} k_m(\lambda) e^{-2\pi i \lambda t} d\lambda, \quad m \in \mathbb{Z}_+, \quad K_{-1} \equiv 0,$$

where

$$k_m(\lambda) = \begin{cases} 1, & |\lambda| < 2^{m-1}, \\ 2\left(1 - \dfrac{|\lambda|}{2^m}\right), & 2^{m-1} \leqslant |\lambda| \leqslant 2^m, \\ 0, & |\lambda| > 2^m, \end{cases} \quad k_0(\lambda) = \begin{cases} 1 - |\lambda|, & 0 \leqslant |\lambda| \leqslant 1, \\ 0, & |\lambda| > 1. \end{cases}$$

For any vector $\boldsymbol{s} = (s_1, \ldots, s_d)$, $s_j \in \mathbb{Z}_+$, $j = \overline{1, d}$, we define

$$A_{\boldsymbol{s}}^*(\boldsymbol{x}) = \prod_{j=1}^{d} \left(K_{s_j}(x_j) - K_{s_j - 1}(x_j)\right),$$

$$A_{\boldsymbol{s}}^*(f, \boldsymbol{x}) = f(\boldsymbol{x}) * A_{\boldsymbol{s}}^*(\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{y}) A_{\boldsymbol{s}}^*(\boldsymbol{x} - \boldsymbol{y}) d\boldsymbol{y}.$$

Note that $A_{\boldsymbol{s}}^*(f, \boldsymbol{x})$ is the analog of the de la Vallée Poussin block of sum of periodic function of several variables (see, e.g., [26]).

The following statement is true.

Lemma 10.1 ([10]). *Let $1 \leqslant p \leqslant \infty$, then for any $f \in L_p(\mathbb{R}^d)$ we have*

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{s}} A_{\boldsymbol{s}}^*(f, \boldsymbol{x})$$

and $\operatorname{supp} \mathfrak{F} A_{\boldsymbol{s}}(f, \boldsymbol{x}) \subseteq Q_{2\boldsymbol{s}}^*$.

In the accepted notation, the spaces $S_{p,\theta}^{\boldsymbol{r}} B(\mathbb{R}^d)$, $1 \leqslant p, \theta \leqslant \infty$, $\boldsymbol{r} > 0$, can be defined as follows (see, e.g., [10, 27]):

$$S_{p,\theta}^{\boldsymbol{r}} B(\mathbb{R}^d) := \{f \in L_p(\mathbb{R}^d) \colon \ \|f\|_{S_{p,\theta}^{\boldsymbol{r}} B(\mathbb{R}^d)} < \infty\},$$

where

$$\|f\|_{S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)} \asymp \left(\sum_{\boldsymbol{s}\geqslant\boldsymbol{0}} 2^{(\boldsymbol{s},\boldsymbol{r})\theta}\|A^*_{\boldsymbol{s}}(f,\cdot)\|^{\theta}_{L_p(\mathbb{R}^d)}\right)^{1/\theta} \tag{10.3}$$

for $1 \leqslant \theta < \infty$ and

$$\|f\|_{S^{\boldsymbol{r}}_{p}H(\mathbb{R}^d)} \asymp \sup_{\boldsymbol{s}\geqslant\boldsymbol{0}} 2^{(\boldsymbol{s},\boldsymbol{r})}\|A^*_{\boldsymbol{s}}(f,\cdot)\|_{L_p(\mathbb{R}^d)}. \tag{10.4}$$

As can be seen from (10.1), (10.2), (10.3), (10.4), for any $f \in S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$, $1 < p < \infty$, the following relation holds:

$$\|\delta^*_{\boldsymbol{s}}(f,\cdot)\|_{L_p(\mathbb{R}^d)} \asymp \|A^*_{\boldsymbol{s}}(f,\cdot)\|_{L_p(\mathbb{R}^d)}. \tag{10.5}$$

$S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$ is defined as a set of functions $f \in L_p(\mathbb{R}^d)$ with $\|f\|_{S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)} \leqslant 1$. We conserve the same notations for the classes $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$ as for the spaces $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$.

Note that, for the value of the parameter $\theta = \infty$, the spaces of functions $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$ coincide with the spaces $S^{\boldsymbol{r}}_{p}H(\mathbb{R}^d)$ that were, for the first time, considered by S. M. Nikol'skii [13], and in the case $1 \leqslant \theta < \infty$ they were introduced by T. I. Amanov [1]. These function spaces are called the spaces of functions with dominant mixed smoothness (derivative). In the literature they are also called Nikol'skii–Besov spaces of functions with dominant mixed smoothness (derivative) or, for short, spaces of functions with mixed smoothness.

In the one-dimensional case, the Nikol'skii–Besov spaces with mixed smoothness $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$ coincide with isotropic Nikol'skii–Besov spaces $B^{r}_{p,\theta}(\mathbb{R}^d)$. The isotropic spaces $B^{r}_{p,\theta}(\mathbb{R}^d)$ were introduced by S. M. Nikol'skii [15] in the case $\theta = \infty$ ($B^{r}_{p,\infty}(\mathbb{R}^d) \equiv H^{r}_{p}(\mathbb{R}^d)$) and O. V. Besov [4], when $1 \leqslant \theta < \infty$.

S. M. Nikol'skii and T. I. Amanov obtained for the function spaces $S^{\boldsymbol{r}}_{p}H(\mathbb{R}^d)$ and $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$ the direct and inverse theorems of the representation of functions from these spaces using entire functions of exponential type. These theorems are the main means for obtaining embedding theorems for given function spaces. The main results of studies of the spaces $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$ are presented in the monograph of T. I. Amanov [2], where he also considered the corresponding classes of periodic functions.

We assume that coordinates of the vector $\boldsymbol{r} = (r_1, \ldots, r_d)$, as the parameter of the defined classes $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$, are ordered such that

$$0 < r_1 = r_2 = \ldots = r_\nu < r_{\nu+1} \leqslant \ldots \leqslant r_d.$$

The vector $\boldsymbol{r} = (r_1, \ldots, r_d)$ is associated with the vector $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_d)$, $\gamma_j = \frac{r_j}{r_1}$, $j = \overline{1,d}$, and the vector $\boldsymbol{\gamma}$ is, in turn, associated, with the vector $\boldsymbol{\gamma}'$, where $\gamma'_j = \gamma_j$ if $j = \overline{1,\nu}$ and $1 < \gamma'_j < \gamma_j$, $j = \overline{\nu+1,d}$.

Moreover, for the vectors $\boldsymbol{a} = (a_1, \dots, a_d)$ and $\boldsymbol{b} = (b_1, \dots, b_d)$, the inequalities of the type $\boldsymbol{a} \leqslant \boldsymbol{b}$ ($\boldsymbol{a} > \boldsymbol{b}$) are understood in the coordinate-wise: $a_j \leqslant b_j$ ($a_j > b_j$), $j = \overline{1, d}$. We also use $\boldsymbol{t} \geqslant 0$ ($\boldsymbol{t} > 0$) if $t_j \geqslant 0$ ($t_j > 0$), $j = \overline{1, d}$, and $\boldsymbol{a} \neq \boldsymbol{b}$ if $a_i \neq b_i$ at least for one i, $i = \overline{1, d}$.

Let us formulate one of the embedding theorems for the spaces $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$. This theorem is of great importance in studying the approximation properties of these spaces and was obtained by T. I. Amanov [1, Theorem 3.2], [2, Theorem 3.1].

Proposition 10.1. *Let $1 \leqslant p, \theta \leqslant \infty$, $1 \leqslant p \leqslant q \leqslant \infty$, and we have a vector $\boldsymbol{\rho}$ such that $\rho_j = r_j - (\frac{1}{p} - \frac{1}{q}) > 0$, $j = \overline{1, d}$. If $f \in S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$, then $f \in S^{\boldsymbol{\rho}}_{q,\theta}B(\mathbb{R}^d)$ and*

$$\|f\|_{S^{\boldsymbol{\rho}}_{q,\theta}B(\mathbb{R}^d)} \leqslant C_3 \|f\|_{S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)}, \quad C_3 > 0.$$

We note that for the spaces $S^{\boldsymbol{r}}_{p}H(\mathbb{R}^d)$ ($\theta = \infty$) Proposition 10.1 was established by S. M. Nikol'skii [13, Theorem 5].

The result of Proposition 10.1 determines the conditions on the coordinates of the vector $\boldsymbol{r}$ under which, for a function $f \in S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$, we can assert that f will also belong to the space $L_q(\mathbb{R}^d)$.

10.2 Approximation by entire functions with the supports of their Fourier transforms in the step hyperbolic cross

First, we give definitions of the approximation characteristics, which are investigated in this section.

Let $\mathcal{L} \subset \mathbb{Z}^d_+$ be a finite set. We define the set $Q(\mathcal{L}) = \bigcup_{\boldsymbol{s} \in \mathcal{L}} Q^*_{2^{\boldsymbol{s}}}$ and denote

$$G\big(Q(\mathcal{L})\big) := \{f \in L_q(\mathbb{R}^d)\colon \ \operatorname{supp} \mathfrak{F} f \subseteq Q(\mathcal{L})\}.$$

It is known that the elements of the set $G\big(Q(\mathcal{L})\big)$ are entire functions of the exponential type and the supports of their Fourier transforms lies in $Q(\mathcal{L})$ (see, e.g., [14, § 3.1]).

For $f \in L_q(\mathbb{R}^d)$, $1 \leqslant q \leqslant \infty$, we define

$$E\Big(f, G\big(Q(\mathcal{L})\big)\Big)_{L_q(\mathbb{R}^d)} := E_{Q(\mathcal{L})}\big(f\big)_{L_q(\mathbb{R}^d)} := \inf_{g \in G(Q(\mathcal{L}))} \|f(\cdot) - g(\cdot)\|_{L_q(\mathbb{R}^d)}. \quad (10.6)$$

This quantity is called the best approximation of the function f by entire functions from the set $G\big(Q(\mathcal{L})\big)$.

If $F \subset L_q(\mathbb{R}^d)$ is a functional class, then we set

$$E_{Q(\mathcal{L})}\big(F\big)_{L_q(\mathbb{R}^d)} := \sup_{f \in F} E_{Q(\mathcal{L})}\big(f\big)_{L_q(\mathbb{R}^d)}. \quad (10.7)$$

Furthermore, for $f \in L_q(\mathbb{R}^d)$, $1 \leqslant q \leqslant \infty$, we will consider the function of the form

$$S_{Q(\mathcal{L})}(f,\boldsymbol{x}) = \sum_{\boldsymbol{s}\in\mathcal{L}} \delta_{\boldsymbol{s}}^*(f,\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^d.$$

Thus, we have defined the entire function $S_{Q(\mathcal{L})}(f,\boldsymbol{x})$ that belongs to the space $L_q(\mathbb{R}^d)$ (see, for example, [11]) and the support of its Fourier transform is concentrated on the set $Q(\mathcal{L})$, i.e., supp $S_{\mathfrak{M}}(f,\boldsymbol{x}) \subseteq Q(\mathcal{L})$.

Consider the following approximative characteristic:

$$\begin{aligned} \mathcal{E}_{Q(\mathcal{L})}\big(f\big)_{L_q(\mathbb{R}^d)} &:= \|f(\cdot) - S_{Q(\mathcal{L})}(f,\cdot)\|_{L_q(\mathbb{R}^d)}, \\ \mathcal{E}_{Q(\mathcal{L})}\big(F\big)_{L_q(\mathbb{R}^d)} &:= \sup_{f\in F} \mathcal{E}_{Q(\mathcal{L})}\big(f\big)_{L_q(\mathbb{R}^d)}. \end{aligned} \tag{10.8}$$

Depending on the choice of elements of the set $\mathcal{L}$ and, accordingly, the construction of the set $Q(\mathcal{L})$, we can consider several approximation characteristics of the form (10.7) and (10.8).

One of such sets, which plays an important role in the theory of approximation of classes of functions $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$, as already noted, is the step hyperbolic cross. Let the set $\mathcal{L}$ contain vectors $\boldsymbol{s}$ for which the condition $(\boldsymbol{s},\boldsymbol{\gamma}) \leqslant n$ is satisfied, where $n \in \mathbb{N}$. That is, we obtain $\mathcal{L} = \{\boldsymbol{s}\colon (\boldsymbol{s},\boldsymbol{\gamma}) \leqslant n\}$. In this case, as the set $Q(\mathcal{L})$, we will consider the set $\tilde{Q}_n^{\gamma}$, which is denoted as follows:

$$\tilde{Q}_n^{\gamma} = \bigcup_{(\boldsymbol{s},\boldsymbol{\gamma})\leqslant n} Q_{2^{\boldsymbol{s}}}^*.$$

The set $\tilde{Q}_n^{\gamma}$ is called a stepwise hyperbolic cross and, moreover, mes $\tilde{Q}_n^{\gamma} \asymp 2^n n^{d-1}$ (see, e.g., [12]), where mes $\tilde{Q}_n^{\gamma}$ is the Lebesgue measure of the set $\tilde{Q}_n^{\gamma}$.

For the quantity (10.6) we will use the notation

$$E_{\tilde{Q}_n^{\gamma}}\big(f\big)_{L_q(\mathbb{R}^d)} := \inf_{g\in G(\tilde{Q}_n^{\gamma})} \|f(\cdot) - g(\cdot)\|_{L_q(\mathbb{R}^d)}.$$

This quantity is called the best approximation of the function f by entire functions of the exponential type with the supports of their Fourier transform in the step hyperbolic cross.

If $F = S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$ is a functional class, then, according to (10.7), we put

$$E_{\tilde{Q}_n^{\gamma}}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)} := \sup_{f\in S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)} E_{\tilde{Q}_n^{\gamma}}\big(f\big)_{L_q(\mathbb{R}^d)}. \tag{10.9}$$

For (10.8) in this case we will use the following notation:

$$\mathcal{E}_{\tilde{Q}_n^{\gamma}}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)} := \sup_{f\in S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)} \mathcal{E}_{\tilde{Q}_n^{\gamma}}\big(f\big)_{L_q(\mathbb{R}^d)}, \tag{10.10}$$

where

$$\mathcal{E}_{\tilde{Q}_n^\gamma}(f)_{L_q(\mathbb{R}^d)} := \|f(\cdot) - S_{\tilde{Q}_n^\gamma}(f,\cdot)\|_{L_q(\mathbb{R}^d)},$$
$$S_{\tilde{Q}_n^\gamma}(f,\boldsymbol{x}) = \sum_{(\boldsymbol{s},\gamma)\leqslant n} \delta_{\boldsymbol{s}}^*(f,\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

Note that$S_{\tilde{Q}_n^\gamma}(f,\boldsymbol{x})$ is an entire function and the support of its Fourier transform is concentrated in a step hyperbolic cross, i.e., on the set $\tilde{Q}_n^\gamma$.

We note that, for $1 < q < \infty$ and $f \in L_q(\mathbb{R}^d)$, the following relation is true:

$$E_{\tilde{Q}_n^\gamma}(f)_{L_q(\mathbb{R}^d)} \leqslant \mathcal{E}_{\tilde{Q}_n^\gamma}(f)_{L_q(\mathbb{R}^d)} \leqslant C_4 E_{\tilde{Q}_n^\gamma}(f)_{L_q(\mathbb{R}^d)},$$

where $C_4 \geqslant 1$ is a constant (see, e.g., [12]).

The problem of approximating classes of functions of one and many variables defined on Rd by entire functions with spectrum in the step hyperbolic cross was investigated, in particular, by Ya. S. Bugrov [6], H.-J. Schmeisser and W. Sickel [23], W. Sickel and T. Ullrich [24], Sun Yongsheng, Liu Yongping, Chen Dirong [25], Wang Heping and Sun Yongsheng [28] and others.

For the quantities (10.9) and (10.10), Sun Yongsheng and Wang Heping [28] established the following statements.

Theorem 10.1. *If* $1 < p < q < \infty$, $1 \leqslant \theta \leqslant \infty$ *and* $r_1 > \frac{1}{p} - \frac{1}{q}$, *then*

$$\mathcal{E}_{\tilde{Q}_n^\gamma}\left(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\right)_{L_q(\mathbb{R}^d)} \asymp E_{\tilde{Q}_n^\gamma}\left(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\right)_{L_q(\mathbb{R}^d)} \asymp 2^{-n\left(r_1-\frac{1}{p}+\frac{1}{q}\right)} n^{(\nu-1)\left(\frac{1}{q}-\frac{1}{\theta}\right)_+},$$

where $a_+ = \max\{a;\ 0\}$.

Theorem 10.2. *Let* $1 < p < \infty$, $\boldsymbol{r} > \boldsymbol{0}$. *Then the following relations holds for* $1 \leqslant \theta \leqslant \infty$

$$\mathcal{E}_{\tilde{Q}_n^\gamma}\left(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\right)_{L_p(\mathbb{R}^d)} \asymp E_{\tilde{Q}_n^\gamma}\left(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\right)_{L_p(\mathbb{R}^d)} \asymp$$
$$\asymp \begin{cases} 2^{-nr_1} n^{(d-1)\left(\frac{1}{p}-\frac{1}{\theta}\right)_+}, & \text{for} \quad 1 < p \leqslant 2, \\ 2^{-nr_1} n^{(d-1)\left(\frac{1}{2}-\frac{1}{\theta}\right)_+}, & \text{for} \quad 2 < p < \infty. \end{cases}$$

Theorem 10.3. *Let* $\boldsymbol{\gamma}$ *and* $\boldsymbol{\gamma}'$ *be vectors specified as follows:* $1 = \gamma_1 = \gamma_1' = \ldots = \gamma_\nu = \gamma_\nu'$ *and* $1 < \gamma_j' < \gamma_j$, $j = \overline{\nu+1, d}$ $(1 \leqslant \nu \leqslant d)$, $\boldsymbol{r} = r_1 \cdot \boldsymbol{\gamma}$. *Then the following relations holds for* $1 \leqslant \theta \leqslant \infty$:

a)

$$\mathcal{E}_{\tilde{Q}_n^{\gamma'}}\left(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\right)_{L_p(\mathbb{R}^d)} \asymp E_{\tilde{Q}_n^{\gamma'}}\left(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\right)_{L_p(\mathbb{R}^d)} \asymp$$
$$\asymp \begin{cases} 2^{-nr_1} n^{(\nu-1)\left(\frac{1}{p}-\frac{1}{\theta}\right)_+}, & \text{for} \quad 1 < p \leqslant 2, \\ 2^{-nr_1} n^{(\nu-1)\left(\frac{1}{2}-\frac{1}{\theta}\right)_+}, & \text{for} \quad 2 < p < \infty, \end{cases} \quad \text{where } r_1 > 0;$$

b)

$$\mathcal{E}_{\tilde{Q}_n^{\gamma'}}(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R})^d)_{L_q(\mathbb{R}^d)} \asymp E_{\tilde{Q}_n^{\gamma'}}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)} \asymp 2^{-n\left(r_1-\frac{1}{p}+\frac{1}{q}\right)} n^{(\nu-1)\left(\frac{1}{q}-\frac{1}{\theta}\right)_+},$$
$$\textit{for} \quad 1<p<q<\infty \quad \textit{and} \quad r_1 > \frac{1}{p}-\frac{1}{q}$$

The set $\tilde{Q}_n^{\gamma'}$ is called an extended step hyperbolic cross.

For the limiting value of the parameter $p=1$ in the definition of classes, Sun Yongsheng and Wang Heping [10] obtained exact values of quantity (10.9) only for Nikol'skii classes, i.e., $S_1^{\boldsymbol{r}}H(\mathbb{R}^d)$.

Theorem 10.4. *The following relation holds*

$$E_{\tilde{Q}_n^{\gamma}}(S_1^{\boldsymbol{r}}H(\mathbb{R}^d))_{L_q(\mathbb{R}^d)} \asymp \begin{cases} 2^{-n\left(r_1-1+\frac{1}{q}\right)} n^{\frac{\nu-1}{q}}, & 1<q<\infty, \quad r_1>1-\dfrac{1}{q}, \\ 2^{-nr_1} n^{d-1}, & q=1, \quad r_1>0. \end{cases}$$

We have managed to extend the result of Theorem 10.4 to Besov classes, i.e. for $1 \leqslant \theta < \infty$.

Let us formulate an auxiliary statements that we will use multiple times in the proofs.

Proposition 10.2 ([14], § 3.3.4). *If* $1 \leqslant p \leqslant q \leqslant \infty$*, then the inequality*

$$\|g_{\boldsymbol{\nu}}\|_{L_q(\mathbb{R}^d)} \leqslant 2^d \left(\prod_{j=1}^d \nu_j\right)^{\frac{1}{p}-\frac{1}{q}} \|g_{\boldsymbol{\nu}}\|_{L_p(\mathbb{R}^d)}$$

is true for an entire function of exponential type $g_{\boldsymbol{\nu}} \in L_p(\mathbb{R}^d)$, $\boldsymbol{\nu}=(\nu_1,\dots,\nu_d)$, $\nu_j \geqslant 0$, $j=\overline{1,d}$.

This inequality is commonly called the "Nikol'skii inequality of different metricss" for entire functions of exponential type.

Proposition 10.3 ([14], §. 1.5.6). *Let* $1<p<\infty$*. There exist positive numbers* C_5 *and* C_6 *such that, for any function* $f \in L_p(\mathbb{R}^d)$*, the following relations are true:*

$$C_5\|f(\cdot)\|_{L_p(\mathbb{R}^d)} \leqslant \left\| \left(\sum_{\boldsymbol{s}\geqslant \boldsymbol{0}} |\delta_{\boldsymbol{s}}^*(f,\cdot)|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)} \leqslant C_6\|f(\cdot)\|_{L_p(\mathbb{R}^d)}.$$

In the literature, theorems of the type of Proposition 10.3 are called Littlewood–Paley theorems.

Lemma 10.2 ([28]). *Suppose that* $1 < p < q < \infty$ *and* $f \in L_p(\mathbb{R}^d)$ *are given. Then*

$$\|f(\cdot)\|_{L_q(\mathbb{R}^d)} \ll \left(\sum_{\boldsymbol{s} \geqslant \boldsymbol{0}} \|\delta^*_{\boldsymbol{s}}(f, \cdot)\|^q_{L_p(\mathbb{R}^d)} 2^{\|\boldsymbol{s}\|_1 \left(\frac{1}{p} - \frac{1}{q}\right) q} \right)^{1/q},$$

where $\|\boldsymbol{s}\|_1 = s_1 + \ldots + s_d$, $s_j \in \mathbb{Z}_+$, $j = \overline{1, d}$.

Lemma 10.3. *Suppose that* $1 < q < p < \infty$ *and* $f \in L_q(\mathbb{R}^d)$ *are given. Then*

$$\|f(\cdot)\|_{L_q(\mathbb{R}^d)} \gg \left(\sum_{\boldsymbol{s} \geqslant \boldsymbol{0}} \|\delta^*_{\boldsymbol{s}}(f, \cdot)\|^q_{L_p(\mathbb{R}^d)} 2^{\|\boldsymbol{s}\|_1 \left(\frac{1}{p} - \frac{1}{q}\right) q} \right)^{1/q}.$$

Lemmas 10.2, 10.3 are analogous to the lemmas, which were first proved, in the periodic case by V. M. Temlyakov (see, e.g., [26, Ch. 1, § 3]).

Lemma 10.4 ([26], Introduction). *Let* $\boldsymbol{s} \in \mathbb{N}^d$, $\boldsymbol{\gamma} \in \mathbb{R}^d$, $\gamma_j > 0$, $j = \overline{1, d}$. *Then for* $\alpha > 0$, *the following estimate holds:*

$$\sum_{(s, \boldsymbol{\gamma}) \geqslant l} 2^{-\alpha(s, \boldsymbol{\gamma})} \asymp 2^{-\alpha l} l^{d-1}.$$

If $\boldsymbol{\gamma}' \in \mathbb{R}^d$ *is such that* $\gamma_j = \gamma'_j = 1$ *for* $j = \overline{1, \nu}$ *and* $1 < \gamma'_j < \gamma_j$ *for* $j = \overline{\nu + 1, d}$. *Then for* $\alpha > 0$, *it holds:*

$$\sum_{(\boldsymbol{s}, \boldsymbol{\gamma}') \geqslant l} 2^{-\alpha(\boldsymbol{s}, \boldsymbol{\gamma})} \asymp 2^{-\alpha l} l^{\nu-1}.$$

Before proceeding to the formulation and proof of the main results of this section, let us formulate several statements that we obtained (see [33]) and concern the estimation of the norms $A^*_{\boldsymbol{s}}(\cdot)$ in the metric space $L_p(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$.

Lemma 10.5. *The following estimate holds:*

$$\|A^*_{\boldsymbol{s}}(\cdot)\|_{L_\infty(\mathbb{R}^d)} \asymp 2^{\|\boldsymbol{s}\|_1},$$

where $\|\boldsymbol{s}\|_1 = s_1 + \ldots + s_d$, $s_j \in \mathbb{Z}_+$, $j = \overline{1, d}$.

Lemma 10.6. *Let* $1 \leqslant p < \infty$ *and* $\boldsymbol{s} \in \mathbb{Z}^d_+$, *then the estimate holds:*

$$\|A^*_{\boldsymbol{s}}(\cdot)\|_{L_p(\mathbb{R}^d)} \asymp 2^{\|\boldsymbol{s}\|_1 \left(1 - \frac{1}{p}\right)}.$$

Lemma 10.7. *The following estimate holds:*

$$\left\| \sum_{(\boldsymbol{s}, \boldsymbol{1}) = n+1} A^*_{\boldsymbol{s}}(\cdot) \right\|_{L_\infty(\mathbb{R}^d)} \asymp 2^n n^{d-1}. \tag{10.11}$$

Note that it is on the basis of the de la Vallée Poussin sums that the extremal functions that implement the lower estimates in the theorems formulated below were constructed. The estimates obtained in the lemmas made it possible to show that this functions belong to the corresponding classes of functions with a dominant mixed derivative $S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)$.

Further, we establish estimates for the quantities (10.9) and (10.10) in the metric space $L_p(\mathbb{R}^d)$, $1 < p \leqslant \infty$.

Theorem 10.5. *Let $r_1 > 1$, $1 \leqslant \theta \leqslant \infty$. Then the ordinal relation holds*

$$\mathcal{E}_{\tilde{Q}^{\gamma}_n}(S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d))_{L_\infty(\mathbb{R}^d)} \asymp 2^{-n(r_1-1)}n^{(\nu-1)(1-1/\theta)}. \tag{10.12}$$

Remark 10.1. *Note that in each of the formulated theorems, the corresponding condition on the coordinates of the vector $\boldsymbol{r}$ ensures the embedding of the spaces $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$ into the space $L_q(\mathbb{R}^d)$ (space in whose metric the approximation error is estimated). Thus, in Theorem 10.5, since $r_1 > 1$, then by Proposition 10.1 there exists a vector $\boldsymbol{\rho}$, $\rho_j = r_j - 1 > 0$, $j = \overline{1,d}$, such that for $f \in S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)$ we have $f \in S^{\boldsymbol{\rho}}_{\infty,\theta}B(\mathbb{R}^d)$, i.e. $f \in L_\infty(\mathbb{R}^d)$. In addition, we can state that for some $1 < q_0 < \infty$, $f \in S^{\boldsymbol{\rho}}_{q_0,\theta}B$, where $\rho_j = r_j - \left(1 - \frac{1}{q_0}\right) > 0$, $j = \overline{1,d}$.*

Proof. We first find the upper estimates in (10.12). Let $f \in S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)$. Then, taking into account Remark 10.1, using Minkowski's inequality, the inequality of different metrics (Proposition 10.2), and also relation (10.5), we can write

$$\mathcal{E}_{\tilde{Q}^{\gamma}_n}(f)_{L_\infty(\mathbb{R}^d)} = \|f(\cdot) - S_{\tilde{Q}^{\gamma}_n}(f,\cdot)\|_{L_\infty(\mathbb{R}^d)} = \Bigg\| f(\cdot) - \sum_{(\boldsymbol{s},\boldsymbol{\gamma})\leqslant n} \delta^*_{\boldsymbol{s}}(f,\cdot)\Bigg\|_{L_\infty(\mathbb{R}^d)} \leqslant$$

$$\leqslant \sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} \|\delta^*_{\boldsymbol{s}}(f,\cdot)\|_{L_\infty(\mathbb{R}^d)} \ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} 2^{\frac{\|\boldsymbol{s}\|_1}{q_0}} \|\delta^*_{\boldsymbol{s}}(f,\cdot)\|_{L_{q_0}(\mathbb{R}^d)} \asymp$$

$$\asymp \sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} 2^{\frac{\|\boldsymbol{s}\|_1}{q_0}} \|A^*_{\boldsymbol{s}}(f,\cdot)\|_{L_{q_0}(\mathbb{R}^d)} \ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} 2^{\frac{\|\boldsymbol{s}\|_1}{q_0}} 2^{\|\boldsymbol{s}\|_1\left(1-\frac{1}{q_0}\right)} \|A^*_{\boldsymbol{s}}(f,\cdot)\|_{L_1(\mathbb{R}^d)} =$$

$$= \sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} 2^{\|\boldsymbol{s}\|_1} \|A^*_{\boldsymbol{s}}(f,\cdot)\|_{L_1(\mathbb{R}^d)}. \tag{10.13}$$

To continue estimate (10.13), we first consider the case $1 \leqslant \theta < \infty$. Hence, by using the Hölder inequality, we get

$$\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} 2^{\|\boldsymbol{s}\|_1} \|A^*_{\boldsymbol{s}}(f,\cdot)\|_{L_1(\mathbb{R}^d)} \leqslant \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|A^*_{\boldsymbol{s}}(f,\cdot)\|^{\theta}_{L_1(\mathbb{R}^d)}\right)^{1/\theta} \times$$

$$\times \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} 2^{-(\boldsymbol{s},\boldsymbol{r}-\mathbf{1})\frac{\theta}{\theta-1}} \right)^{1-1/\theta} \ll \|f(\cdot)\|_{S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)} \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} 2^{-(\boldsymbol{s},\boldsymbol{r}-\mathbf{1})\frac{\theta}{\theta-1}} \right)^{1-1/\theta} \leqslant$$

$$\leqslant \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} 2^{-(\boldsymbol{s},\bar{\boldsymbol{\gamma}})(r_1-1)\frac{\theta}{\theta-1}} \right)^{1-1/\theta} =: J_1,$$

where $\bar{\boldsymbol{\gamma}} = (\bar{\gamma}_1, \ldots, \bar{\gamma}_d)$ is the vector with coordinates $\bar{\gamma}_j = \frac{r_j-1}{r_1-1}$, $j = \overline{1,d}$, and $\boldsymbol{r}-\mathbf{1}$ denotes the vector with coordinates $r_j - 1$, $j = \overline{1,d}$. If $j = \overline{1,\nu}$, then $\bar{\gamma}_j = \gamma_j$ and $1 < \gamma_j \leqslant \bar{\gamma}_j$ for $j = \overline{\nu+1,d}$. By using Lemma 10.4, we obtain

$$J_1 \ll 2^{-n(r_1-1)} n^{(\nu-1)(1-1/\theta)}. \tag{10.14}$$

Comparing (10.13) and (10.14), we get

$$\sup_{f\in S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)} \|f(\cdot) - S_{\tilde{Q}^{\gamma}_n}(f,\cdot)\|_{L_\infty(\mathbb{R}^d)} \ll 2^{-n(r_1-1)} n^{(\nu-1)(1-\frac{1}{\theta})}.$$

Now let $\theta = \infty$. Thus, according to the definition of the classes $S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)$, we obtain $\|A^*_{\boldsymbol{s}}(f,\cdot)\|_{L_p(\mathbb{R}^d)} \ll 2^{-(\boldsymbol{s},\boldsymbol{r})}$. By using Lemma 10.4, for relation (10.13), we can write

$$\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} 2^{\|\boldsymbol{s}\|_1} \|A^*_{\boldsymbol{s}}(f,\cdot)\|_{L_1(\mathbb{R}^d)} \ll \sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} 2^{-(\boldsymbol{s},\boldsymbol{r}-\mathbf{1})} =$$

$$= \sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} 2^{-(\boldsymbol{s},\bar{\boldsymbol{\gamma}})(r_1-1)} \ll 2^{-n(r_1-1)} n^{\nu-1}. \tag{10.15}$$

Combining (10.13) and (10.15), we get the upper estimate in relation (10.12)

It is sufficient to get the lower estimate in (10.12) for the case $\nu = d$. Below we construct the functions that realizes the obtained upper order estimate. Consider the functions

$$f_1(\boldsymbol{x}) = C_7 2^{-nr_1} n^{-\frac{d-1}{\theta}} \sum_{(\boldsymbol{s},\mathbf{1})=n+1} A^*_{\boldsymbol{s}}(\boldsymbol{x}), \quad C_7 > 0,$$

for $1 \leqslant \theta < \infty$ and

$$f_2(\boldsymbol{x}) = C_8 2^{-nr_1} \sum_{(\boldsymbol{s},\mathbf{1})=n+1} A^*_{\boldsymbol{s}}(\boldsymbol{x}), \quad C_8 > 0,$$

for $\theta = \infty$.

It will be proved that these functions belong to the classes $S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)$ and $S^{\boldsymbol{r}}_{1,\infty}B(\mathbb{R}^d)$, respectively. Since the estimate $\left\|A^*_{\boldsymbol{s}}(\cdot)\right\|_{L_1(\mathbb{R}^d)} \asymp C_9$, $C_9 > 0$, is true, we find

$$\|f_1(\cdot)\|_{S^{\boldsymbol r}_{1,\theta}B(\mathbb{R}^d)} \asymp \left(\sum_{(\boldsymbol s,\boldsymbol 1)=n+1} 2^{(\boldsymbol s,\boldsymbol r)\theta} \|A^*_{\boldsymbol s}(f_1,\cdot)\|^{\theta}_{L_1(\mathbb{R}^d)} \right)^{1/\theta} \asymp$$

$$\asymp 2^{-nr_1} n^{-\frac{d-1}{\theta}} \left(\sum_{(\boldsymbol s,\boldsymbol 1)=n+1} 2^{(\boldsymbol s,\boldsymbol r)\theta} \|A^*_{\boldsymbol s}(\cdot)\|^{\theta}_{L_1(\mathbb{R}^d)} \right)^{1/\theta} \asymp$$

$$\asymp 2^{-nr_1} n^{-\frac{d-1}{\theta}} \left(\sum_{(\boldsymbol s,\boldsymbol 1)=n+1} 2^{r_1(\boldsymbol s,1)\theta} \right)^{\frac{1}{\theta}} \ll n^{-\frac{d-1}{\theta}} \left(\sum_{(\boldsymbol s,\boldsymbol 1)=n+1} 1 \right)^{1/\theta} \ll 1.$$

For f_2, we can write

$$\|f_2(\cdot)\|_{S^{\boldsymbol r}_{1,\infty}(\mathbb{R}^d)} \asymp \sup_{(\boldsymbol s,\boldsymbol 1)=n+1} 2^{(\boldsymbol s,\boldsymbol r)} \|A^*_{\boldsymbol s}(f_2,\cdot)\|_{L_1(\mathbb{R}^d)} \asymp$$

$$\asymp 2^{-nr_1} \sup_{(\boldsymbol s,\boldsymbol 1)=n+1} 2^{(\boldsymbol s,\boldsymbol r)} \|A^*_{\boldsymbol s}(\cdot)\|_{L_1(\mathbb{R}^d)} \asymp 2^{-nr_1} \sup_{(\boldsymbol s,\boldsymbol 1)=n+1} 2^{(\boldsymbol s,\boldsymbol r)} \ll 1.$$

Further, in view of the choice of the functions f_1 and f_2, we obtain $S_{\tilde Q^{\gamma}_n}(f_1,\boldsymbol x) = 0$ and $S_{\tilde Q^{\gamma}_n}(f_2,\boldsymbol x) = 0$.

By virtue of estimate (10.11), we get

$$\mathcal{E}_{\tilde Q^{\gamma}_n}\big(S^{\boldsymbol r}_{1,\theta}B(\mathbb{R}^d)\big)_{L_\infty(\mathbb{R}^d)} \gg \mathcal{E}_{\tilde Q^{\gamma}_n}\big(f_1\big)_{L_\infty(\mathbb{R}^d)} = \|f_1(\cdot) - S_{\tilde Q^{\gamma}_n}(f_1,\cdot)\|_{L_\infty(\mathbb{R}^d)} =$$

$$= \|f_1(\cdot)\|_{L_\infty(\mathbb{R}^d)} \asymp 2^{-nr_1} n^{-\frac{d-1}{\theta}} \left\| \sum_{(\boldsymbol s,\boldsymbol 1)=n+1} A^*_{\boldsymbol s}(\cdot) \right\|_{L_\infty(\mathbb{R}^d)} \asymp 2^{-n(r_1-1)} n^{(d-1)\left(1-\frac{1}{\theta}\right)}.$$

Similarly for f_2 we have

$$\mathcal{E}_{\tilde Q^{\gamma}_n}\big(S^{\boldsymbol r}_{1,\infty}B(\mathbb{R}^d)\big)_{L_\infty(\mathbb{R}^d)} \gg 2^{-n(r_1-1)} n^{d-1}.$$

The lower estimates are established.

Theorem 10.5 is proved. □

Theorem 10.6. *Let $1 < q < \infty$ and $r_1 > 1 - \frac{1}{q}$. Then, for $1 \leqslant \theta \leqslant \infty$, he following order relations are true:*

$$E_{\tilde Q^{\gamma}_n}(S^{\boldsymbol r}_{1,\theta}B(\mathbb{R}^d))_{L_q(\mathbb{R}^d)} \asymp \mathcal{E}_{\tilde Q^{\gamma}_n}\big(S^{\boldsymbol r}_{1,\theta}B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)} \asymp 2^{-n\left(r_1-1+\frac{1}{q}\right)} n^{(\nu-1)\left(\frac{1}{q}-\frac{1}{\theta}\right)_+}, \tag{10.16}$$

where $a_+ = \max\{a;\ 0\}$.

Proof. We first establish the upper estimates. Let $f \in S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)$. Then, see Remark 10.1, we have $f \in L_q(\mathbb{R}^d)$. For $1 < q_0 < q$, we now apply Lemma 10.2 and then the inequality of different metrics (Proposition 10.2). This yields

$$\mathcal{E}_{\tilde{Q}_n^\gamma}(f)_{L_q(\mathbb{R}^d)} = \|f(\cdot) - S_{\tilde{Q}_n^\gamma}(f,\cdot)\|_{L_q(\mathbb{R}^d)} = \left\| \sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} \delta^*_{\boldsymbol{s}}(f,\cdot) \right\|_{L_q(\mathbb{R}^d)} \ll$$

$$\ll \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} \|\delta^*_{\boldsymbol{s}}(f,\cdot)\|^q_{L_{q_0}(\mathbb{R}^d)} 2^{\|s\|_1\left(\frac{1}{q_0}-\frac{1}{q}\right)q} \right)^{1/q} \asymp$$

$$\asymp \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} \|A^*_{\boldsymbol{s}}(f,\cdot)\|^q_{L_{q_0}(\mathbb{R}^d)} 2^{\|s\|_1\left(\frac{1}{q_0}-\frac{1}{q}\right)q} \right)^{1/q} \ll$$

$$\ll \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} \|A^*_{\boldsymbol{s}}(f,\cdot)\|^q_{L_1(\mathbb{R}^d)} 2^{\|s\|_1\left(1-\frac{1}{q_0}\right)q} 2^{\|s\|_1\left(\frac{1}{q_0}-\frac{1}{q}\right)q} \right)^{1/q} =$$

$$= \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} \|A^*_{\boldsymbol{s}}(f,\cdot)\|^q_{L_1(\mathbb{R}^d)} 2^{\|s\|_1\left(1-\frac{1}{q}\right)q} \right)^{1/q} =: J_2.$$

In order to continue the estimate J_2, we consider several cases.

Let $1 < q < \theta < \infty$. We now apply the Hölder inequality with exponent $\frac{\theta}{q}$ to J_2 and take into account the fact that $r_1 > 1 - \frac{1}{q}$. This yields

$$J_2 = \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} \|A^*_{\boldsymbol{s}}(f,\cdot)\|^q_{L_1(\mathbb{R}^d)} 2^{(\boldsymbol{s},\boldsymbol{r})q} 2^{-(\boldsymbol{s},\boldsymbol{r})q} 2^{\|s\|_1\left(1-\frac{1}{q}\right)q} \right)^{\frac{1}{q}} \ll$$

$$\ll \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} \|A^*_{\boldsymbol{s}}(f,\cdot)\|^\theta_{L_1(\mathbb{R}^d)} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \right)^{\frac{1}{\theta}} \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} \left(2^{-(\boldsymbol{s},\boldsymbol{r})q} 2^{\|s\|_1\left(1-\frac{1}{q}\right)q} \right)^{\frac{\theta}{\theta-q}} \right)^{\frac{1}{q}-\frac{1}{\theta}} \leqslant$$

$$\leqslant \|f(\cdot)\|_{S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)} \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} \left(2^{-\left((\boldsymbol{s},\boldsymbol{r})-\left(1-\frac{1}{q}\right)\|s\|_1\right)} \right)^{\frac{q\theta}{\theta-q}} \right)^{\frac{1}{q}-\frac{1}{\theta}} \leqslant$$

$$\leqslant \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} \left(2^{-\left(\boldsymbol{s},\boldsymbol{r}-\left(1-\frac{1}{q}\right)\right)} \right)^{\frac{q\theta}{\theta-q}} \right)^{\frac{1}{q}-\frac{1}{\theta}} = \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} 2^{-(\boldsymbol{s},\bar{\boldsymbol{\gamma}})\left(r_1-1+\frac{1}{q}\right)\frac{q\theta}{\theta-q}} \right)^{\frac{1}{q}-\frac{1}{\theta}},$$

where $\bar{\boldsymbol{\gamma}}$ is the vector with coordinates $\bar{\gamma}_j = \frac{r_j-1+1/q}{r_1-1+1/q}$, $j=\overline{1,d}$. It is easy to see that $\bar{\gamma}_j = \gamma_j$ for $j=\overline{1,\nu}$ and $\bar{\gamma}_j \geqslant \gamma_j$ for $j=\overline{\nu+1,d}$. Applying Lemma 10.4 to the last sum, we obtain

$$J_2 \ll 2^{-n\left(r_1-1+\frac{1}{q}\right)} n^{(\nu-1)\left(\frac{1}{q}-\frac{1}{\theta}\right)}.$$

For $1 \leqslant \theta \leqslant q < \infty$, $q \neq 1$, by using the inequality

$$\left(\sum_k |a_k|^{v_2}\right)^{1/v_2} \leqslant \left(\sum_k |a_k|^{v_1}\right)^{1/v_1}, \quad 0 < v_1 \leqslant v_2 < \infty\,,$$

(see [9, p. 43]) and the Hölder inequality and taking into account the fact that $r_1 > 1-\frac{1}{q}$, we can continue the estimate J_2 as follows:

$$J_2 \leqslant \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} \|A_{\boldsymbol{s}}^*(f,\cdot)\|_{L_1(\mathbb{R}^d)}^{\theta} 2^{\|s\|_1\left(1-\frac{1}{q}\right)\theta}\right)^{1/\theta} =$$

$$= \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} \|A_{\boldsymbol{s}}^*(f,\cdot)\|_{L_1(\mathbb{R}^d)}^{\theta} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} 2^{-(\boldsymbol{s},\bar{\boldsymbol{\gamma}})\left(r_1-1+\frac{1}{q}\right)\theta}\right)^{1/\theta} \ll$$

$$\ll \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|A_{\boldsymbol{s}}^*(f,\cdot)\|_{L_1(\mathbb{R}^d)}^{\theta}\right)^{1/\theta} \sup_{(\boldsymbol{s},\boldsymbol{\gamma})>n} 2^{-(\boldsymbol{s},\bar{\boldsymbol{\gamma}})\left(r_1-1+\frac{1}{q}\right)} \leqslant$$

$$\leqslant \|f(\cdot)\|_{S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)} 2^{-n\left(r_1-1+\frac{1}{q}\right)} \leqslant 2^{-n\left(r_1-1+\frac{1}{q}\right)},$$

where, as in the previous case, the vector $\bar{\boldsymbol{\gamma}}$ is defined in a similar way and $\bar{\boldsymbol{\gamma}} \geqslant \boldsymbol{\gamma}$.

Let now $\theta=\infty$. Then, for $f \in S^{\boldsymbol{r}}_{1,\infty}B(\mathbb{R}^d)$, according to the definition of the norm of the function (10.4), we have that

$$\|A_{\boldsymbol{s}}^*(f,\cdot)\|_{L_1(\mathbb{R}^d)} \ll 2^{-(\boldsymbol{s},\boldsymbol{r})}.$$

By using Lemma 10.4, we get

$$J_2 \ll \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} 2^{-(\boldsymbol{s},\boldsymbol{r})q} 2^{\|s\|_1\left(1-\frac{1}{q}\right)q}\right)^{1/q} = \left(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>n} 2^{-(\boldsymbol{s},\bar{\boldsymbol{\gamma}})\left(r_1-1+\frac{1}{q}\right)q}\right)^{1/q} \asymp$$

$$\asymp 2^{-n\left(r_1-1+\frac{1}{q}\right)} n^{\frac{\nu-1}{q}}.$$

The required upper estimates are established.

We now establish the lower estimates. To this end, for some values of the parameters q and θ, it is sufficient to indicate the functions $f \in S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)$ for which the lower estimates of the quantities $\mathcal{E}_{\tilde{Q}_n^\gamma}\big(f\big)_{L_q(\mathbb{R}^d)}$ coincide (in order) with the upper estimates of the quantities $\mathcal{E}_{\tilde{Q}_n^\gamma}\big(S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)}$ in (10.16). Note that, it is sufficient to consider the case $\nu = d$. Thus, we assume that $\gamma_j = 1$, $j = \overline{1, d}$.

Let $1 \leqslant \theta \leqslant q$, $q \neq 1$. Consider a function

$$f_3(\boldsymbol{x}) = 2^{-r_1 n} A^*_{\tilde{\boldsymbol{s}}}(\boldsymbol{x}), \quad \text{where} \quad \|\tilde{\boldsymbol{s}}\|_1 = n+1.$$

Let us show that $f_3 \in S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)$. Then we can write

$$\|f_3(\cdot)\|_{S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)} \asymp \left(\sum_{\boldsymbol{s}\geqslant \boldsymbol{0}} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|A^*_{\boldsymbol{s}}(f_3,\cdot)\|^\theta_{L_1(\mathbb{R}^d)} \right)^{1/\theta} \asymp$$

$$\asymp 2^{-r_1 n} \left(2^{(\tilde{\boldsymbol{s}},\boldsymbol{r})\theta} \|A^*_{\tilde{\boldsymbol{s}}}(\cdot)\|^\theta_{L_1(\mathbb{R}^d)} \right)^{1/\theta} \ll 2^{-r_1 n}\, 2^{r_1 n} = 1.$$

Since the functions $f_3(\boldsymbol{x})$ satisfy $S_{\tilde{Q}_n^\gamma}(f_3, \boldsymbol{x}) = 0$, by virtue of Lemma 10.6, we obtain

$$\mathcal{E}_{\tilde{Q}_n^\gamma}(S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d))_{L_q(\mathbb{R}^d)} \gg \mathcal{E}_{\tilde{Q}_n^\gamma}(f_3)_{L_q(\mathbb{R}^d)} = \|f_3(\cdot) - S_{\tilde{Q}_n^\gamma}(f_3,\cdot)\|_{L_q(\mathbb{R}^d)} =$$

$$= \|f_3(\cdot)\|_{L_q(\mathbb{R}^d)} \asymp 2^{-nr_1} \|A^*_{\tilde{\boldsymbol{s}}}(\cdot)\|_{L_q(\mathbb{R}^d)} \asymp 2^{-nr_1} 2^{n\left(1-\frac{1}{q}\right)} = 2^{-n\left(r_1 - 1 + \frac{1}{q}\right)}.$$

For $1 < q < \theta < \infty$, we consider a function

$$f_4(\boldsymbol{x}) = 2^{-nr_1} n^{-\frac{d-1}{\theta}} \sum_{\|\boldsymbol{s}\|_1 = n+1} A^*_{\boldsymbol{s}}(\boldsymbol{x}).$$

Let us show that $f_4 \in S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)$. Then we can write

$$\|f_4(\cdot)\|_{S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)} \asymp \left(\sum_{\boldsymbol{s}\geqslant \boldsymbol{0}} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|A^*_{\boldsymbol{s}}(f_4,\cdot)\|^\theta_{L_1(\mathbb{R}^d)} \right)^{1/\theta} =$$

$$= 2^{-nr_1} n^{-\frac{d-1}{\theta}} \left(\sum_{\|\boldsymbol{s}\|_1 = n+1} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|A^*_{\boldsymbol{s}}(\cdot)\|^\theta_{L_1(\mathbb{R}^d)} \right)^{1/\theta} \ll$$

$$\ll 2^{-nr_1} n^{-\frac{d-1}{\theta}} \left(\sum_{\|\boldsymbol{s}\|_1 = n+1} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \right)^{1/\theta} \ll 1.$$

Since, in view of the choice of the functions f_4 the equality $S_{\tilde{Q}_n^\gamma}(f_4, \boldsymbol{x}) = 0$ is true, we conclude that

$$\mathcal{E}_{Q_n^\gamma}(S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d))_{L_q(\mathbb{R}^d)} \gg \|f_4(\cdot) - S_{\tilde{Q}_n^\gamma}(f_4,\cdot)\|_{L_q(\mathbb{R}^d)} = \|f_4(\cdot)\|_{L_q(\mathbb{R}^d)}.$$

Since, as shown above, $f_4 \in S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)$ and, by the conditions of the theorem, $r_1 > 1 - \frac{1}{q}$, according to Proposition 10.1, we get $f_4 \in L_q(\mathbb{R}^d)$. For $\boldsymbol{s} \in \mathbb{Z}^d_+$, we now set

$$\Delta(\boldsymbol{s}) = \{\boldsymbol{x}\colon\ 2^{-s_j-1} \leqslant x_j < 2^{-s_j},\ j = \overline{1,d}\},$$

$\Delta(\boldsymbol{s}) \cap \Delta(\boldsymbol{s}') = \varnothing$ for $\boldsymbol{s} \neq \boldsymbol{s}'$. Then, by Proposition 10.3 (Littlewood–Paley theorem) and estimates for $A^*_{\boldsymbol{s}}(\boldsymbol{x})$ (see formula (15) in [33]), we have

$$\|f_4(\cdot)\|_{L_q(\mathbb{R}^d)} \gg$$

$$\gg \left\| \left(\sum_{\|\boldsymbol{s}\|_1=n+1} |\delta^*_{\boldsymbol{s}}(f_4,\cdot)|^2 \right)^{1/2} \right\|_{L_q(\mathbb{R}^d)} \geqslant \left(\sum_{\|\boldsymbol{s}\|_1=n+1} \int\limits_{\Delta(\boldsymbol{s})} |\delta^*_{\boldsymbol{s}}(f_4,\boldsymbol{x})|^q d\boldsymbol{x} \right)^{1/q} \asymp$$

$$\asymp 2^{-nr_1} n^{-\frac{d-1}{\theta}} \left(\sum_{\|\boldsymbol{s}\|_1=n+1} \int\limits_{\Delta(\boldsymbol{s})} |A^*_{\boldsymbol{s}}(\boldsymbol{x})|^q d\boldsymbol{x} \right)^{1/q} \gg 2^{-nr_1} n^{-\frac{d-1}{\theta}} \left(\sum_{\|\boldsymbol{s}\|_1=n+1} 2^{\|\boldsymbol{s}\|_1(q-1)} \right)^{1/q} \asymp$$

$$\asymp 2^{-nr_1} n^{-\frac{d-1}{\theta}} 2^{n\frac{q-1}{q}} n^{(d-1)/q} = 2^{-n(r_1-(1-1/q))} n^{(d-1)\left(\frac{1}{q}-\frac{1}{\theta}\right)}.$$

Finally, we consider the case $\theta = \infty$ and, hence, the function

$$f_5(\boldsymbol{x}) = 2^{-nr_1} \sum_{\|\boldsymbol{s}\|_1=n+1} A^*_{\boldsymbol{s}}(\boldsymbol{x}).$$

It is necessary to show that $f_5 \in S^{\boldsymbol{r}}_{1,\infty}B(\mathbb{R}^d)$. Thus, we get

$$\|f_5(\cdot)\|_{S^{\boldsymbol{r}}_{1,\infty}B(\mathbb{R}^d)} \asymp \sup_{\boldsymbol{s}\geqslant \boldsymbol{0}} 2^{(\boldsymbol{s},\boldsymbol{r})} \|A^*_{\boldsymbol{s}}(f_5,\cdot)\|_{L_1(\mathbb{R}^d)} = 2^{-nr_1} \sup_{\|\boldsymbol{s}\|_1=n+1} 2^{(\boldsymbol{s},\boldsymbol{r})} \|A^*_{\boldsymbol{s}}(\cdot)\|_{L_1(\mathbb{R}^d)} \ll 1.$$

In view of the fact that the functions f_5 satisfy the relation $S_{\tilde{Q}^\gamma_n}(f_5,\boldsymbol{x}) = 0$, as in the previous case, we get

$$\mathcal{E}_{\tilde{Q}^\gamma_n}\big(S^{\boldsymbol{r}}_{1,\theta}B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)} \gg \mathcal{E}_{Q^\gamma_n}(f_5)_{L_q(\mathbb{R}^d)} = \|f_5(\cdot) - S_{\tilde{Q}^\gamma_n}(f_5,\cdot)\|_{L_q(\mathbb{R}^d)} =$$

$$= \|f_5(\cdot)\|_{L_q(\mathbb{R}^d)} \gg \left\| \left(\sum_{\|\boldsymbol{s}\|_1=n+1} |\delta^*_{\boldsymbol{s}}(f_5,\cdot)|^2 \right)^{\frac{1}{2}} \right\|_{L_q(\mathbb{R}^d)} \geqslant$$

$$\geqslant \left(\sum_{\|\boldsymbol{s}\|_1=n+1} \int\limits_{\Delta(\boldsymbol{s})} |\delta^*_{\boldsymbol{s}}(f_5,\boldsymbol{x})|^q d\boldsymbol{x} \right)^{\frac{1}{q}} \asymp 2^{-nr_1} \left(\sum_{\|\boldsymbol{s}\|_1=n+1} \int\limits_{\Delta(\boldsymbol{s})} |A^*_{\boldsymbol{s}}(\boldsymbol{x})|^q d\boldsymbol{x} \right)^{\frac{1}{q}} \gg$$

$$\gg 2^{-nr_1}\left(\sum_{\|s\|_1=n+1} 2^{\|s\|_1(q-1)}\right)^{\frac{1}{q}} \asymp 2^{-nr_1}2^{n\frac{q-1}{q}}n^{\frac{d-1}{q}} = 2^{-n\left(r_1-1+\frac{1}{q}\right)}n^{\frac{d-1}{q}}.$$

The required lower estimates are established.

Theorem 10.6 is proved. □

Note that the methods used to establish the estimates from Theorem 10.6 for $\theta=\infty$ are somewhat differ from the methods used Wang Heping and Sun Youngsheng in Theorem 10.4. Theorem 10.5 is new for the Nikol'skii classes $S_1^{\boldsymbol{r}}H(\mathbb{R}^d)$, i.e., in the case $\theta=\infty$.

Let us formulate another result in which we establish exact order estimates, as in Theorem 10.5, only for the quantity (10.10), but for other values of the parameter p in the metric $L_\infty(\mathbb{R}^d)$.

Theorem 10.7. *Let $1<p<\infty$, $r_1>\frac{1}{p}$, $1\leqslant\theta\leqslant\infty$. Then the ordinal relation holds*

$$\mathcal{E}_{\tilde{Q}_n^\gamma}\left(S_{p,\theta}^{\boldsymbol{r}}B(\mathbb{R}^d)\right)_{L_\infty(\mathbb{R}^d)} \asymp 2^{-n\left(r_1-\frac{1}{p}\right)}n^{(\nu-1)\left(1-\frac{1}{\theta}\right)}.$$

Theorem 10.7 in the one-dimensional case was proved in [29], and for $d\geqslant 2$ in [31].

Note that for $\theta=\infty$, i.e. for Nikol'skii classes $S_p^{\boldsymbol{r}}H(\mathbb{R}^d)$, Theorem 10.7 is also new. Based on the obtained results, we formulate the following corollary.

Corollary 10.1. *Let $1\leqslant p<\infty$, $r_1>\frac{1}{p}$. Then the ordinal relation holds*

$$\mathcal{E}_{\tilde{Q}_n^\gamma}\left(S_p^{\boldsymbol{r}}H(\mathbb{R}^d)\right)_{L_\infty(\mathbb{R}^d)} \asymp 2^{-n\left(r_1-\frac{1}{p}\right)}n^{\nu-1}.$$

Remark 10.2. *Note that the Nikol'skii–Besov classes of periodic functions of one and many variables were studied more extensively. Thus, the order estimates for the approximation of functions from these classes by trigonometric polynomials with numbers of harmonics in the step hyperbolic cross were established in [8, 17, 20, 21]. For the results of more detailed investigations of the Nikol'skii–Besov classes of periodic functions with a dominant mixed smoothness from the point of view of finding of the exact-order estimates for various approximating characteristics, see the monographs V. N. Temlyakov [26], A. S. Romanyuk [18] and D. Dũng, V. N. Temlyakov and T. Ullrich [7].*

10.3 Approximation of functions from $S_{p,\theta}^{\mathbf{r}}B(\mathbb{R}^d)$ by entire functions of a special form

In this section, we study the approximation of functions from the classes $S_{p,\theta}^{\boldsymbol{r}}B(\mathbb{R}^d)$ by entire functions of exponential type with certain restrictions on the support of their Fourier transform. Namely, with the support of their Fourier transform concentrated on sets whose Lebesgue measure is finite and does not exceed M (which we also call as approximation by entire functions of a special form). Let us define the corresponding characteristic of the approximation.

For any $f \in L_q(\mathbb{R}^d)$, $1 \leqslant q \leqslant \infty$, we put

$$e_M^{\mathfrak{F}}(f)_{L_q(\mathbb{R}^d)} := \inf_{\mathcal{L}\colon \operatorname{mes}\mathfrak{M}\leqslant M} \|f(\cdot) - S_{\mathfrak{M}}(f,\cdot)\|_{L_q(\mathbb{R}^d)},$$

where in this case

$$S_{\mathfrak{M}}(f,\boldsymbol{x}) = \sum_{\boldsymbol{s}\in\mathcal{L}} \delta_{\boldsymbol{s}}^*(f,\boldsymbol{x}). \tag{10.17}$$

That is, as the set $Q(\mathcal{L})$ we take the set, which we will denote by $\mathfrak{M}$, namely

$$\mathfrak{M} = \mathfrak{M}(\mathcal{L}) = \bigcup_{\boldsymbol{s}\in\mathcal{L}} Q_{2^{\boldsymbol{s}}}^*,$$

and this set is chosen in such a way that its measure is finite, $\operatorname{mes}\mathfrak{M} \leqslant M$, where $M \in \mathbb{N}$.

Note that $S_{\mathfrak{M}}(f,\boldsymbol{x})$ is an entire function that belongs to the space $L_q(\mathbb{R}^d)$ and $\operatorname{supp} S_{\mathfrak{M}}(f,\boldsymbol{x}) \subseteq \mathfrak{M}$.

For $S_{p,\theta}^{\boldsymbol{r}}B(\mathbb{R}^d) \subset L_q(\mathbb{R}^d)$ we denote

$$e_M^{\mathfrak{F}}\big(S_{p,\theta}^{\boldsymbol{r}}B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)} := \sup_{f\in S_{p,\theta}^{\boldsymbol{r}}B(\mathbb{R}^d)} e_M^{\mathfrak{F}}(f)_{L_q(\mathbb{R}^d)}. \tag{10.18}$$

It follows directly from the definition of approximation characteristics (10.10) and (10.18) that in the case of $\operatorname{mes}\tilde{Q}_n^{\gamma} \asymp \operatorname{mes}\mathfrak{M}$ the following relation holds:

$$e_M^{\mathfrak{F}}\big(S_{p,\theta}^{\boldsymbol{r}}B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)} \ll \mathcal{E}_{\tilde{Q}_n^{\gamma}}\big(S_{p,\theta}^{\boldsymbol{r}}B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)}.$$

The quantity (10.18) is a non-periodic analogue of the best orthogonal trigonometric approximation and the quantity (10.10) corresponds to the approximation by step hyperbolic Fourier sums (see Remark 10.2). In particular, the mentioned approximation characteristics of classes of periodic functions in Lebesgue subspaces are currently being actively studied, see, for example, [16, 22].

Let us first present a result in which estimates of the quantity corresponding to (10.18) are established for isotropic Nikol'skii–Besov classes [32], which we will use in the proof.

Theorem 10.8. *Let* $1 \leqslant p \leqslant q \leqslant \infty$, $(p,q) \neq \{(1,1),(\infty,\infty)\}$, $1 \leqslant \theta \leqslant \infty$. *If* $r > d(\frac{1}{p} - \frac{1}{q})$, *then the order relation*

$$e_M^{\mathfrak{F}}\big(B_{p,\theta}^{r}(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)} \asymp M^{-\frac{r}{d}+\frac{1}{p}-\frac{1}{q}}$$

is true.

The following statements are true.

Theorem 10.9. *Let* $1 < p \leqslant 2$, $1 \leqslant \theta \leqslant \infty$ *and* $r_1 > 0$. *Then the following relation holds:*

$$e_M^{\mathfrak{F}}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_p(\mathbb{R}^d)} \asymp M^{-r_1}(\log^{\nu-1} M)^{\left(r_1+\frac{1}{p}-\frac{1}{\theta}\right)_+}. \tag{10.19}$$

Proof. First, we establish the upper estimate in (10.19). Note that, for $\theta \geqslant p$ this estimate follows from the corresponding estimate of $\mathcal{E}_{\tilde{Q}_n^{\gamma'}}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_p(\mathbb{R}^d)}$, which is given in Theorem 10.3. Therefore, it is sufficient to obtain estimate (10.19) for the case $1 \leqslant \theta < p$. In this case we will use some considerations and methods proposed in the article [19], devoted to the analysis of similar problems in the periodic case.

Let $f \in S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$, $1 \leqslant \theta < 2$. For a given number M, we choose n from the relation $M(n) = M \asymp 2^n n^{\nu-1}$ and set $n_0 = [n + (\nu-1)\log n]$, where $[a]$ is the integer part of the number a.

We now construct a set $\mathcal{L}'$ and, accordingly, an entire function $S_{\mathfrak{M}}(f, \boldsymbol{x})$ with the help of which we will the approximation of a function $f \in S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$. First, we include in $\mathcal{L}'$ the set of vectors $\boldsymbol{s}$ for which $(\boldsymbol{s}, \boldsymbol{\gamma}') < n$. Accordingly, we will include the set $\tilde{Q}_n^{\gamma'} = \bigcup_{(\boldsymbol{s},\boldsymbol{\gamma}')<n} \tilde{Q}^*_{2^{\boldsymbol{s}}}$ in the set $\mathfrak{M}$, and then we will expand it by including an admissible number of sets $\tilde{Q}^*_{2^{\boldsymbol{s}}}$, $l \leqslant (\boldsymbol{s}, \boldsymbol{\gamma}') < l+1$ and $n \leqslant l < n_0$.

Each $l \in \mathbb{N}$, $n \leqslant l < n_0$, is associated with the quantity

$$B_l' = \left(\sum_{l\leqslant(\boldsymbol{s},\boldsymbol{\gamma}')<l+1} 2^{(\boldsymbol{s},\boldsymbol{r})\theta}\|\delta^*_{\boldsymbol{s}}(f,\cdot)\|^\theta_{L_p(\mathbb{R}^d)}\right)^{1/\theta}. \tag{10.20}$$

We set

$$m_l' = [2^n n^{\nu-1} 2^{-l}(B_l')^\theta] + 1.$$

Further, let $a_i'(f, l)$, $i = 1, 2, \ldots$, be the numbers $\|\delta^*_{\boldsymbol{s}}(f,\cdot)\|_{L_p(\mathbb{R}^d)}$ appearing in (10.20) in the order of decreasing. Note that (10.20) yields the relation

$$a_i'(f,l) \ll i^{-\frac{1}{\theta}} 2^{-lr_1} B_l'.$$

In the sum $\sum_{l\leqslant(\boldsymbol{s},\boldsymbol{\gamma}')<l+1} \delta^*_{\boldsymbol{s}}(f,\boldsymbol{x})$ we now choose m_l' blocks $\delta^*_{\boldsymbol{s}}(f,\boldsymbol{x})$ corresponding to the first m_l' numbers $a_i'(f,l)$. In this way, we also specify m_l' sets $\tilde{Q}^*_{2^{\boldsymbol{s}}}$. These sets are generated by m_l' vectors $\boldsymbol{s}$ corresponding to the chosen $\delta^*_{\boldsymbol{s}}(f,\boldsymbol{x})$. If we perform this procedure for each $l \in [n, n_0)$, $l \in \mathbb{N}$, then we get a collection of sets $\tilde{Q}^*_{2^{\boldsymbol{s}}}$ and, in addition, a set of vectors $\boldsymbol{s}$ denoted by $\mathcal{L}_l'$. Thus, we denote $\tilde{Q}_l' = \bigcup_{\boldsymbol{s}\in\mathcal{L}_l'} \tilde{Q}^*_{2^{\boldsymbol{s}}}$. Hence, $\mathfrak{M} = \tilde{Q}_n^{\gamma'} \cup \tilde{Q}_l'$ and the function $S_{\mathfrak{M}}(f,\boldsymbol{x})$ is chosen as follows:

$$S_{\mathfrak{M}}(f,\boldsymbol{x}) = \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')<n} \delta^*_{\boldsymbol{s}}(f,\boldsymbol{x}) + R_1(\boldsymbol{x}),$$

where $R_1(\boldsymbol{x}) = \sum_{\boldsymbol{s}\in\mathcal{L}'_l} \delta^*_{\boldsymbol{s}}(f,\boldsymbol{x})$, i.e., supp $R_1(\boldsymbol{x}) \subseteq \tilde{Q}'_l$.

We now show that mes $\mathfrak{M} \ll M$. By using Lemma 10.4 and taking into account the choice of the numbers m_l, we find

$$\text{mes}\,\mathfrak{M} \ll 2^n n^{\nu-1} + \sum_{l=n}^{n_0} 2^l m'_l \ll$$

$$\ll 2^n n^{\nu-1} + 2^n n^{\nu-1} \sum_{l=n}^{n_0} \sum_{l\leqslant(\boldsymbol{s},\boldsymbol{\gamma}')<l+1} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|\delta^*_{\boldsymbol{s}}(f,\cdot)\|^{\theta}_{L_p(\mathbb{R}^d)} \leqslant$$

$$\leqslant 2^n n^{\nu-1} + 2^n n^{\nu-1} \|f(\cdot)\|^{\theta}_{S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)} \ll 2^n n^{\nu-1} \asymp M.$$

We now directly proceed to the estimation of approximation.

Assume that the function $S_{\mathfrak{M}}(f,\boldsymbol{x})$ is already constructed. We now find the upper estimate of the quantity $\|f(\cdot) - S_{\mathfrak{M}}(f,\cdot)\|_{L_p(\mathbb{R}^d)}$, $f \in S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$.

By $\tilde{\mathcal{L}}'_l$ we denote the set of vectors $\boldsymbol{s}\colon n \leqslant (\boldsymbol{s},\boldsymbol{\gamma}') < n_0$ for which $\delta^*_{\boldsymbol{s}}(f,\boldsymbol{x})$ do not belong to $R_1(\boldsymbol{x})$. Then

$$\|f(\cdot) - S_{\mathfrak{M}}(f,\cdot)\|_{L_p(\mathbb{R}^d)} \leqslant \left\| f(\cdot) - \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')<n_0} \delta^*_{\boldsymbol{s}}(f,\cdot) + \sum_{\boldsymbol{s}\in\tilde{\mathcal{L}}'_l} \delta^*_{\boldsymbol{s}}(f,\cdot) \right\|_{L_p(\mathbb{R}^d)} \leqslant$$

$$\leqslant \left\| f(\cdot) - \sum_{(\boldsymbol{s},\boldsymbol{\gamma}')<n_0} \delta^*_{\boldsymbol{s}}(f,\cdot) \right\|_{L_p(\mathbb{R}^d)} + \left\| \sum_{\boldsymbol{s}\in\tilde{\mathcal{L}}'_l} \delta^*_{\boldsymbol{s}}(f,\cdot) \right\|_{L_p(\mathbb{R}^d)} =: J_3 + J_4. \tag{10.21}$$

By using Theorem 10.3, we get the following estimate for J_3:

$$J_3 \ll 2^{-n_0 r_1} = 2^{-nr_1} n^{-(\nu-1)r_1} \asymp M^{-r_1}. \tag{10.22}$$

To estimate J_4, we first use Theorem 10.3 (the Littlewood–Paley theorem) and then the inequality $|a+b|^c \leqslant |a|^c + |b|^c$, $0 \leqslant c \leqslant 1$. This yields

$$J_4 = \left\| \sum_{\boldsymbol{s}\in\tilde{\mathcal{L}}'_l} \delta^*_{\boldsymbol{s}}(f,\cdot) \right\|_{L_p(\mathbb{R}^d)} \ll \left\| \left(\sum_{\boldsymbol{s}\in\tilde{\mathcal{L}}'_l} |\delta^*_{\boldsymbol{s}}(f,\cdot)|^2 \right)^{\frac{1}{2}} \right\|_{L_p(\mathbb{R}^d)} \leqslant \left(\sum_{\boldsymbol{s}\in\tilde{\mathcal{L}}'_l} \|\delta^*_{\boldsymbol{s}}(f,\cdot)\|^p_{L_p(\mathbb{R}^d)} \right)^{\frac{1}{p}}. \tag{10.23}$$

To continue the estimate (10.23), we will substitute the values of the numbers $a'_i(f,l)$:

$$J_4 \ll \left(\sum_{l=n}^{n_0} \sum_{i>m'_l} (a'_i(f,l))^p \right)^{1/p} = \left(\sum_{l=n}^{n_0} \sum_{i>m'_l} (a'_i(f,l))^{\theta} (a'_i(f,l))^{p-\theta} \right)^{1/p} \ll$$

$$\ll \left(\sum_{l=n}^{n_0}\sum_{i>m'_l}(a'_i(f,l))^{\theta} i^{-\frac{p-\theta}{\theta}} 2^{-l(p-\theta)r_1}(B'_l)^{p-\theta}\right)^{1/p} \leqslant$$

$$\leqslant \left(\sum_{l=n}^{n_0}(m'_l)^{-\frac{p-\theta}{\theta}} 2^{-l(p-\theta)r_1}(B'_l)^{p-\theta}\sum_{i>m'_l}(a'_i(f,l))^{\theta}\right)^{1/p}$$

$$\ll \left(\sum_{l=n}^{n_0}(m'_l)^{-\frac{p-\theta}{\theta}} 2^{-l(p-\theta)r_1}(B'_l)^{p-\theta}\sum_{l\leqslant(\boldsymbol{s},\boldsymbol{\gamma}')<l+1}\|\delta^*_{\boldsymbol{s}}(f,\cdot)\|^{\theta}_{L_p(\mathbb{R}^d)}\right)^{1/p} \ll$$

$$\ll \left(\sum_{l=n}^{n_0}(m'_l)^{-\frac{p-\theta}{\theta}} 2^{-l(p-\theta)r_1}(B'_l)^{p-\theta}2^{-l\theta r_1}\sum_{l\leqslant(\boldsymbol{s},\boldsymbol{\gamma}')<l+1}2^{(\boldsymbol{s},\boldsymbol{r})\theta}\|\delta^*_{\boldsymbol{s}}(f,\cdot)\|^{\theta}_{L_p(\mathbb{R}^d)}\right)^{1/p} \ll$$

$$\ll \left(\sum_{l=n}^{n_0}(m'_l)^{-\frac{p-\theta}{\theta}} 2^{-lr_1(p-\theta)}(B'_l)^{p-\theta}2^{-lr_1\theta}(B'_l)^{\theta}\right)^{\frac{1}{p}} = \left(\sum_{l=n}^{n_0}(m'_l)^{-\frac{p-\theta}{\theta}} 2^{-plr_1}(B'_l)^{p}\right)^{\frac{1}{p}}. \tag{10.24}$$

Further, substituting the values of m'_l in the last sum (10.24), we get

$$J_4 \ll (2^n n^{\nu-1})^{\frac{1}{p}-\frac{1}{\theta}}\left(\sum_{l=n}^{n_0}2^{-pl\left(r_1-\frac{1}{\theta}+\frac{1}{p}\right)}(B'_l)^{\theta}\right)^{1/p}. \tag{10.25}$$

To continue the estimate for J_4, we consider the following two cases:

a) $r_1 \geqslant \frac{1}{\theta}-\frac{1}{p}$;

b) $0 < r_1 < \frac{1}{\theta}-\frac{1}{p}$.

If $r_1 \geqslant \frac{1}{\theta}-\frac{1}{p}$, then it follows from (10.25) that

$$J_4 \ll (2^n n^{\nu-1})^{\left(\frac{1}{p}-\frac{1}{\theta}\right)}2^{-n\left(r_1-\frac{1}{\theta}+\frac{1}{p}\right)}\left(\sum_{l=n}^{n_0}(B'_l)^{\theta}\right)^{1/p} \ll$$
$$\ll (2^n n^{\nu-1})^{\left(\frac{1}{p}-\frac{1}{\theta}\right)}2^{-n\left(r_1-\frac{1}{\theta}+\frac{1}{p}\right)}\|f(\cdot)\|^{\theta/p}_{S^{\boldsymbol{r}}_{p,\theta}(\mathbb{R}^d)} \leqslant$$
$$\leqslant 2^{-nr_1}n^{(\nu-1)\left(\frac{1}{p}-\frac{1}{\theta}\right)} \asymp M^{-r_1}\left(\log^{\nu-1}M\right)^{r_1+\frac{1}{p}-\frac{1}{\theta}}. \tag{10.26}$$

Substituting (10.22) and (10.26) into (10.21), for $r_1 \geqslant \frac{1}{\theta}-\frac{1}{p}$, we obtain

$$\|f(\cdot)-S_{\mathfrak{M}}(f,\cdot)\|_{L_p(\mathbb{R}^d)} \ll M^{-r_1}(\log^{\nu-1}M)^{r_1+\frac{1}{p}-\frac{1}{\theta}}.$$

Now let $0 < r_1 < \frac{1}{\theta} - \frac{1}{p}$. In this case, estimate (10.25) can be continued as follows:

$$J_4 \ll (2^n n^{\nu-1})^{\left(\frac{1}{p}-\frac{1}{\theta}\right)} 2^{-n_0\left(r_1-\frac{1}{\theta}+\frac{1}{p}\right)} \left(\sum_{l=n}^{n_0} (B_l')^{\theta}\right)^{1/p} \ll$$

$$\ll (2^n n^{\nu-1})^{\left(\frac{1}{p}-\frac{1}{\theta}\right)} 2^{-n\left(r_1-\frac{1}{\theta}+\frac{1}{p}\right)} n^{-(\nu-1)\left(r_1-\frac{1}{\theta}+\frac{1}{p}\right)} \|f(\cdot)\|^{\theta/p}_{S^{\boldsymbol{r}}_{p,\theta}(\mathbb{R}^d)} \asymp$$

$$\asymp 2^{-nr_1} n^{-(\nu-1)r_1} \asymp M^{-r_1}. \tag{10.27}$$

Now, substituting estimates (10.22) and (10.27) into (10.21), we obtain

$$\|f(\cdot) - S_{\mathfrak{M}}(f,\cdot)\|_{L_p(\mathbb{R}^d)} \ll M^{-r_1}.$$

Thus, the required upper estimate in Theorem 10.9 is established.

We now establish the lower estimate. It suffices to obtain this estimate for $\nu = d$. To this end, we use the reasoning proposed by V. M. Temlyakov [26, p. 94]. Given number M, we choose n such that $M \asymp 2^n n^{d-1}$ and the number of points with integer-valued coordinates in the set

$$F_n = \bigcup_{(\boldsymbol{s},\boldsymbol{1})=n} \rho^*_+(\boldsymbol{s})$$

is greater than $4M$.

Further, to establish lower bounds, we will consider extreme functions that are built on the basis of some standard function.

We put

$$D_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^{d} D_{k_j}(x_j), \quad \boldsymbol{k} \in \mathbb{Z}^d_+, \tag{10.28}$$

where

$$D_{k_j}(x_j) = \sqrt{\frac{2}{\pi}} \left(2\sin\frac{x_j}{2}\cos\frac{2k_j+1}{2}x_j\right) \cdot x_j^{-1},$$

$$D_{\frac{1}{2}}(x_j) = D_0(x_j) = \sqrt{\frac{2}{\pi}}\,\frac{\sin x_j}{x_j}.$$

In further considerations, of significant importance is the fact that, as indicated in [28], for the Fourier transform of the function $D_{\boldsymbol{k}}(\boldsymbol{x})$, the equality holds:

$$\mathfrak{F} D_{\boldsymbol{k}}(\boldsymbol{x}) = \chi_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^{d} \chi_{k_j}(x_j),$$

where

$$\chi_{k_j}(x_j) = \begin{cases} 1, & k_j < |x_j| < k_j + 1, \\ \frac{1}{2}, & |x_j| = k_j \quad \text{or} \quad |x_j| = k_j + 1, \\ 0, & \text{otherwise,} \end{cases} \quad \chi_0(x_j) = \begin{cases} 1, & |x_j| < 1, \\ \dfrac{1}{2}, & |x_j| = 1, \\ 0, & |x_j| > 1. \end{cases}$$

Hence, for the inverse transform, we obtain $\mathfrak{F}^{-1}\chi_{\boldsymbol{k}}(\boldsymbol{t}) = D_{\boldsymbol{k}}(\boldsymbol{x})$.

Note that for $1 < p < \infty$ the following estimate holds [28]:

$$\left\| \sum_{\boldsymbol{k} \in \rho_+^*(\boldsymbol{s})} D_{\boldsymbol{k}}(\cdot) \right\|_{L_p(\mathbb{R}^d)} \asymp 2^{\|\boldsymbol{s}\|_1 (1-1/p)}, \tag{10.29}$$

where

$$\rho_+^*(\boldsymbol{s}) := \{\boldsymbol{k} = (k_1, \ldots, k_d) \colon \ \eta(s_j) 2^{s_j - 1} \leqslant k_j < 2^{s_j}, k_j \in \mathbb{Z}_+, \ j = \overline{1, d}\},$$

$\eta(0) = 0$ and $\eta(t) = 1$, $t > 0$, that is, $\rho_+^*(\boldsymbol{s}) = Q_{2^{\boldsymbol{s}}}^* \cap \mathbb{Z}_+^d$.

Depending on the value of the parameter θ, we will consider functions constructed on the basis of function (10.28), namely

$$f_6(\boldsymbol{x}) = C_{10} 2^{-n\left(r_1 + 1 - \frac{1}{p}\right)} n^{-\frac{d-1}{\theta}} \sum_{\boldsymbol{k} \in F_n} D_{\boldsymbol{k}}(\boldsymbol{x}), \quad C_{10} > 0,$$

for $1 \leqslant \theta < \infty$ and

$$f_7(x) = C_{11} 2^{-n\left(r_1 + 1 - \frac{1}{p}\right)} \sum_{\boldsymbol{k} \in F_n} D_{\boldsymbol{k}}(\boldsymbol{x}), \quad C_{11} > 0,$$

for $\theta = \infty$.

Let us show that the functions f_6 and f_7 belong to the class $S_{p,\theta}^{\boldsymbol{r}} B(\mathbb{R}^d)$ and $S_{p,\infty}^{\boldsymbol{r}} B(\mathbb{R}^d)$ respectively. For the function f_6 , the following estimates hold:

$$\|f_6(\cdot)\|_{S_{p,\theta}^{\boldsymbol{r}} B(\mathbb{R}^d)} = \left(\sum_{(\boldsymbol{s},\mathbf{1})=n} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|\delta_{\boldsymbol{s}}^*(f_6, \cdot)\|_{L_p(\mathbb{R}^d)}^\theta \right)^{\frac{1}{\theta}} \ll$$

$$\ll 2^{nr_1} 2^{-n(r_1+1-1/p)} n^{-\frac{d-1}{\theta}} \left(\sum_{(\boldsymbol{s},\mathbf{1})=n} \|\delta_{\boldsymbol{s}}^*(F, \cdot)\|_{L_p(\mathbb{R}^d)}^\theta \right)^{\frac{1}{\theta}} =: J_5,$$

where

$$F(\boldsymbol{x}) = \sum_{(\boldsymbol{s},\mathbf{1})=n} \sum_{\boldsymbol{k} \in \rho_+^*(\boldsymbol{s})} D_{\boldsymbol{k}}(\boldsymbol{x}).$$

For J_5, using (10.29), we can continue the estimate as follows:

$$J_5 \asymp 2^{-n(1-1/p)} n^{-\frac{d-1}{\theta}} \left(\sum_{(\boldsymbol{s},\boldsymbol{1})=n} 2^{\theta\|\boldsymbol{s}\|_1\left(1-\frac{1}{p}\right)} \right)^{\frac{1}{\theta}} \asymp 2^{-n\left(1-\frac{1}{p}\right)} n^{-\frac{d-1}{\theta}} 2^{n\left(1-\frac{1}{p}\right)} n^{\frac{d-1}{\theta}} = 1.$$

We make a conclusion that for an appropriate choice of the constant $C_{10} > 0$, the function $f_6 \in S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$.

For f_7 we will have:

$$\|f_7(\cdot)\|_{S^{\boldsymbol{r}}_{p,\infty}B(\mathbb{R}^d)} = \sup_{(\boldsymbol{s},\boldsymbol{1})=n} 2^{(\boldsymbol{s},\boldsymbol{r})} \|\delta^*_{\boldsymbol{s}}(f_7,\cdot)\|_{L_p(\mathbb{R}^d)} \ll$$

$$\ll 2^{-n\left(r_1+1-\frac{1}{p}\right)} \sup_{(\boldsymbol{s},\boldsymbol{1})=n} 2^{(\boldsymbol{s},\boldsymbol{r})} \|\delta^*_{\boldsymbol{s}}(F,\cdot)\|_{L_p(\mathbb{R}^d)} \asymp 2^{nr_1} 2^{-n\left(r_1+1-\frac{1}{p}\right)} 2^{n\left(1-\frac{1}{p}\right)} = 1.$$

Therefore $f_7 \in S^{\boldsymbol{r}}_{p,\infty}B(\mathbb{R}^d)$ with some constant $C_{11} > 0$.

Further, let $\mathcal{L}$ be an arbitrary set of vectors $\boldsymbol{s} = (s_1, \ldots, s_d)$ such that for the set

$$\mathfrak{M} = \bigcup\nolimits_{\boldsymbol{s}\in\mathcal{L}} \tilde{Q}^*_{2^{\boldsymbol{s}}}$$

the relation $\operatorname{mes}\mathfrak{M} \leqslant M$ holds. We consider set $\mathcal{L}' = \{\boldsymbol{s} \in \mathbb{Z}^d_+ : \ \boldsymbol{k} \in F_n, \ (\boldsymbol{s},\boldsymbol{1}) = n\}$. According to the choice of the number n, for the number of elements of the set $\mathcal{L}' \setminus \mathcal{L}$ we will have the relation $|\mathcal{L}' \setminus \mathcal{L}| \asymp n^{d-1}$.

Let

$$S_{\mathfrak{M}}(f_6, x) = \sum_{\boldsymbol{s}\in\mathcal{L}} \delta^*_{\boldsymbol{s}}(f_6, \boldsymbol{x}).$$

Consider the case when $p = 2$. Then, according to Proposition 10.3 for f_6, in the case $r_1 > \max\left\{0, \frac{1}{\theta} - \frac{1}{2}\right\}$, we obtain

$$e^{\mathfrak{F}}_M\left(S^{\boldsymbol{r}}_{2,\theta}B(\mathbb{R}^d)\right)_{L_2(\mathbb{R}^d)} \geqslant e^{\mathfrak{F}}_M\left(f_6\right)_{L_2(\mathbb{R}^d)} = \inf_{\mathcal{L}:\ \operatorname{mes}\mathfrak{M}\leqslant M} \|f_6(\cdot) - S_{\mathfrak{M}}(f_6,\cdot)\|_{L_2(\mathbb{R}^d)} \geqslant$$

$$\geqslant \left(\sum_{\boldsymbol{s}} \|\delta^*_{\boldsymbol{s}}\left(f_6(\cdot) - S_{\mathfrak{M}}(f_6,\cdot),\cdot\right)\|^2_{L_2(\mathbb{R}^d)} \right)^{1/2} =$$

$$= 2^{-n\left(r_1+\frac{1}{2}\right)} n^{-\frac{d-1}{\theta}} \left(\sum_{\boldsymbol{s}\in\mathcal{L}'\setminus\mathcal{L}} \|\delta^*_{\boldsymbol{s}}(f_6,\cdot)\|^2_{L_2(\mathbb{R}^d)} \right)^{\frac{1}{2}} \asymp 2^{-n(r_1+\frac{1}{2})} n^{-\frac{d-1}{\theta}} 2^{\frac{n}{2}} \left(\sum_{\boldsymbol{s}\in\mathcal{L}'\setminus\mathcal{L}} 1 \right)^{\frac{1}{2}} \asymp$$

$$\asymp 2^{-nr_1} n^{-\frac{d-1}{\theta}} n^{\frac{d-1}{2}} \asymp M^{-r_1}\left(\log^{d-1} M\right)^{r_1+\frac{1}{2}-\frac{1}{\theta}}.$$

Similarly, for f_7 at $p = 2$ we get

$$e^{\mathfrak{F}}_M\left(S^{\boldsymbol{r}}_{2,\infty}B(\mathbb{R}^d)\right)_{L_2(\mathbb{R}^d)} \geqslant \ e^{\mathfrak{F}}_M\left(f_7\right)_{L_2(\mathbb{R}^d)} =$$

$$= \inf_{\mathcal{L}:\ \mathrm{mes}\mathfrak{M}\leqslant M} \|f_7(\cdot)-S_{\mathfrak{M}}(f_7,\cdot)\|_{L_2(\mathbb{R}^d)} \geqslant \left(\sum_{\boldsymbol{s}} \|\delta^*_{\boldsymbol{s}}\big(f_7(\cdot)-S_{\mathfrak{M}}(f_7,\cdot),\cdot\big)\|^2_{L_2(\mathbb{R}^d)}\right)^{1/2} =$$

$$= 2^{-n\left(r_1+\frac{1}{2}\right)}\left(\sum_{\boldsymbol{s}\in\mathcal{L}'\setminus\mathcal{L}} \|\delta^*_{\boldsymbol{s}}(f_7,\cdot)\|^2_{L_2(\mathbb{R}^d)}\right)^{1/2} \asymp 2^{-n(r_1+\frac{1}{2})}2^{\frac{n}{2}}\left(\sum_{\boldsymbol{s}\in\mathcal{L}'\setminus\mathcal{L}} 1\right)^{1/2} \asymp$$

$$\asymp 2^{-nr_1} n^{\frac{d-1}{2}} \asymp M^{-r_1}\big(\log^{d-1} M\big)^{r_1+\frac{1}{2}}.$$

When $0 < r_1 < \frac{1}{\theta} - \frac{1}{2}$, the lower estimate in (10.19), for the corresponding values of the parameters, follows from Theorem 10.8 in the one-dimensional case ($d = 1$). In the case $1 < p < 2$, we similarly obtain the necessary estimates for both f_6 and f_7 using Lemma 10.3. The lower estimates in (10.19) are established.

Theorem 10.9 is proved. □

After analyzing and comparing Theorem 10.9 with the corresponding result of Theorem 10.3, we can make the following conclusion:

• In the cases $1 < p \leqslant \theta \leqslant 2$ and $1 < p \leqslant 2$, $2 < \theta \leqslant \infty$, the estimates of the quantities $e^{\mathfrak{F}}_M\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_p(\mathbb{R}^d)}$ (10.18) and $\mathcal{E}_{\tilde{Q}^{\gamma'}_n}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_p(\mathbb{R}^d)}$ (10.10) coincide in order for $M \asymp 2^n n^{\nu-1}$;

• In the case $1 \leqslant \theta < p \leqslant 2$, the estimates of $e^{\mathfrak{F}}_M\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_p(\mathbb{R}^d)}$ and $\mathcal{E}_{\tilde{Q}^{\gamma'}_n}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_p(\mathbb{R}^d)}$ differ in order, namely:

– if $0 < r_1 < \frac{1}{\theta} - \frac{1}{p}$, then the relation holds

$$e^{\mathfrak{F}}_M\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_p(\mathbb{R}^d)} \asymp \mathcal{E}_{\tilde{Q}^{\gamma'}_n}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_2(\mathbb{R}^d)} n^{-(\nu-1)r_1}, \quad M \asymp 2^n n^{\nu-1};$$

– if $r_1 \geqslant \frac{1}{\theta} - \frac{1}{p}$, then the relation holds

$$e^{\mathfrak{F}}_M\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_p(\mathbb{R}^d)} \asymp \mathcal{E}_{\tilde{Q}^{\gamma'}_n}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_p(\mathbb{R}^d)} n^{-(\nu-1)\left(\frac{1}{\theta}-\frac{1}{p}\right)}, \quad M \asymp 2^n n^{\nu-1}.$$

Theorem 10.10. *Let $1 < p < q < \infty$ and $r_1 > \frac{1}{p} - \frac{1}{q}$. Then the order relation*

$$e^{\mathfrak{F}}_M\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)} \asymp M^{-\left(r_1-\frac{1}{p}+\frac{1}{q}\right)}\big(\log^{\nu-1} M\big)^{\left(r_1-\frac{1}{p}+\frac{2}{q}-\frac{1}{\theta}\right)_+} \tag{10.30}$$

is true for $1 \leqslant \theta \leqslant \infty$.

Proof. First, we establish the upper estimate in (10.30). Note that, for $\theta \geqslant q$, this estimate follows from the corresponding estimate for the approximation of the classes $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$ by entire functions $S_{Q^{\gamma}_n}(f,\boldsymbol{x})$, which is given in Theorem 10.1, under the condition that the numbers n are chosen from the relation $M \asymp 2^n n^{\nu-1}$.

We now establish the upper estimate for $1 \leqslant \theta \leqslant q$. In this case, we use the same reasoning as in the proof of the upper estimate in Theorem 10.9. Thus, we only indicate on the differences in the proof.

For $f \in S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$, we choose a function used for approximation as follows:

$$S_{\mathfrak{M}}(f, \boldsymbol{x}) = \sum_{(\boldsymbol{s},\boldsymbol{\gamma})<n} \delta^*_{\boldsymbol{s}}(f, \boldsymbol{x}) + R_2(\boldsymbol{x}),$$

where $R_2(\boldsymbol{x})$ and, hence, the set on which the support of its Fourier transform is concentrated are constructed by analogy with R_1 and $\widetilde{Q}'_l$ in Theorem 10.9 with the only difference that, instead of the quantity B'_l, we use the quantity

$$B_l = \left(\sum_{l\leqslant(\boldsymbol{s},\boldsymbol{\gamma})<l+1} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|\delta^*_{\boldsymbol{s}}(f,\cdot)\|^{\theta}_{L_p(\mathbb{R}^d)} \right)^{\frac{1}{\theta}},$$

i.e., $\mathfrak{M} = \tilde{Q}^{\gamma}_n \cup \tilde{Q}_l$, $R_2(\boldsymbol{x}) = \sum_{\boldsymbol{s}\in\mathcal{L}_l} \delta^*_{\boldsymbol{s}}(f, \boldsymbol{x})$ and supp $R_2(\boldsymbol{x}) \subseteq \tilde{Q}_l$.

Assume that the function $S_{\mathfrak{M}}(f, \boldsymbol{x})$ is already constructed. We now find the upper estimate of the quantity $\|f(\cdot) - S_{\mathfrak{M}}(f, \cdot)\|_{L_q(\mathbb{R}^d)}$, $f \in S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$. By $\tilde{\mathcal{L}}_l$ we denote the set of all vectors $\boldsymbol{s}\colon n \leqslant (\boldsymbol{s}, \boldsymbol{\gamma}) < n_0$ for which $\delta^*_{\boldsymbol{s}}(f, \boldsymbol{x})$ do not belong to $R_2(\boldsymbol{x})$. Then for $f \in S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$, we obtain

$$\|f(\cdot) - S_{\mathfrak{M}}(f, \cdot)\|_{L_q(\mathbb{R}^d)} = \left\| f(\cdot) - \sum_{(\boldsymbol{s},\boldsymbol{\gamma})<n_0} \delta^*_{\boldsymbol{s}}(f, \cdot) + \sum_{\boldsymbol{s}\in\tilde{\mathcal{L}}_l} \delta^*_{\boldsymbol{s}}(f, \cdot) \right\|_{L_q(\mathbb{R}^d)} \leqslant$$

$$\leqslant \left\| f(\cdot) - \sum_{(\boldsymbol{s},\boldsymbol{\gamma})<n_0} \delta^*_{\boldsymbol{s}}(f, \cdot) \right\|_{L_q(\mathbb{R}^d)} + \left\| \sum_{\boldsymbol{s}\in\tilde{\mathcal{L}}_l} \delta^*_{\boldsymbol{s}}(f, \cdot) \right\|_{L_q(\mathbb{R}^d)} =: J_6 + J_7. \qquad (10.31)$$

According to Theorem 10.1 and the choice of the number n_0, we can write

$$J_6 \ll 2^{-n_0\left(r_1 - \frac{1}{p} + \frac{1}{q}\right)} \leqslant 2^{-n\left(r_1 - \frac{1}{p} + \frac{1}{q}\right)} n^{-(\nu-1)\left(r_1 - \frac{1}{p} + \frac{1}{q}\right)} \asymp M^{-\left(r_1 - \frac{1}{p} + \frac{1}{q}\right)}. \qquad (10.32)$$

For the estimate J_7, using Lemma 10.2 and reasoning similarly to the estimate of the quantity J_4 (see (10.24) and (10.25)), we obtain

$$J_7 \ll (2^n n^{\nu-1})^{\frac{1}{q} - \frac{1}{\theta}} \left(\sum_{l=n}^{n_0} 2^{-lq\left(r_1 - \frac{1}{p} + \frac{2}{q} - \frac{1}{\theta}\right)} B^{\theta}_l \right)^{\frac{1}{q}} = (2^n n^{\nu-1})^{\frac{1}{q} - \frac{1}{\theta}} J_8. \qquad (10.33)$$

To estimate J_8, we consider the following two cases:

a) $r_1 \geqslant \frac{1}{p} - \frac{2}{q} + \frac{1}{\theta}$;

b) $\frac{1}{p} - \frac{1}{q} < r_1 < \frac{1}{p} - \frac{2}{q} + \frac{1}{\theta}$.

If $r_1 \geqslant \frac{1}{p} - \frac{2}{q} + \frac{1}{\theta}$, then

$$J_8 \leqslant 2^{-n\left(r_1 - \frac{1}{p} + \frac{2}{q} - \frac{1}{\theta}\right)} \left(\sum_{l=n}^{n_0} B_l^\theta\right)^{\frac{1}{q}} =$$

$$= 2^{-n\left(r_1 - \frac{1}{p} + \frac{2}{q} - \frac{1}{\theta}\right)} \left(\sum_{l=n}^{n_0} \sum_{l \leqslant (\boldsymbol{s}, \boldsymbol{\gamma}) < l+1} 2^{(\boldsymbol{s}, \boldsymbol{r})\theta} \|\delta_{\boldsymbol{s}}^*(f, \cdot)\|_{L_p(\mathbb{R}^d)}^\theta\right)^{\frac{1}{q}} \ll$$

$$\ll 2^{-n\left(r_1 - \frac{1}{p} + \frac{2}{q} - \frac{1}{\theta}\right)} \|f\|_{S_{p,\theta}^{\boldsymbol{r}} B(\mathbb{R}^d)}^{\theta/q} \leqslant 2^{-n\left(r_1 - \frac{1}{p} + \frac{2}{q} - \frac{1}{\theta}\right)}.$$

Comparing (10.33) and(10.3), we find

$$J_7 \ll M^{-\left(r_1 - \frac{1}{p} + \frac{1}{q}\right)} \left(\log^{\nu - 1} M\right)^{\left(r_1 - \frac{1}{p} + \frac{2}{q} - \frac{1}{\theta}\right)}. \tag{10.34}$$

Let $\frac{1}{p} - \frac{1}{q} < r_1 < \frac{1}{p} - \frac{2}{q} + \frac{1}{\theta}$. In this case,

$$J_8 \leqslant 2^{-n_0\left(r_1 - \frac{1}{p} + \frac{2}{q} - \frac{1}{\theta}\right)} \left(\sum_{l=n}^{n_0} B_l^\theta\right)^{\frac{1}{q}} =$$

$$= 2^{-n_0\left(r_1 - \frac{1}{p} + \frac{2}{q} - \frac{1}{\theta}\right)} \left(\sum_{l=n}^{n_0} \sum_{l \leqslant (\boldsymbol{s}, \boldsymbol{\gamma}) < l+1} 2^{(\boldsymbol{s}, \boldsymbol{r})\theta} \|\delta_{\boldsymbol{s}}^*(f, \cdot)\|_{L_p(\mathbb{R}^d)}^\theta\right)^{\frac{1}{q}} \ll$$

$$\ll 2^{-n_0\left(r_1 - \frac{1}{p} + \frac{2}{q} - \frac{1}{\theta}\right)} \|f\|_{S_{p,\theta}^{\boldsymbol{r}} B(\mathbb{R}^d)}^{\theta/q} \leqslant 2^{-n\left(r_1 - \frac{1}{p} + \frac{2}{q} - \frac{1}{\theta}\right)} n^{-(\nu-1)\left(r_1 - \frac{1}{p} + \frac{2}{q} - \frac{1}{\theta}\right)}. \tag{10.35}$$

By using (10.33) and (10.35), we conclude that

$$J_7 \ll M^{-\left(r_1 - \frac{1}{p} + \frac{1}{q}\right)}. \tag{10.36}$$

Finally, substituting, in turn, estimates (10.32) and (10.34) and then (10.32) and (10.36) in (10.31), we establish the estimate for $e_M^{\mathfrak{F}}\left(S_{p,\theta}^{\boldsymbol{r}} B(\mathbb{R}^d)\right)_{L_q(\mathbb{R}^d)}$ in the case where $1 \leqslant \theta < q$. The upper estimate in Theorem 10.10 is obtained.

We now establish the lower estimate. It suffices to obtain this estimate for $\nu = d$. To do this, based on the function $D_{\boldsymbol{k}}(\boldsymbol{x})$, which is defined according to formula (10.28), we construct extremal functions $f \in S_{p,\theta}^{\boldsymbol{r}} B(\mathbb{R}^d)$ that realize it.

We use the following well-known relation (see, e.g., [5, § 2.6]):

Let $f \in L_q(\mathbb{R}^d)$ and $g \in L_{q'}(\mathbb{R}^d)$. Then with $\frac{1}{q} + \frac{1}{q'} = 1$ one has

$$\|f(\cdot)\|_{L_q(\mathbb{R}^d)} = \sup_{\|g(\cdot)\|_{L_{q'}(\mathbb{R}^d)} \leqslant 1} \int_{\mathbb{R}^d} |f(\boldsymbol{x})\, g(\boldsymbol{x})| d\boldsymbol{x}.$$

Let $f \in S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$ and let $S_{\mathfrak{M}}(f,\boldsymbol{x})$ is an entire function of the form (10.17), such that the support of its Fourier transform is contained in the set $\mathfrak{M} = \bigcup_{\boldsymbol{s}\in\mathcal{L}} Q^*_{2^{\boldsymbol{s}}}$, mes $\mathfrak{M} \leqslant M$. Then, according to the relations presented above, we can write

$$\|f(\cdot) - S_{\mathfrak{M}}(f,\cdot)\|_{L_q(\mathbb{R}^d)} = \sup_{\|g(\cdot)\|_{L_{q'}(\mathbb{R}^d)}\leqslant 1} \int_{\mathbb{R}^d} \left|\big(f(\boldsymbol{x}) - S_{\mathfrak{M}}(f,\boldsymbol{x})\big)g(\boldsymbol{x})\right| d\boldsymbol{x}. \quad (10.37)$$

Further, let M and $l = l(M) \in \mathbb{N}$ be numbers satisfying $2^l l^{d-1} \asymp M$ and $2^l l^{d-1} \geqslant 2M$. Consider a function

$$F(\boldsymbol{x}) = \sum_{(\boldsymbol{s},\boldsymbol{1})\leqslant l} \sum_{\boldsymbol{k}\in\rho^*_+(\boldsymbol{s})} D_{\boldsymbol{k}}(\boldsymbol{x}).$$

By using the estimate (10.29) for $1 \leqslant \theta < \infty$, we get

$$\|F\|_{S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)} = \left(\sum_{(\boldsymbol{s},\boldsymbol{1})\leqslant l} 2^{(\boldsymbol{s},\boldsymbol{r})\theta}\|\delta^*_{\boldsymbol{s}}(F,\cdot)\|^\theta_{L_p(\mathbb{R}^d)}\right)^{\frac{1}{\theta}} =$$

$$= \left(\sum_{(\boldsymbol{s},\boldsymbol{1})\leqslant l} 2^{(\boldsymbol{s},\boldsymbol{r})\theta}\left\|\sum_{\boldsymbol{k}\in\rho^*_+(\boldsymbol{s})} D_{\boldsymbol{k}}(\cdot)\right\|^\theta_{L_p(\mathbb{R}^d)}\right)^{\frac{1}{\theta}} \asymp \left(\sum_{(\boldsymbol{s},\boldsymbol{1})\leqslant l} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} 2^{(\boldsymbol{s},\boldsymbol{1})\left(1-\frac{1}{p}\right)\theta}\right)^{\frac{1}{\theta}} =$$

$$= \left(\sum_{(\boldsymbol{s},\boldsymbol{1})\leqslant l} 2^{(\boldsymbol{s},\boldsymbol{1})\left(r_1+1-\frac{1}{p}\right)\theta}\right)^{\frac{1}{\theta}} \ll 2^{l\left(r_1+1-\frac{1}{p}\right)} l^{\frac{d-1}{\theta}}.$$

Hence,

$$f_8(\boldsymbol{x}) = C_{12} 2^{-l\left(r_1+1-\frac{1}{p}\right)} l^{-\frac{d-1}{\theta}} F(\boldsymbol{x}) \quad (10.38)$$

belongs to the class $S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)$ with a constant $C_{12} > 0$.

If $\theta = \infty$, then, by using (10.29), we find

$$\|F\|_{S^{\boldsymbol{r}}_{p,\infty}B(\mathbb{R}^d)} = \sup_{(\boldsymbol{s},\boldsymbol{1})\leqslant l} 2^{(\boldsymbol{s},\boldsymbol{r})}\|\delta^*_{\boldsymbol{s}}(F,\cdot)\|_{L_p(\mathbb{R}^d)} \ll 2^{l\left(r_1+1-\frac{1}{p}\right)}.$$

Thus, the function

$$f_9(\boldsymbol{x}) = C_{13} 2^{-l\left(r_1+1-\frac{1}{p}\right)} F(\boldsymbol{x}) \quad (10.39)$$

belongs to the class $S^{\boldsymbol{r}}_{p,\infty}B(\mathbb{R}^d)$ with a constant $C_{13} > 0$.

We now show that the function

$$g(\boldsymbol{x}) = C_{14} 2^{-\frac{l}{q}} l^{-\frac{d-1}{q'}} F(\boldsymbol{x}), \quad (10.40)$$

satisfies the condition $\|g\|_{L_{q'}(\mathbb{R}^d)} \leqslant 1$ with a constant $C_{14} > 0$.

To this end, it is necessary to show that

$$\|F\|_{L_{q'}(\mathbb{R}^d)} \ll 2^{\frac{l}{q}} l^{\frac{d-1}{q'}}, \quad 1 < q' < \infty. \tag{10.41}$$

We consider two possible cases.

Then, by virtue of Theorem B and the inequality

Let $1 < q' \leqslant 2$. Then, by virtue of Proposition 10.3 and the inequality $|a+b|^\alpha \leqslant |a|^\alpha + |b|^\alpha$, $0 \leqslant \alpha \leqslant 1$, we can write

$$\|F\|_{L_{q'}(\mathbb{R}^d)} \ll \left\| \left(\sum_{(\boldsymbol{s},\boldsymbol{1})\leqslant l} |\delta_{\boldsymbol{s}}^*(F,\cdot)|^2 \right)^{1/2} \right\|_{L_{q'}(\mathbb{R}^d)} = \left(\sum_{(\boldsymbol{s},\boldsymbol{1})\leqslant l} \|\delta_{\boldsymbol{s}}^*(F,\cdot)\|_{L_{q'}(\mathbb{R}^d)}^{q'} \right)^{\frac{1}{q'}}. \tag{10.42}$$

By using (10.29), we continue estimate (10.42) as follows:

$$\|F\|_{L_{q'}(\mathbb{R}^d)} \ll \left(\sum_{(\boldsymbol{s},\boldsymbol{1})\leqslant l} 2^{(\boldsymbol{s},\boldsymbol{1})\left(1-\frac{1}{q'}\right)q'} \right)^{\frac{1}{q'}} \ll 2^{\frac{l}{q}} \left(\sum_{(\boldsymbol{s},\boldsymbol{1})\leqslant l} 1 \right)^{\frac{1}{q'}} \ll 2^{\frac{l}{q}} l^{\frac{d-1}{q'}}.$$

If $2 < q' < \infty$, then, by using Lemma 10.2 and relation (10.29), we find

$$\|F\|_{L_{q'}(\mathbb{R}^d)} \ll \left(\sum_{(\boldsymbol{s},\boldsymbol{1})\leqslant l} \|\delta_{\boldsymbol{s}}^*(F,\cdot)\|_{L_q(\mathbb{R}^d)}^{q'} 2^{\|\boldsymbol{s}\|_1\left(\frac{1}{q}-\frac{1}{q'}\right)q'} \right)^{\frac{1}{q'}} \ll$$

$$\ll \left(\sum_{(\boldsymbol{s},\boldsymbol{1})\leqslant l} 2^{\|\boldsymbol{s}\|_1\left(1-\frac{1}{q}\right)q'} 2^{\|\boldsymbol{s}\|_1\left(\frac{1}{q}-\frac{1}{q'}\right)q'} \right)^{\frac{1}{q'}} \leqslant 2^{\frac{l}{q}} \left(\sum_{(s,1)\leqslant l} 1 \right)^{\frac{1}{q'}} \ll 2^{\frac{l}{q}} l^{\frac{d-1}{q'}}.$$

Let $1 \leqslant \theta < \infty$. Thus, substituting (10.38) and (10.40) in (10.37) and taking into account the fact that $M \asymp 2^l l^{d-1}$ and $\|g(\cdot)\|_{L_{q'}(\mathbb{R}^d)} \leqslant 1$, we conclude that

$$e_M^{\mathfrak{F}}\big(S_{p,\theta}^{\boldsymbol{r}} B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)} \geqslant e_M^{\mathfrak{F}}(f_8)_{L_q(\mathbb{R}^d)} = \inf_{\mathcal{L}:\ \mathrm{mes}\,\mathfrak{M}\leqslant M} \|f_8(\cdot) - S_{\mathfrak{M}}(f_8,\cdot)\|_{L_q(\mathbb{R}^d)} \geqslant$$

$$\geqslant \inf_{\mathcal{L}:\ \mathrm{mes}\,\mathfrak{M}\leqslant M} \int\limits_{\mathbb{R}^d} \Big| \Big(f_8(\boldsymbol{x}) - S_{\mathfrak{M}}(f_8,\boldsymbol{x}) \Big) g(\boldsymbol{x}) \Big| d\boldsymbol{x} \geqslant$$

$$\geqslant \inf_{\mathcal{L}:\ \mathrm{mes}\,\mathfrak{M}\leqslant M} \int_{\mathbb{R}^d} \left| \left(2^{-l\left(r_1+1-\frac{1}{p}\right)} l^{-\frac{d-1}{\theta}} F(\boldsymbol{x}) - S_{\mathfrak{M}}(f_8,\boldsymbol{x}) \right) 2^{-\frac{l}{q}} l^{-\frac{d-1}{q'}} F(\boldsymbol{x}) \right| d\boldsymbol{x} \gg$$

$$\gg 2^{-l\left(r_1+1-\frac{1}{p}+\frac{1}{q}\right)} l^{-(d-1)\left(\frac{1}{\theta}+\frac{1}{q'}\right)} \inf_{\mathcal{L}:\ \mathrm{mes}\,\mathfrak{M}\leqslant M} \left(\int\limits_{\mathbb{R}^d} |F(\boldsymbol{x})|^2 d\boldsymbol{x} - \int_{\mathbb{R}^d} |S_{\mathfrak{M}}(F,\boldsymbol{x}) F(\boldsymbol{x})| d\boldsymbol{x} \right). \tag{10.43}$$

We now estimate each integral in relation (10.43) separately.

According to (10.41), for $q = q' = 2$ we obtain

$$\int\limits_{\mathbb{R}^d} |F(\boldsymbol{x})|^2 d\boldsymbol{x} = \|F(\cdot)\|^2_{L_2(\mathbb{R}^d)} \asymp 2^l l^{d-1}.$$

To estimate the second integral, we introduce additional notation. By $\mathcal{L}_l$ we denote a set of vectors $\boldsymbol{s}$ such that $(\boldsymbol{s}, \mathbf{1}) \leqslant l$ and, correspondingly, $\tilde{\mathcal{L}}$ denote the set of vectors $\{\boldsymbol{s} \colon \boldsymbol{s} \in \mathcal{L} \cap \mathcal{L}_l\}$. Let $\tilde{\mathfrak{M}} = \bigcup_{\boldsymbol{s} \in \tilde{\mathcal{L}}} Q^*_{2^{\boldsymbol{s}}}$. Then we can write $\tilde{\mathfrak{M}} \subseteq \mathfrak{M}$ and $\operatorname{mes} \tilde{\mathfrak{M}} \leqslant M$.

Thus, in view of the fact that the product $S_{\mathfrak{M}}(F, x) \cdot F(x)$ differ from zero only in the set $\tilde{\mathfrak{M}}$, we can write

$$\int\limits_{\mathbb{R}^d} |S_{\mathfrak{M}}(F, \boldsymbol{x}) F(\boldsymbol{x})| d\boldsymbol{x} = \int\limits_{\mathbb{R}^d} |S_{\tilde{\mathfrak{M}}}(F, \boldsymbol{x})|^2 d\boldsymbol{x} = \|S_{\tilde{\mathfrak{M}}}(F, \cdot)\|^2_{L_2(\mathbb{R}^d)} \leqslant$$

$$\leqslant \left\| \left(\sum_{\boldsymbol{s} \in \tilde{\mathcal{L}}} |\delta^*_{\boldsymbol{s}}(F, \cdot)|^2 \right)^{1/2} \right\|^2_{L_2(\mathbb{R}^d)} = \sum_{\boldsymbol{s} \in \tilde{\mathcal{L}}} \|\delta^*_{\boldsymbol{s}}(F, \cdot)\|^2_{L_2(\mathbb{R}^d)} \ll \sum_{\boldsymbol{s} \in \tilde{\mathcal{L}}} \left(2^{\|\boldsymbol{s}\|_1 (1 - \frac{1}{2})} \right)^2 =$$

$$= \operatorname{mes} \tilde{\mathfrak{M}} \leqslant M.$$

Hence, estimate (10.43) can be continued as follows:

$$e^{\mathfrak{F}}_M \big(S^{\boldsymbol{r}}_{p,\theta} B(\mathbb{R}^d) \big)_{L_q(\mathbb{R}^d)} \gg 2^{-l \left(r_1 + 1 - \frac{1}{p} + \frac{1}{q} \right)} l^{-(d-1) \left(\frac{1}{\theta} + \frac{1}{q'} \right)} \big(2^l l^{d-1} - M \big) \gg$$
$$\gg 2^{-l \left(r_1 + 1 - \frac{1}{p} + \frac{1}{q} \right)} l^{-(d-1) \left(\frac{1}{\theta} + \frac{1}{q'} \right)} 2^l l^{d-1} \asymp M^{-\left(r_1 - \frac{1}{p} + \frac{1}{q} \right)} \big(\log^{d-1} M \big)^{r_1 - \frac{1}{p} + \frac{2}{q} - \frac{1}{\theta}}. \tag{10.44}$$

Now let $\theta = \infty$. Substituting functions (10.39) and (10.40) in equality (10.37), we get

$$e^{\mathfrak{F}}_M \big(S^{\boldsymbol{r}}_{p,\infty} B(\mathbb{R}^d) \big)_{L_q(\mathbb{R}^d)} \geqslant e^{\mathfrak{F}}_M \big(f_9 \big)_{L_q(\mathbb{R}^d)} = \inf_{\mathcal{L} \colon \operatorname{mes} \mathfrak{M} \leqslant M} \|f_9(\cdot) - S_{\mathfrak{M}}(f_9, \cdot)\|_{L_q(\mathbb{R}^d)} \gg$$
$$\gg M^{-\left(r_1 - \frac{1}{p} + \frac{1}{q} \right)} \big(\log^{d-1} M \big)^{r_1 - \frac{1}{p} + \frac{2}{q}}. \tag{10.45}$$

Note that the obtained estimates (10.44) and (10.45) coincide (in order) with the upper estimates in the case where $r_1 \geqslant \frac{1}{p} - \frac{1}{q} + \frac{1}{\theta}$.

To get the required lower estimate, it remains to consider the case where $\frac{1}{p} - \frac{1}{q} < r_1 < \frac{1}{p} - \frac{2}{q} + \frac{1}{\theta}$. Since, under the imposed conditions, the upper estimate of the quantity $e^{\mathfrak{F}}_M \big(S^{\boldsymbol{r}}_{p,\theta} B(\mathbb{R}^d) \big)_{L_q(\mathbb{R}^d)}$ is independent of the dimension of the space $\mathbb{R}^d$, it suffices to establish this estimate for $d = 1$. However, in the one-dimensional case, estimate (10.30) coincides (in order) with the corresponding upper estimate with Theorem 10.8.

Theorem10.10 is proved. □

If we compare the result of Theorem 10.10 with the corresponding result of Theorem 10.1, we can make the following conclusion:

• In the case $1 < p < q < \infty$, $q \leqslant \theta \leqslant \infty$ and $r_1 > \frac{1}{p} - \frac{1}{q}$ the estimates of the quantities $e_M^{\mathfrak{F}}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)}$ and $\mathcal{E}_{\tilde{Q}_n^\gamma}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)}$ coincide in order for $M \asymp 2^n n^{\nu-1}$;

• In the case $1 \leqslant \theta < q < \infty$ the estimates of $e_M^{\mathfrak{F}}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)}$ and $\mathcal{E}_{\tilde{Q}_n^\gamma}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)}$ differ in order, namely:

– if $\frac{1}{p} - \frac{1}{q} < r_1 < \frac{1}{p} - \frac{2}{q} + \frac{1}{\theta}$, then the relation holds

$$e_M^{\mathfrak{F}}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)} \asymp \mathcal{E}_{\tilde{Q}_n^\gamma}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)} n^{-(\nu-1)\left(r_1 - \frac{1}{p} + \frac{1}{q}\right)},$$
$$M \asymp 2^n n^{\nu-1};$$

– if $r_1 \geqslant \frac{1}{p} - \frac{2}{q} + \frac{1}{\theta}$, then the relation holds

$$e_M^{\mathfrak{F}}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)} \asymp \mathcal{E}_{\tilde{Q}_n^\gamma}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_q(\mathbb{R}^d)} n^{-(\nu-1)\left(\frac{1}{\theta} - \frac{1}{q}\right)},$$
$$M \asymp 2^n n^{\nu-1}.$$

As we can see from the above comparisons, the corresponding estimates of quantity (10.10) (Theorems 10.9 and 10.10) and quantity (10.18) (Theorems 10.1, 10.3) on the classes $S^{\boldsymbol{r}}_p H(\mathbb{R}^d)$ coincide in order for all values of the parameter $\boldsymbol{r}$.

In what follows, we formulate the another obtained theorem in the following form.

Theorem 10.11. *Let $1 \leqslant p < \infty$, $1 \leqslant \theta \leqslant \infty$, $r_1 > \frac{1}{p}$ and $d \geqslant 1$. Then the following relation holds:*

$$e_M^{\mathfrak{F}}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_\infty(\mathbb{R}^d)} \asymp \big(M^{-1}\log^{\nu-1} M\big)^{r_1 - \frac{1}{p}}\big(\log^{\nu-1} M\big)^{1-\frac{1}{\theta}}. \tag{10.46}$$

In the case $1 < p < \infty$, the estimate of Theorem 10.11 was established in [31], and for $p = 1$ in [30].

Remark 10.3. *Comparing the ordinal estimates of $\mathcal{E}_{\tilde{Q}_n^\gamma}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_\infty(\mathbb{R}^d)}$, $1 \leqslant p < \infty$, $d \geqslant 1$ obtained in Theorems 10.5 and 10.7 with the estimates of $e_M^{\mathfrak{F}}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_\infty(\mathbb{R}^d)}$ from Theorem 10.11, we conclude that $\mathcal{E}_{\tilde{Q}_n^\gamma}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_\infty(\mathbb{R}^d)} \asymp e_M^{\mathfrak{F}}\big(S^{\boldsymbol{r}}_{p,\theta}B(\mathbb{R}^d)\big)_{L_\infty(\mathbb{R}^d)}$, $M \asymp 2^n n^{\nu-1}$.*

The results of Theorems 10.9, 10.10 and 10.11 are also new for Nikol'skii classes $S^{\boldsymbol{r}}_p H(\mathbb{R}^d)$. As you can see, the estimates (10.19), (10.30) and (10.46) for $d = 1$ does not depend on the parameter θ unlike to the corresponding estimate in the case $d \geqslant 2$.

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RESUME

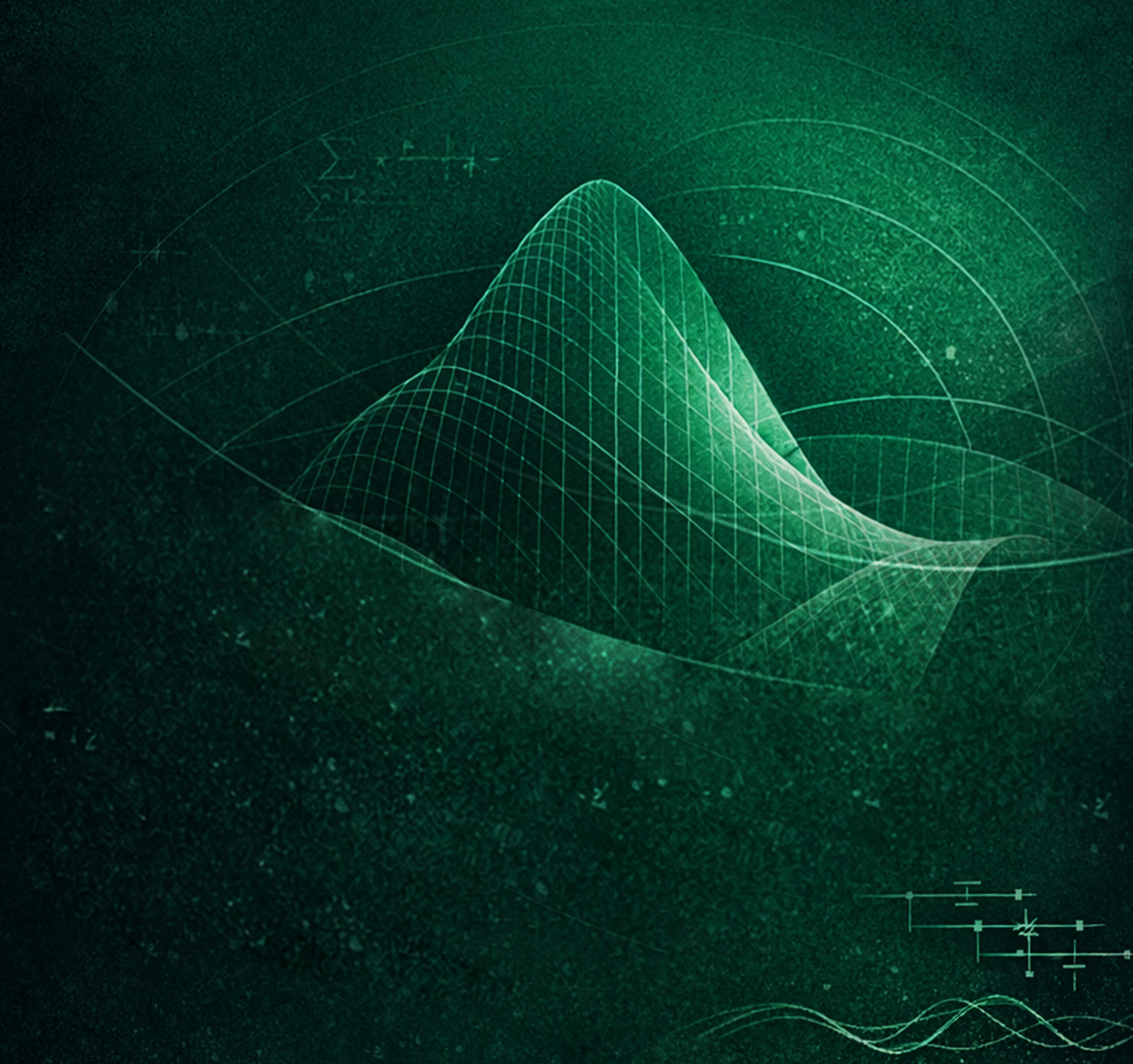
Roman Dmytryshyn is a Professor of Department of Mathematical and Functional Analysis at the Vasyl Stefanyk Carpathian National University, where he coordinates the Master's program in Mathematics. He is an expert on accreditation of educational programs of the National Agency for Higher Education Quality Assurance and an expert of the National Research Foundation of Ukraine.

Roman Dmytryshyn has a significant experience as scientific secretary of the Specialized Academic Council with the right to accept for consideration and defense the dissertations for the degree of Doctor of Physical and Mathematical Sciences and chairman of the Specialized Academic Councils with the right to accept for consideration and conduct a one-time defense of dissertations for PhD degree in Mathematics.

He has participated in different national and international research projects and was the organizer of various scientific events, including the ICMU Winter School "Construction Approximation and its Applications.

Roman Dmytryshyn is the founder and director of the Charitable Foundation "Ivano-Frankivsk Mathematical Society".

He is the author of numerous scientific works and his research interests include approximation theory, continued fractions and their generalizations, special functions, and numerical analysis.



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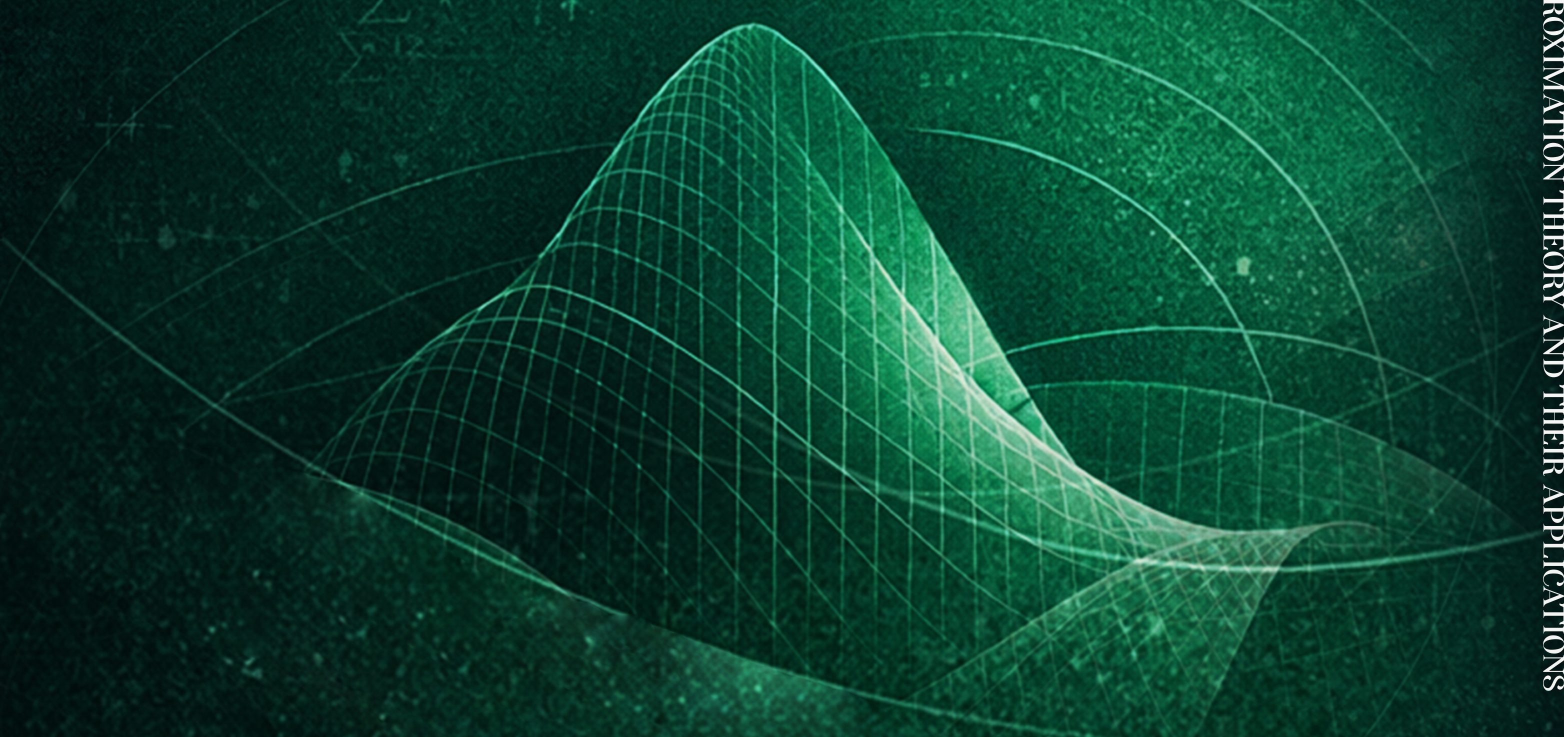

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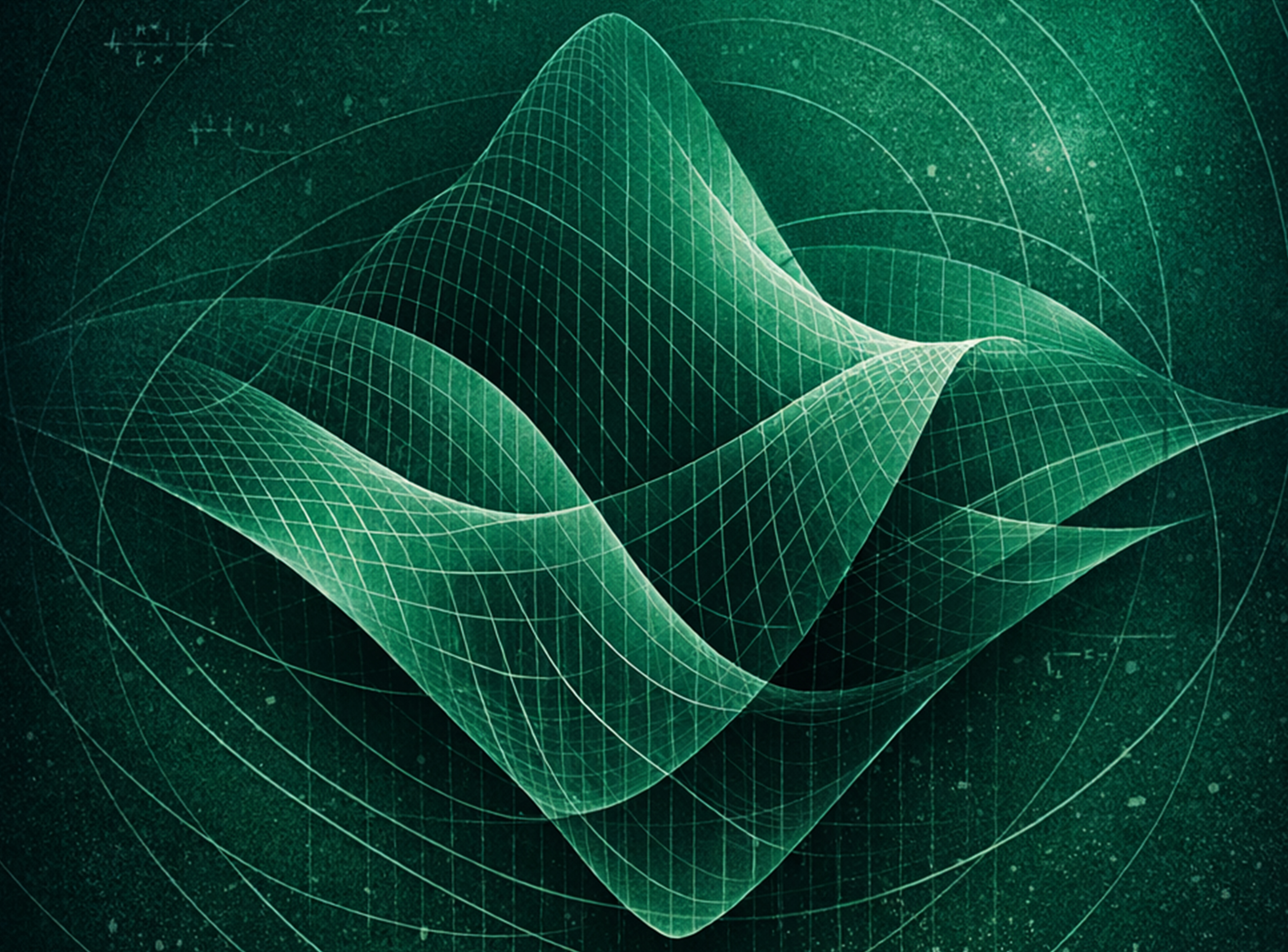

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