RECENT TRENDS IN FRACTIONAL INEQUALITIES AND COMPLEX INEQUALITIES AND THEIR APPLICATIONS

PROF. DR. PRAVEEN AGARWAL
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CONTENTS

On the domains of the existence of multidimensional lacunary Hartogs series with Ostrovsky lacunae

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Abstract. This paper is devoted to multidimensional analogues of Ostrovsky's theorem on lacunary series. The work examines the domains of existence of Hartogs lacunary series with Ostrovsky lacunae and series in homogeneous polynomials. Analogues of Ostrovsky's theorem for such series are given and the domains of convergence of these series are described.

Key words: plurisubharmonic function, singular point, nowhere dense set, power series, lacunary Hartogs series, series in homogeneous polynomials.

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Introduction

The work is devoted to the study of domain of convergence of lacunary series. The main problem discussed in this article is the following question: *what can be said about the features of a power series if the characteristics of the lacunae are known?*

This problem attracted the attention of many mathematicians at the end of the 19th century, such as Hadamard [1], Fabry [2], Faber [3], Mandelbroit [4], G. Polya [5], Ostrovsky [6], etc. In the twenties In the 20th century, remarkable results were obtained that allowed the development of this direction to be considered almost complete. Almost all of these results are associated with the name of the outstanding Hungarian mathematician G. Pólya [5]. All classical works on lacunar series are given in detail in the monograph by L. Bieberbach [7]. Multidimensional analogues of lacunary series have been studied in [8-11] and others.

In the first part of the article, in the one-dimensional case, it is clarified under what conditions the disc of convergence of a power series is a natural Weierstass domain of existence for the sum of the series and under what conditions the domain of existence of a lacunary series will be wider than its disc of convergence. The second part of the work is devoted to a multidimensional analogue of Ostrovsky's theorem on lacunary series. It studies the domain of convergence of multidimensional lacunary Hartogs series. An analogue of Ostrovsky's theorem for Hartogs series is given and the domains of convergence of these series are described. In the last part of the work, an analogue of Ostrovsky's theorem for series in homogeneous polynomials is proved.

1. The domain of existence of one-dimensional lacunary series

Let the following power series be given on the plane

$$f(z)=\sum_{n=0}^{\infty} c_n z^n, \qquad \overline{\lim_{n\to\infty}} |c_n|^{\frac{1}{n}}=1. \tag{1}$$

Then it is clear that, according to the Cauchy-Hadamard formula, the radius of convergence of this series is 1 and this series converges inside the unit disc. Moreover, on the boundary ∂U of the unit disc there lies at least one singular point of the function $f(z)$.

An interesting case is *when each boundary point of the unit disc is singular for* $f(z)$*, i.e. is the unit disc* U *a natural Weierstrass domain of existence for the function* $f(z)$*?*

For a power series

$$f(z)=\sum_{k=0}^{\infty} z^k$$

radius of convergence $R=1$ and at the boundary $\partial U=\{z:|z|=1\}$ of the disc of convergence there is only one singular point $z=1$, i.e. the disc U is not a natural Weierstrass domain of existence for this function. And for a row

$$g(z)=\sum_{k=0}^{\infty} z^{2^k}$$

each boundary point of the disc of convergence is singular (see [12]), and, therefore, the disc of convergence $U=\{z:|z|<1\}$ is a natural Weierstrass domain of existence for the function $g(z)$.

The second series differs from the first in that it is lacunar, with large intervals of zero terms.

Definition 1. *Series (1) is called lacunary if there is a strictly increasing sequence of natural numbers* $\{n_k\}$ *such that*

$$c_n=0 \text{ if } n\notin\{n_k\}.$$

Let the following lacunary series be given on the plane $\mathbb{C}$

$$f(z)=\sum_{k=0}^{\infty} c_{n_k} z^{n_k}, \quad \overline{\lim_{k\to\infty}} |c_{n_k}|^{\frac{1}{n_k}}=1. \tag{2}$$

The following Hadamard theorem on lacunae holds [1]: *if for the lacunary series (2) the condition is satisfied*

$$\frac{n_{k+1}-n_k}{n_k}>\theta_0, \quad k=0,1,2,..., \tag{3}$$

θ_0 *is some positive number independent of* k, *then* $f(z)$ *has a circle* $|z|=1$ *by its natural boundary, i.e.* $f(z)$ *is single-valued and holomorphic in* $\{|z|<1\}$ *and each point of the circle* $|z|=1$ *is a singular point for the function* $f(z)$. *Thus, the natural domain of Weierstrass existence is* $W_f(z)=\{|z|<1\}$.

Hadamard's theorem on lacunae was strengthened a little later by Fabry [2] (see also [7]): let the condition be satisfied for series (2)

$$\overline{\lim_{k\to\infty}} \frac{k}{n_k}=0. \tag{4}$$

Then $f(z)$ has circle $|z|=1$ as its natural boundary.

Remark. Fabry's theorem on lacunae is a far-reaching generalization of Hadamard's theorem on lacunae, i.e. Fabry condition for lacunae $\overline{\lim_{k\to\infty}} \frac{k}{n_k}=0$ is significantly weaker than the Hadamard condition $\underline{\lim}_{k\to\infty} \frac{n_{k+1},-n_k}{n_k}>0$.

◀ Indeed, if the sequence $\{n_k\}$ satisfies the Hadamard condition (3): $\frac{n_{k+1}-n_k}{n_k}>\theta_0$, $k=0,1,2,...,$, where θ_0 is some positive number independent of k, then the inequality holds

$$n_{k+1}>(1+\theta_0)n_k \tag{5}$$

Using inequality (5) successively k times we obtain the inequality $n_k>(1+\theta_0)^k$ and, therefore, according to Newton's binomial $n_k>(1+\theta_0)^k>ck^2$, where c is some constant. Then

$$0<\frac{k}{n_k}<\frac{k}{ck^2}=\frac{1}{ck}$$

and hence we have $0\le\lim\limits_{k\to\infty}\frac{k}{n_k}\le\lim\limits_{k\to\infty}\frac{1}{ck}=0$, i.e. on the properties of number sequences $\lim\limits_{k\to\infty}\frac{k}{n_k}=0$. ▶

Example 1. Let $n_k=k^2$. For this sequence, we check the fulfillment of the Fabry and Hadamard conditions:

$$\overline{\lim_{k\to\infty}}\frac{k}{n_k}=\overline{\lim_{k\to\infty}}\frac{k}{k^2}=\overline{\lim_{k\to\infty}}\frac{1}{k}=0,$$

But

$$\lim_{k\to\infty}\frac{n_{k+1}-n_k}{n_k}=\lim_{k\to\infty}\frac{(k+1)^2-k^2}{k^2}=\lim_{k\to\infty}\frac{2k+1}{k^2}=0.$$

The Fabry condition is satisfied, but the Hadamard condition is not satisfied. ▶

This example once again proves that the Fabry condition is weaker than the Hadamard condition.

It follows the next question: is it possible to replace condition (4) with a weaker condition, say

$$\overline{\lim_{k\to\infty}}(n_{k+1}-n_k)=\infty \tag{6}$$

or

$$\underline{\lim_{k\to\infty}}\frac{k}{n_k}=0, \tag{7}$$

so that the conclusion about the non-continuity of series (2) through the unit disc still remains valid?

Fabry [2] and Faber [3] constructed examples showing that conditions (6) and (7) no longer lead to the desired result. Fabry for condition (6), and Faber for condition (7) constructed series that satisfy these conditions and have circle $|z|=1$ only one singular point. Mandelbroit [4] constructed an example of a series that satisfies condition (6) and has only one singular point in the entire extended plane. Let's take Faber's example.

Example 2. Let a power series be given

$$f(z)=\sum_{k=1}^{\infty}\left(\frac{z+z^2}{2}\right)^{n_k}=\sum_{n=1}^{\infty}c_n z^n \quad . \tag{8}$$

Here $\{n_k\}$ is some increasing sequence of natural numbers. This series converges uniformly inside the lemniscate

$$|z(z+1)|=2 \tag{9}$$

and under the condition $k=o(n_k)$ this lemniscate, by Fabry's theorem on lacunae, is a natural boundary for it. Lemniscate (9) contains the disc $|z|<1$ inside itself and touches it at the point $z=1$.

Let's take $\{n_k\}=\{k!\}$. Then, $\frac{n_{k+1}}{n_k}\to\infty$ and the power series

$$f(z)=\sum_{k=0}^{\infty}\left(\frac{z+z^2}{2}\right)^{n_k}=\sum_{k=0}^{\infty}\frac{1}{2^{n_k}}\left[z^{n_k}+n_k z^{n_k+1}+\ldots+z^{2n_k}\right].$$

Since, $2n_k<n_{k+1}$, $k>1$, then this series is a lacunary series Ostrovsky (see definition 2) $f(z)=\sum_{n=1}^{\infty}c_n z^n$, with convergence radius $R=1$ with lacunae $c_n=0\ \in[2n_k+1,n_{k+1}-1]$. In addition, it is easy to calculate that

$$\underline{\lim}_{k\to\infty}\frac{k}{n_k}=0,$$

But

$$\overline{\lim}_{k\to\infty}\frac{k}{n_k}=\frac{1}{2}. \blacktriangleright$$

This issue was brought to full clarity by Pólya [5], who proved the following statement (Pólya's theorem on lacunae): *if condition (7) is satisfied for a lacunary series (2), then this series defines a single-valued holomorphic function with a univalent domain of existence, i.e. its natural domain of existence is* $W_f\subset\mathbb{C}$.

An analogue result holds for power series with Ostrovsky lakunae.

Definition 2. *A power series (1) is said to have Ostrovsky lacunae if there exist two strictly increasing sequences* $\{n_k\}$ *and* $\{n'_k\}$ *natural numbers such that*

$$c_n = 0 \ \ if \ \ n_k < n \le n'_k \quad (k=0,1,2,...).$$

and

$$\lim_{k\to\infty} \frac{n_k}{n'_k} = 0. \qquad (10)$$

Example 3. If $\{n_k\}=\{k^k\}$ and $\{n'_k\}=\{k^{k+1}\}$, then these sequences satisfy condition (10) and series (1) with lacunae on the intervals $(n_k, n'_k]$, $(k=2,3,4,...)$ will be Ostrovsky's lacunary series.

Ostrovsky's lacunar series can always be written in the form $f(z) = \sum_{k=0}^{\infty} c_{n_k} z^{n_k}$ assuming $n_{k+1} = \max\left\{ j > n_k : , c_j = 0 \right\} + 1$. Then, $c_n = 0$, for $n_k < n < n_{k+1}$ $(k\ ,=0,1,2,...)$, $n_{k+1} \ge n'_k$ and $\underline{\lim}_{k\to\infty} \frac{n_k}{n_{k+1}} = 0$.

Note that Pólya's lacunar series is Ostrovsky's lacunar series, but the converse is not true: Ostrovsky's condition $\underline{\lim}_{k\to\infty} \frac{n_k}{n_{k+1}} = 0$ is much weaker than the Polya condition.

For a power series (1) with Ostrovsky lacunae, the following Ostrovsky theorem holds [6]: *let series (1) have Ostrovsky lacunae. Then the lacunary power series (1) defines a single-valued holomorphic function with a simply connected domain of existence* $W_f \subset \mathbb{C}$.

In [5], Pólya obtained the following inversion of Pólya’s theorem on lacunae: *let the inequality (2) be satisfied for the lacunary series*

$$\underline{\lim}_{k\to\infty} \frac{k}{n_k} > 0,$$

Then it is always possible to select the coefficients c_{n_k} *in such a way that series (2) gives, upon analytical continuation, a multi-valued analytic function.*

Many scientific works are devoted to the study of singular points of the lacunary series (2). They are presented in detail in the monograph by L. Bieberbach [7].

2. Multidimensional lacunary Hartogs series with Ostrovsky lacunae

It is known that many problems of multidimensional complex analysis are solved using Hartogs series and Jacobi-Hartogs series (see, for example, [13-19], etc.). In this regard, it is of interest to study the domain of convergence of such series. In this work we will study Hartogs series with Ostrovsky lacunae.

The domains of convergence and multidimensional analogues of the Fabry and Pólya theorems for Hartogs series and series in homogeneous polynomials were studied in [8-11] and others. This article is devoted to the study of the domains of existence of Hartogs series with Ostrovsky lacunae.
Consider the Hartogs series

$$f(z,w)=\sum_{n=0}^{\infty} c_n(z)w^n \tag{11}$$

with holomorphic coefficients $c_n(z)\in\mathcal{O}(D)$, $n=0,1,2,...$, where $D\subset\mathbb{C}^n$ is some domain. In this case, we will require the natural condition that the radius of convergence $R(z)$ of series (11) is positive for each fixed point $z\in D$.

Let's start with the following result, which is a multidimensional analogue of Ostrovsky's theorem on lacunae.

Theorem 1. *Let series (11) satisfy the following conditions:*

1) converges uniformly inside the domain $D\times\{|w|<r\}\subset\mathbb{C}_z^n\times\mathbb{C}_w$, $r>0$;

2) has Ostrovsky lacunae.

Then series (11) defines a unique holomorphic function $f(z,w)$ *in the domain* W^0, *where* W^0 *is the open kernel of the set* $W=\bigcup_{z\in D} W_f(z)$ *and* $W_f(z)$ *is the maximum domain into which* $f(z,w)$ *extends for a fixed* $z\in D$.

To prove this theorem we need the following statement.

Theorem 2 [16]. *Consider a holomorphic one in the domain* $D\times\{|w|<r\}\subset\mathbb{C}_z^n\times\mathbb{C}_w$, $r>0$, *function* $f(z,w)$ *such that for every fixed* $z\in D$ *a function* $f(z,w)$ *that is holomorphic in the disk* $|w|<r$, *is unique in* $\mathbb{C}_w$, *i.e.* $W_f(z)\subset\mathbb{C}_w$.

Let $W=\bigcup_{z\in D} W_f(z)$. *Then*

1) the open kernel W^0 *is a domain (i.e., it is connected), and the set* $P=np_D\left(W\setminus W^0\right)$ *is pluripolar in* D;

2) the function $f(z,w)$ *extends holomorphically into* W^0;

Proof of Theorem 1. From condition 1) of Theorem 1 it follows that $f(z,w)\in\mathcal{O}\left(D\times\{|w|<r\}\right)\subset\mathbb{C}_z^n\times\mathbb{C}_w$, $r>0$. We fix an arbitrary point $z\in D$. Then series (11) turns into an ordinary one-dimensional power lacunary series and by condition 2) according to the classical Ostrovsky theorem on lacunae we obtain that for each fixed $z \in D$ the function $f(z,w)$ is holomorphic in the disk $|w| < r$, uniquely continues to $\mathbb{C}_w$, i.e. $W_f(z) \subset \mathbb{C}_w$.

According to Theorem 2, the open kernel W^0 of the set $W = \bigcup_{z \in G} W_f(z)$ is domain and the function $f(z,w)$ extends holomorphically into W^0 and this completes the proof of Theorem 1.►

Using Theorems 1 and 2 we obtain the following multidimensional analogue of Ostrovsky's theorem for a lacunary Hartogs series with a variable radius of convergence (the main result of the work).

Theorem 3. *Let series (11) satisfy the following conditions:*

1) $c_n(z) \in \mathcal{O}(D)$, $n = 0,1,2,,...$

2) for each fixed $z \in D$ *the series converges in the disk* $|w| < R(z)$, $R(z) > r > 0$;

3) has Ostrovsky lacunae, i.e. there are two strictly increasing sequences $\{n_k\}$ *and* $\{n'_k\}$ *of natural numbers such that*

$$c_n = 0 \text{ if } n_k < n \le n'_k \quad (k=0,1,2,...).$$

and

$$\lim_{k\to\infty} \frac{n_k}{n'_k} = 0.$$

Then there exists a closed nowhere dense set $S \subset D$ *such that series (11) defines a unique holomorphic function* $f(z,w)$ *in the domain* $W^0 \setminus (S \times \mathbb{C})$, *where* W^0 *is the open core of the set* $W = \bigcup_{z \in D} W_f(z)$.

To prove this theorem we need the following statement.

An analogue of Hartogs' lemma on the upper limit [19]. *Let a sequence of plurisubharmonic functions* $u_k(z)$ *be given locally uniformly bounded above in the domain* $D \subset \mathbb{C}^n$ *and a real-valued function* $A(z) \in C(D)$ *such that at each fixed point* $z \in D$ *the inequality*

$$\overline{\lim_{k\to\infty}}\, u_k(z) \le A(z). \tag{12}$$

Then for any compact set $K \subset\subset D$ and any number $\varepsilon > 0$ there is a number $k_0 \in N$ such that

$$u_k(z) \leq A(z) + \varepsilon \tag{13}$$

for all $z \in K$ and $k > k_0$.

Proof of Theorem 3.

1^0. From the conditions of the theorem it follows that series (11) for each fixed $z^0 \in D$ defines the germ $\sum_{n=0}^{\infty} c_n\left(z^0\right) w^n$ in point $w=0$, i.e. the radius of convergence of this series $R(z^0) > 0$. Let's consider the sets

$$E_m = \left\{ z \in D : \left|c_n(z)\right|^{\frac{1}{n}} \leq m, \quad n = 1, 2, \ldots \right\}, m = 1, 2, \ldots .$$

By construction and from the holomorphy of the coefficients $c_n(z)$ in D it follows that these sets are closed, $E_1 \subset E_2 \subset \ldots$ and since $R(z^0) > 0$ for all $z^0 \in D$, then $D = \bigcup_{m=1}^{\infty} \mathrm{E}_m$. According to Baire's theorem on categories [20], the domain D cannot be represented as a union of nowhere dense sets. Therefore, there exists $m \in N$ such that E_m has an interior point z^0, i.e. there is a ball $B(z^0, \varepsilon)$ such that $B(z^0, \varepsilon) \subset E_m$. By the definition of E_m, for all $z \in B(z^0, \varepsilon)$ the inequalities are satisfied

$$\frac{1}{n} \ln\left|c_n\left(z\right)\right| \leq \ln m, \; n = 1, , 2, \ldots .$$

Let us now denote by D_1 the set of all points $z^0 \in D$ such that for $z^0 \in D_1$ there exists a ball $B(z^0, \varepsilon)$ and a number $M(z^0) \in N$, for which the inequalities $\frac{1}{n} \ln\left|c_n\left(z\right)\right| \leq \ln M(z^0), z \in B(z^0, \varepsilon) \; (n = 1, 2, \ldots)$, i.e. in some neighborhood of the point z^0 all functions $\frac{1}{n} \ln\left|c_n\left(z\right)\right|$are bounded from above by some constant $M(z^0)$. Then the set $S = D \backslash D_1$ is closed and nowhere dense in D .

Indeed, if not so, then S contains some neighborhood $U : U \subset S$. Then, by what was proved above, there exists a ball $B \subset U$ such that in this ball the functions $\frac{1}{n}\ln\left|c_n(z)\right|$ are uniformly bounded from above, i.e. for some M inequality

$$\frac{1}{n}\ln\left|c_n(z)\right|\le\ln M$$

holds for all $z\in B$ and $k=1,2,\ldots$. The resulting contradiction proves that the set S is nowhere dense.

2^0. Let now

$$R(z)=\frac{1}{\overline{\lim\limits_{n\to\infty}}\left|c_n(z)\right|^{\frac{1}{n}}}$$

radius of convergence of series (11). Then

$$-\ln R(z)=\overline{\lim_{n\to\infty}}\frac{1}{n}\ln\left|c_n(z)\right|$$

and due to the plurisubharmonicity and local boundedness from above in D_1 of the functions $\frac{1}{n}\ln\left|c_n(z)\right|$ we obtain that the function $-\ln R_*(z)$ will be plurisubharmonic in D_1, where $R_*(z)=\underline{\lim}_{\xi\to z}R(\xi)$ is the lower regularization of the radius-function $R(z)$. $R_*(z)\le R(z)$ and outside some pluripolar set $\subset D$ the equality $R_*(z)=R(z)$ [21].

Note that in D_1 the function $R(z)$ is locally uniformly bounded from zero; this fact easily follows from the fact that in D_1 the functions $\frac{1}{n}\ln\left|c_n(z)\right|$ are locally uniformly bounded from above. It follows that $R_*(z)$ is also locally delimited from zero. Let us prove the uniform convergence of series (11) inside the domain $\left\{(z,w):z\in D\backslash S,\left|w\right|<R_*(z)\right\}$. To do this, we take an arbitrary compact $K\subset\subset D_1$. From the definition of $R(z)$ for any fixed $z\in D_1$ correct relation

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\ln\left|c_n(z)\right|=-\ln R(z)\le-\ln R_*(z).$$

Since $-\ln R_*(z)\in psh(D_1)$, then for an arbitrary domain G satisfying the condition $K\subset G\subset\subset D_1$, there exists monotonically decreasing sequence

$v_j(z) \in psh(G) \bigcap C^{\infty}(G)$ such that $v_j(z) \searrow -\ln R_*(z)$ at $j \to \infty$. If you enter the designation

$$w_j(z) = \exp\left\{-v_j(z)\right\},$$

then $w_j \in C^{\infty}(G)$ and $w_j(z) \nearrow R_*(z)$ at $j \to \infty$. Then, according to the analogue of the Hartogs lemma on the upper limit, for any number $j \in N$ and any number $\varepsilon > 0$ there is a number n_0 such that inequality holds

$$\frac{1}{n} \ln\left|c_n(z)\right| \leq v_j(z) - \ln(1-\varepsilon) = -\ln(1-\varepsilon) w_j(z),\ n \geq n_0, z \in K.$$

From here we have

$$\left|c_n(z)\right| \leq \frac{1}{\left[(1-\varepsilon) w_j(z)\right]^n}, n \geq n_0, z \in, K.$$

This proves that series (11) converges absolutely and uniformly on the set

$$\{(z,w): z \in K, |w| < (1-\varepsilon) w_j(z)\}.$$

Due to the arbitrariness of the compact set $K \subset D \setminus S$ and the number $\varepsilon > 0$ and the fact that $w_j(z) \nearrow R_*(z)$ with $j \to \infty$, follows absolute and uniform convergence of series (11) inside the domain

$$\left\{(z,w): z \in D \setminus S, |w| < R_*(z)\right\}.$$

3^0. Let $D_0 \subset\subset D \setminus S = D_1$ arbitrary domain. Then series (11) satisfies the following conditions:

1) series (11) converges uniformly inside the domain $D_0 \times \left\{|w| < R_*(z)\right\}$,

2) the coefficients of this series have Ostrovsky lacunae,

that is series (11) satisfies the conditions of Theorem 1.

Then, by Theorem 1, series (11) defines a unique holomorphic function $f(z,w)$ in the domain $\left(\bigcup\limits_{z \in D_0 \setminus S} W_f(z)\right)^0$. From the arbitrariness of the domain D_0 we find that the sum of series (11) defines a unique holomorphic function $f(z,w)$ in the domain $W^0 \setminus (S \times \mathbb{C})$, where W^0 is the open core of the set $W = \bigcup\limits_{z \in D} W_f(z)$. Theorem 3 is proven. ►

Remark. If in Theorem 3 we do not require the condition $R(z) > 0$ for any fixed $z \in D$, then the statement of the theorem will not be true, i.e. Condition 2 of Theorem 3 is necessary (see [18]).

3. An analogue of Ostrovsky's theorem for series in homogeneous polynomials.

Consider the formal lacunary series

$$\sum_{k=0}^{\infty} \mathbb{Q}_{n_k}(z) \tag{14}$$

over homogeneous polynomials $\mathbb{Q}_{n_k}(z)$ and we will investigate the domain of convergence of such a series.

In [8] J. Sichak considered the following situation. Let the function n of complex variables $f(z)$ be holomorphic in a neighborhood of zero and expanded into a lacunary series in homogeneous polynomials

$$f(z) = \sum_{k=0}^{\infty} \mathbb{Q}_{n_k}(z) \tag{15}$$

Let $\psi(z) := \overline{\lim_{k\to\infty}} \left|\mathbb{Q}_{n_k}(z)\right|^{\frac{1}{n_k}}$ and $h(z) := \psi^*(z)$ is a regularization of the function ψ. Then it is known that the circular domain

$$D := \{z \in \mathbb{C}^n : h(z) < 1\}$$

is the domain of convergence of series (15). For this series, the following multidimensional analogue of Fabry's theorem holds.

Theorem (Siciak [8]). *If for series (15)*

$$\overline{\lim_{k\to\infty}} \frac{k}{n_k} = \lim_{k\to\infty} \frac{k}{n_k} = 0,$$

then the domain of convergence D of series (15) coincides with the natural domain of existence W_f of the function $f(z)$.

In [9], using the complex theory of pluripotential, a simpler proof of this theorem is given and an analogue of the theorem is considered in more general situations; an analogue of Pólya's theorem is given (the case when $\underline{\lim}_{k\to\infty} \frac{k}{n_k} = 0$). In the case when $\underline{\lim}_{k\to\infty} \frac{k}{n_k} = 0$, series (15) can converge outside the domain D, i.e. D will not be a natural domain of existence of the function f. However, in [9] it was proven that f is a single-valued function in $\mathbb{C}^n$, i.e. $W_f \subset \mathbb{C}^n$ and a simple description of W_f is given in terms of $W_f\big|_l$, $\ell \in \mathbb{P}^{n-1}$. In this section we will prove an analogue of Pólya's theorem for the lacunary Ostrovsky series in homogeneous polynomials with a variable radius of convergence.

Theorem 4. *Let the series (15) and the family of complex lines* $L \subset \mathbb{P}^{n-1}$ *satisfy the following conditions*

1) L *is not pluripolar,*

2) *series (15) has Ostrovsky lacunae,*

3) for each straight line $\ell \in L$, *series (15) converges in a disc of positive radius* $R(\ell) \geq r > 0$.

Then series (15) defines a unique holomorphic function $f(z)$ *in* $\mathbb{C}^n$, *i.e. the natural domain of existence of* W_f *of the function* f *is univalent,* $W_f \subset \mathbb{C}^n$. *Moreover,*

1) $0 \in W_f$,

2) W_f *coincides with the open kernel of the union* $\bigcup_{\ell \ni 0} W_{f,\ell}$, *i.e.*

$$W_f = (\bigcup_{\ell \in \mathbb{P}^{n-1}} W_{f,\ell})^0,$$

3) for all complex lines $\ell \in \mathbb{P}^{n-1}$ *with the exception of some pluripolar set* $\mathrm{P} \subset \mathbb{P}^{n-1}$,

$$\ell \bigcap W_f = W_{f,\ell}\ ,\ \ell \notin \mathrm{P}.$$

To prove this theorem, we need the following statement, proven in [9].

Theorem 5. *Let a formal series be given*

$$\sum_{s=0}^{\infty} \mathbb{Q}_s(z)$$

by homogeneous polynomials $\mathbb{Q}_s$ *and the family* $L \subset \mathbb{P}^{n-1}$ *of complex lines* ℓ. *If for each complex line* $\ell \in L$ *the series* $\sum_{s=0}^{\infty} \mathbb{Q}_s(z)\ \Big|_l$ *converges in the disc* $\ell \cap B(0, r(l)), 0 < r(l) \leq 1$, *then it converges uniformly inside the domain*

$$\Omega = \{z \in \mathbb{C}^n : |z| \cdot \exp V^*(\frac{z}{|z|}, E) < 1\}, \tag{16}$$

where $E=\bigcup_{l\in L}(l\cap S(0,r(l)))$ *,* $S(0,r(l))=\partial B(0,r(l))$ *and* $V^{*}(z,E)$ *is the Green's function of the set* E *(or the Sichak-Zakharyuta extremal function).*

Since the generalized Green's function $V^{*}(z,E)\equiv+\infty$ if and only if E is a pluripolar set in $\mathbb{C}^{n}$, then from equality (16) it is clear that Theorem 5 is meaningful only if the set E is non-pluripolar in $\mathbb{C}^{n}$, which we will also assume. We also exclude the trivial case when $\Omega=\mathbb{C}^{n}$.

Proof of Theorem 4. From the conditions of Theorem 4 it follows that series (15) satisfies the conditions of Theorem 5. Then, according to Theorem 5, series (15) converges uniformly in a certain neighborhood of zero, and its sum determines the function $f(z)$ holomorphic in this neighborhood .

$$f(z):=\sum_{s=0}^{\infty}\mathbb{Q}_{s}(z). \tag{17}$$

Further using fractional - linear transformation

$$('z,z_{n})\to('w,z_{n}),\ \text{where}\ w_{\nu}=\frac{z_{\nu}}{z_{n}}\,(\nu=1,2,\dots,n-1),$$

Let us map the domain under consideration into the Hartogs domain (see [9]). Then the straight lines $\ell\ni 0$ will go into straight lines parallel to the axis z_{n}, and the neighborhood of zero into some polydisc $'U\times\{|z_{n}|<r\},r>0$. Consequently, the series in homogeneous polynomials (17) turns into a Hartogs series

$$g('z,z_{n})=\sum_{k=0}^{\infty}c_{k}('z)z_{n}^{k} \tag{18}$$

and this series satisfies the conditions of Theorem 1:

Let series (18) be such that

1) series (18) converges uniformly in a polydisc.

$$'U\times\{|z_{n}|<r\}\subset\mathbb{C}_{'z}^{n-1}\times\mathbb{C}_{z_{n}},\quad r>0\ ,$$

2) series (18) has Ostrovsky lacunae.

Then, according to Theorem 1, series (18) defines a single-valued holomorphic function with a univalent domain of existence, i.e.

$$g('z,z_n)\in\mathcal{O}\{('z,z_n)\in\mathbb{C}^{n-1}_{'z}\times\mathbb{C}_{z_n}:'z\in 'U,\ |z_n|<R_*('z)\}$$

where $R_*('z)=\underline{\lim}_{'w\to 'z} R('w)$ is a lower regularization radius-functions $R\ ('z)$ and $R\ ('z)$ radius of the maximum disc, where series (18) converges, for a fixed $'z\in' D$.

Next, using Theorem 2, we obtain the proof of Theorem 4.►

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Currents H_u^k and their convergence in the class of m-convex functions

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Annotation. The work is devoted to m-convex $(m-cv)$ functions in Euclidean space $\mathbb{R}^n, (1 \le m \le n)$. We use the connection $m-cv$ functions with strongly m-subharmonic (sh_m) functions in complex space $\mathbb{C}^n$, which are based on differential forms and currents $(dd^c u)^k \wedge \beta^{n-k} \ge 0,\ k = 1,2,\dots,n-m+1,$ where $\beta = dd^c\|z\|^2$ is the standard volume form in $\mathbb{C}^n$. Theory of sh_m $-$function is well developed and is currently the subject of study by many mathematicians.

For a locally bounded m-convex function $u(x) \in m-cv(D),\ D \subset \mathbb{R}^n$, the Hessians $H^k(u),\ k = 1,2,\dots,n-m+1$ are defined as Borel measures and a number of important and fundamental properties of these Hessians are proved.

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Key words. Subharmonic functions, Convex functions, Borel measures, Hessians.

1. Introduction. The work is devoted to m $-$convex $(m-cv)$ functions in Euclidean space $\mathbb{R}^n$, $(1 \le m \le n)$. Let $u(x) \in C^2(D)$ be a doubly smooth function in the domain $D \subset \mathbb{R}^n$. Then the matrix $\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right)$ is orthogonal, $\frac{\partial^2 u}{\partial x_j \partial x_k} = \frac{\partial^2 u}{\partial x_k \partial x_j}$ and after a suitable orthonormal transformation it is converted to diagonal form,

$$\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right) \to \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix},$$

where $\lambda_j = \lambda_j(x) \in \mathbb{R}$ are the eigenvalues of the matrix $\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right)$. Let

$$H^k(u) = H^k(\lambda) = \sum_{1 \le j_1 < \dots < j_k \le n} \lambda_{j_1} \dots \lambda_{j_k}$$

is the Hessian of degree k of the eigenvalue vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$.

Definition 1. *A twice smooth function $u(x) \in C^2(D)$ is called m $-$convex in $D \subset \mathbb{R}^n$, $u \in m-cv(D)$, if its eigenvalue vector $\lambda = \lambda(x) = \big(\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x)\big)$ satisfies the conditions*

$$H^k(u) = H^k\big(\lambda(x)\big) \ge 0, \qquad \forall x \in D,\ k = 1, \dots, n-m+1.$$

When $m = n$ the class $n-cv$ coincides with the class of subharmonic functions $sh = \{\lambda_1 + \lambda_2 + \dots + \lambda_n \ge 0\}$, and when $m = 1$ it coincides with the class of convex functions $cv = \{\lambda_1 \ge 0, \lambda_2 \ge 0, \dots, \lambda_n \ge 0\}$. In general,

$$cv = 1-cv \subset 2-cv \subset \dots \subset n-cv = sh.$$

The theory of subharmonic functions is a developed and important part of Function Theory and Mathematical Physics. The Theory of convex functions is well studied and reflected in the works of A. Aleksandrov, I. Bakelman, V. Pogorelov and others (see [2]-[4], [5]-[6], [14]). Class $m-cv$ when $m > 1$ was studied in a series of works by N. Ivochkina, N. Trudinger, X. Wang, K. Chou, et al. (see [8], [10, [17]-[19]). In the work of N. Trudinger, X. Wang [18], $m-cv$ functions are introduced in the class of upper semi-continuous in the domain $D \subset \mathbb{R}^n$ functions

$u(x)$, using the so-called viscosity definition: that is $H^k(q) \geq 0$, $k = 1,2,\ldots,n-m+1$, for any quadratic polynomial $q(x)$: the difference $u(x) - q(x)$ for which has only a finite number of local maximum in the domain D. In addition, in this work, the maximum degree operator $H^{n-m+1}(u)$,was defined as a Borel measure and, using this operator, the condenser capasity $C(E,D)$ was introduced, a number of potential properties of this capacity were proved.

We have established that the class of m −convex functions has a close connection with m −subharmonic (sh_m) functions in the complex space $\mathbb{C}^n$, which are based on differential forms and currents $(dd^c u)^k \wedge \beta^{n-k} \geq 0$, $k = 1,2,\ldots,n-m+1$, where $\beta = dd^c|z|^2$ is the standard volume form in $\mathbb{C}^n$. Function theory of sh_m −functions is well developed, currently is the subject of study by many mathematicians (Z. Blocki [7], S. Dinew and S. Kolodziej [9], S.Li [11], C.Lu [12]-[13], etc.). A fairly complete review of this theory is available in the article by A.Sadullaev and B. Abdullaev [1] in the Proceedings of MIRAN.

Definition 2. *A twice smooth function $u(z) \in C^2(D)$, $D \subset \mathbb{C}^n$, is called (strongly) m −subharmonic, $u \in sh_m(D)$, if at each point of the domain D*

$$(dd^c u)^k \wedge \beta^{n-k} \geq 0, \quad k = 1,2,\ldots,n-m+1 \tag{1}$$

where $\beta = dd^c\|z\|^2$.

Operators $(dd^c u)^k \wedge \beta^{n-k}$ are functionally related to Hessians. For a doubly smooth function $u \in C^2(D)$, the second order differential $dd^c u = \frac{i}{2}\sum_{j,k}\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\,\bar{z}_k$ (at a fixed point $o \in D$) is a Hermitian quadratic form. After a suitable unitary coordinate transformation, it is reduced to diagonal form $dd^c u = \frac{i}{2}[\lambda_1 dz_1 \wedge d\,\bar{z}_1 + \ldots + \lambda_n dz_n \wedge d\,\bar{z}_n]$, where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of the Hermitian matrix $\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)$, which are real: $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$. Note that the unitary transformation does not change the differential form $\beta = dd^c\|z\|^2$. It is easy to see that

$$(dd^c u)^k \wedge \beta^{n-k} = k!\,(n-k)!\,H^k(u)\beta^n, \tag{2}$$

where $H^k(u) = \sum_{1 \leq j_1 < \ldots < j_k \leq n} \lambda_{j_1} \ldots \lambda_{j_k}$ is the Hessian of the degree k of the vector $\lambda = \lambda(u) \in \mathbb{R}^n$.

Consequently, a doubly smooth function $u(z) \in C^2(D)$, $D \subset \mathbb{C}^n$, is m −subharmonic, if the inequalities

$$H^k(u) = H_o^k(u) \geq 0, \quad k = 1,2,\ldots,n-m+1, \tag{3}$$

hold at each point $o \in D$. Note that the concept of m −subharmonic function in a generalized sense is also defined in the general case, for upper semicontinuous functions.

Definition 3. *A function $u(z)$, defined in a domain $D \subset \mathbb{C}^n$ is called sh_m, if it is upper semi-continuous and for any doubly smooth sh_m functions $v_1, \ldots, v_{n-m} \in C^2(D) \cap sh_m(D)$ the current $dd^c u \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1}$ defined as*

$$\begin{aligned} &[dd^c u \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1}](\omega) = \\ &= \int u\, dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega, \qquad \omega \in F^{0,0}, \end{aligned} \tag{4}$$

positive,

$$\int u\, dd^c v_1 \wedge \ldots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega \geq 0, \qquad \forall\, \omega \in F^{0,0}, \qquad \omega \geq 0.$$

Here $F^{0,0}(D)$ is a family of infinitely smooth, finitely supported in D functions.

In Blocki's work [7] it was proven that this definition is correct, that for functions $u \in C^2(D)$ this definition coincides with the original definition of sh_m −functions. Moreover, in the class of bounded sh_m −functions, operators $(dd^c u)^k \wedge \beta^{n-k} \geq 0$, $k = 1,2,\ldots,n-m+1$ are defined as Borel measures in a domain D (see [1], [7]).

2. m-convex functions. In this work, we propose a completely different approach to the study of $m - cv$ functions, based on relationships $m - cv$ function to sh_m −function, using rich and well-studied properties of sh_m −functions. To do this, we embed real space $\mathbb{R}_x^n$ into complex space $\mathbb{C}_z^n$, $\mathbb{R}_x^n \subset \mathbb{C}_z^n = \mathbb{R}_x^n + i\mathbb{R}_y^n (z = x + iy)$,as a real n −dimensional subspace.

Then, we lift the function $u(x)$,given in the domain $D \subset \mathbb{R}_x^n$ to the domain $\Omega = D \times i\mathbb{R}_y^n \subset \mathbb{C}_z^n$, assuming it is a constant on parallel planes $\Pi_{x^0} = \{z \in \mathbb{C}^n : x = x^0, \ y \in \mathbb{R}_y^n\}$, $u^c(z) = u^c(x+iy) = u(x)$.

The key point in the study of m-functions in a wider class of functions is the following connection between $m-cv$ and sh_m functions (see [16]).

Theorem 1. *A twice smooth function $u(x) \in C^2(D)$, $D \subset \mathbb{R}_x^n$, is in $m-cv$ if and only if the function $u^c(z) = u^c(x+iy) = u(x)$, that does not depend on variables $y \in \mathbb{R}_y^n$, is sh_m –function in the domain $D \times i\mathbb{R}_y^n$.*

To study a m –convex function $u(x)$, we extend it into complex space $\mathbb{C}^n$ as sh_m –function $u^c(z)$, and then apply the known properties of $u^c(z) \in sh_m$ to $u(x)$, obtain similar properties of m –convex functions. All the basic properties of m –convex functions in recent years, obtained by me and my students, have been proven in this way, using connections $m-cv$ and sh_m –functions.

As a result, we significantly complement the previously available results in $m-cv$function theory and obtain a number of new results. In particular, we give a complete construction of the Potential Theory in the class of $m-cv$ functions.

Theorem 1 allows us to define a m-convex function in the class of upper semicontinuous functions.

Definition 3. *An upper semi-continuous function $u(x)$ in a domain $D \subset \mathbb{R}_x^n$ is called m –convex, if the function $u^c(z)$ is (strongly) m –subharmonic, $u^c(z) \in sh_m(D \times i\mathbb{R}_y^n)$.*

This definition is convenient in the study of m –convex functions, transferring known properties of sh_m –functions to the class $m-cv$. We present some non-trivial ones

-- **(Approximation).** We take the standard kernel $K_\delta(x) = \frac{1}{\delta^n} K\left(\frac{x}{\delta}\right)$, $\delta > 0$, where

-- $K(x) = K(|x|)$;

-- $K(x) \in C^\infty(\mathbb{R}^n)$;

-- the content $\operatorname{supp} K = B(0,1)$;

-- $\int_{\mathbb{R}^n} K(x)\ dx = \int_{B(0,1)} K(x)\ dx = 1$.

Then convolution

$$u_\delta(y) = \int_D u(x) K_\delta(x-y)dx = \int_{\mathbb{R}^n} u(x+y) K_\delta(x)dx \tag{5}$$

has the property, that $u_\delta(x) \in m-cv(D_\delta) \cap C^\infty(D_\delta)$, where $D_\delta = \{x \in D: \ \ dist(x, \partial D) > \delta\}$, which as $\delta \downarrow \infty$ decreasing converges pointwise to the function $u(x) \in m-cv(D)$, $u_\delta(x) \downarrow u(x)$.

-- *the limit of uniformly convergent or decreasing sequence of $m-cv$functions is $m-cv$;*

-- *the maximum of a finite number of $m-cv$ functions is $m-cv$ function; for an arbitrary locally uniformly bounded family $\{u_\theta\} \subset m-cv$, the regularization $u*(x)$ of the supremum $u(x) = \left\{\sup_\theta u_\theta(x)\right\}$ will also be a $m-cv$function. Since $m-cv \subset sh$, then the set $\{u(x) < u*(x)\}$ is polar in $\mathbb{C}^n \approx \mathbb{R}^{2n}$. In particular, it has Lebesgue measure zero.*

Similarly, for a locally uniformly bounded sequence $\{u_j\} \subset m-cv$, the regularization $u*(x)$ of the limit $u(x) = \overline{\lim_{j\to\infty}} u_j(x)$ will also be a $m-cv$ function, and the set $\{u(x) < u*(x)\}$ is polar.

Theorem 2. *If $u(x) \in m-cv(D)$, then for any hyperplane $\Pi \subset \mathbb{R}^n$ the restriction$u|_\Pi \in m-cv(D \cap \Pi)$.*

In fact, considering, without loss of generality $\Pi_x = \{x \in \mathbb{R}^n: \ \ x_n = 0\}$, we write the restriction as $u|_\Pi = u('x, 0)$, where as usual $'x = (x_1, \dots, x_{n-1})$. Consider complex hyperplane $\Pi_z = \{z \in \mathbb{C}^n: \ \ z_n = 0\}$ in the space $\mathbb{C}_z^n = \mathbb{R}_x^n \times i\mathbb{R}_y^n$.Raising the function $u(x) \in m-cv(D)$ in $D \times i\mathbb{R}_y^n$ we obtain $u^c(z) \in sh_m(D \times i\mathbb{R}_y^n)$. According to Property 8) [1] the restriction

$u^c(z)|_{\Pi_z} = u('z,0)$ is sh_m −function in $(D \times i\mathbb{R}_y^n) \cap \Pi_z$, $u^c(z,0) \in sh_m(D \times i\mathbb{R}_y^n) \cap \Pi_z$. Since $u^c('z,0) = u('x,0)$,then $u('x,0)$ is a m −convex function in $D \cap \Pi_x$. ⊳

Corollary. *If* $u(x) \in m - cv(D)$, *then* $u|_\Pi \in sh(D \cap \Pi)$ *for any plane* $\Pi \subset \mathbb{R}^n$, $\dim\Pi = m$.

3. Currents $H^k(u)$. In the class of bounded sh_m −functions, operators $(dd^c u)^k \wedge \beta^{n-k} \geq 0$, $k = 1,2,\dots,n-m+1$ are defined as Borel measures in a domain D (see [7], [1]). Using the connection between $m - cv$ functions and $sh_m -$ functions, in this section we give definitions of Hessians $H^k(u)$, $k = 1,\dots,n-m+1$for m-convex functions, as a Borel measures.

Let $u(x)$ be a locally bounded $m - cv$ function in a domain $D \subset \mathbb{R}^n$. Then, according to Theorem 1, the function $u^c(z) = u^c(x+iy) = u(x)$, which does not depend on variables $y \in \mathbb{R}_y^n$, is also a locally bounded and sh_m function in the domain $D \times i\mathbb{R}_y^n \subset \mathbb{C}^n$, $u^c(z) \in sh_m(D \times i\mathbb{R}_y^n) \cap L_{loc}^\infty(D \times i\mathbb{R}_y^n)$. Consequently, the currents $(dd^c u^c)^k \wedge \beta^{n-k}$, $k = 1,2,\dots,n-m+1$, are defined, that are Borel measures in $D \times i\mathbb{R}_y^n \subset \mathbb{C}^n$. If $u_j^c(z) = u_j \circ K(w-z)$ is standard approximation, then $u_j^c(z)$is infinitely smooth and $u_j^c(z) \downarrow u^c(z)$. Moreover, there is weak convergence of currents,

$$\left(dd^c u_j^c\right)^k \wedge \beta^{n-k} \mapsto (dd^c u^c)^k \wedge \beta^{n-k},\ k = 1,2,\dots,n-m+1. \tag{6}$$

Since $\left(dd^c u_j^c\right)^k \wedge \beta^{n-k} = k!\,(n-k)!\,H^k\left(u_j^c\right)\beta^n$, then (6) entails the convergence of Hessins

$$H^k\left(u_j^c\right) \mapsto H^k(u^c),\ k = 1,2,\dots,n-m+1. \tag{7}$$

(7) defines for $u^c(z) \in sh_m(D \times i\mathbb{R}_y^n) \cap L_{loc}^\infty(D \times i\mathbb{R}_y^n)$ Hessians $H^k(u^c)$, $k = 1,2,\dots,n-m+1$, as Borel measures, $H^k(u^c) = \mu^k$.

Since $u^c \in sh_m(D \times \mathbb{R}_y^n)$ does not depend on $y \in \mathbb{R}_y^n$, then for any Borel sets $E_x \subset D$, $E_y \subset\subset \mathbb{R}_y^n$ the measures $\frac{4^k}{mesE_y}\mu_k(E_x \times E_y)$ do not depend on the set $E_y \subset \mathbb{R}_y^n$, i.e. $\frac{4^k}{mesE_y}\mu_k(E_x \times E_y) = \nu_k(E_x)$. Borel measures

$$\nu_k\colon\quad \nu_k(E_x) = \frac{4^k}{mesE_y}\mu_k(E_x \times E_y),\ k = 1,2,\dots,n-m+1, \tag{8}$$

are natural to call them *Hessians* $H^k(u)$, $k = 1,2,\dots,n-m+1$, for a bounded, m-convex function $u(x) \in m - cv(D)$ in the domain $D \subset \mathbb{R}_x^n$, since $H^k(u) = 4^k H^k(u^c)$ for a doubly smooth function $u(x) \in m - cv(D)$ (see. [15]).

4. Properties of currents $H^k(u)$.

Theorem 3. *If a sequence of locally uniformly bounded functions* $\{u_j(x)\} \subset m - cv(D)$ *converges decreasingly to* $u(x)$, *then there is weak convergence of measures*

$$H^k(u_j) \mapsto H^k(u), \qquad k = 1,2,\dots,n-k+1.$$

The proof follows from a similar fact for the class $sh_m(D \times i\mathbb{R}_y^n)$. ⊳

Theorem 4. *Let* $u(x), v(x) \in m - cv(D) \cap C(D)$ *and open set* $F = \{u(x) < v(x)\} \subset\subset D$. *Then*

$$\int_F H^{n-m+1}(u) \geq \int_F H^{n-m+1}(v). \tag{9}$$

The proof of the theorem is carried out in several stages.

1) *If* $D \subset \mathbb{R}^n$ *is a bounded domain with a smooth boundary* ∂D *and*$u, v \in m - cv(D) \cap C^2(D)$: $u|_D \leq v|_D$, $u|_{\partial D} \equiv v|_{\partial D}$, *then*

$$\int_D H^{n-m+1}(u) \geq \int_D H^{n-m+1}(v).$$

In fact, let's embed $\mathbb{R}_x^n$ in $\mathbb{C}_z^n$, $\mathbb{R}_x^n \subset \mathbb{C}_z^n = \mathbb{R}_x^n + i\mathbb{R}_y^n$ $(z = x + iy)$, and construct functions $u^c(z) = u(x) \in sh_m(D \times i\mathbb{R}_y^n)$, $v^c(z) = v(x) \in sh_m(D \times i\mathbb{R}_y^n)$. Take a cylinder $\Omega = \{(x, y) \in D \times \mathbb{R}_y^n: \ x \in D, \ |y| < 1\}$. Boundary of the cylinder $\partial\Omega = S_1 \cup S_2$, where $S_1 = D \times \{|y| = 1\}$, $S_2 = \partial D \times \{|y| < 1\}$.

According to the Stokes formula we have

$$\int_\Omega [(dd^c u^c)^{n-m+1} \wedge \beta^{m-1} - (dd^c v^c)^{n-m+1} \wedge \beta^{m-1}] =$$

$$= \int_\Omega [(dd^c u^c) - (dd^c v^c)] \wedge$$

$$\wedge [(dd^c u^c) \wedge (dd^c v^c)^{n-m} + (dd^c u^c)^2 \wedge (dd^c v^c)^{n-m-1} + \ldots + (dd^c u^c)^{n-m} \wedge (dd^c v^c)] \wedge \beta^{m-1} =$$

$$= \int_{\partial\Omega} [(d^c u^c) - (d^c v^c)] \wedge$$

$$\wedge [(dd^c u^c) \wedge (dd^c v^c)^{n-m} + (dd^c u^c)^2 \wedge (dd^c v^c)^{n-m-1} + \ldots + (dd^c u^c)^{n-m} \wedge (dd^c v^c)] \wedge \beta^{m-1}.$$

Note that the differential form

$[(dd^c u^c) \wedge (dd^c v^c)^{n-m} + (dd^c u^c)^2 \wedge (dd^c v^c)^{n-m-1} + \ldots + (dd^c u^c)^{n-m} \wedge (dd^c v^c)]$ is positive and $[(d^c u^c) - (d^c v^c)] = d^c(u^c - v^c)$ expresses the internal normal derivative $[(d^c u^c) - (d^c v^c)] = d^c(u^c - v^c) = \frac{\partial(u^c - v^c)}{\partial n} d\sigma$, where $d\sigma$ is the area element on $\partial\Omega$. Since the function $u^c - v^c$ does not depend on y, then $\frac{\partial(u^c - v^c)}{\partial n}\big|_{|y=1|} = 0$. Therefore,

$$\int_{S_1} [(d^c u^c) - (d^c v^c)] \wedge$$

$$\wedge [(dd^c u^c) \wedge (dd^c v^c)^{n-m} + (dd^c u^c)^2 \wedge (dd^c v^c)^{n-m-1} + \ldots + (dd^c u^c)^{n-m} \wedge (dd^c v^c)] \wedge \beta^{m-1} = 0.$$

For the integral over S_2

$$\int_{S_2} [(d^c u^c) - (d^c v^c)] \wedge$$

$$\wedge [(dd^c u^c) \wedge (dd^c v^c)^{n-m} + (dd^c u^c)^2 \wedge (dd^c v^c)^{n-m-1} + \ldots + (dd^c u^c)^{n-m} \wedge (dd^c v^c)] \wedge \beta^{m-1} \geq 0,$$

since $u^c - v^c \leq 0$ inside D and $(u^c - v^c)|_{\partial D} = 0$. Therefore, $d^c(u^c - v^c)$ is positive on S_2.

That's why,

$$\int_\Omega [(dd^c u^c)^{n-m+1} \wedge \beta^{m-1} - (dd^c v^c)^{n-m+1} \wedge \beta^{m-1}] =$$

$$= \int_{D \times \|y\| \leq 1} [(dd^c u^c)^{n-m+1} \wedge \beta^{m-1} - (dd^c v^c)^{n-m+1} \wedge \beta^{m-1}] \geq 0.$$

From here,

$$\int_{D \times \|y\| \leq 1} (dd^c u^c)^{n-m+1} \wedge \beta^{m-1} \geq \int_{D \times \|y\| \leq 1} (dd^c v^c)^{n-m+1} \wedge \beta^{m-1}$$

and according to (8) $H_u^{n-m+1}(D) \geq H_v^{n-m+1}(D)$. ▻

2) *If* $u, v \in C^2(D)$ *and open set* $F = \{u(x) < v(x)\} \subset\subset D$, *then*

$$\int_F H^{n-m+1}(u) \geq \int_F H^{n-m+1}(v).$$

T h e p r o o f easily follows from 1).

3) General case $u(x), v(x) \in m - cv(D) \cap C(D)$. Then the set $F = \{u(x) < v(x)\} \subset\subset D$ will be open. We fix a domains $G, \ G'$: $F \subset\subset G \subset\subset G' \subset\subset D$, number $\delta > 0$ and open set $F_\delta = \{u(x) + \delta < v(x)\} \subset\subset F$. We construct a sequences of approximations $u_j, \ v_j \in m - cv(G') \cap C^\infty(G')$, $j = 1,2,\ldots$: $u_j \downarrow u$, $v_j \downarrow v$. Due to the continuity of $u, \ v$ the convergence

$u_j \downarrow u,\ \ v_j \downarrow\ v$ will be uniform in G and, therefore, $\exists\, j_0, k_0$: $F_{3\delta} \subset F' = \{u_k + 2\delta < v_j\} \subset F_\delta$, $j \ge j_0,\ k \ge k_0$. According to 2) we have

$$H_{u_k}^{n-m+1}(F') \ge H_{v_j}^{n-m+1}(F'),\ \ k \ge k_0, \qquad j \ge j_0.$$

Hence for such k and j

$$H_{v_j}^{n-m+1}(F_{3\delta}) \le H_{v_j}^{n-m+1}(F') \le H_{u_k}^{n-m+1}(F') \le H_{u_k}^{n-m+1}(\bar{F}_\delta).$$

When $j \to \infty,\ \ k \to \infty$, according to the properties of Borel measures, we have

$$H_v^{n-m+1}(F_{3\delta}) \le H_u^{n-m+1}(\bar{F}_\delta).$$

Tending $\delta \to 0$ we hence obtain that $H_v^{n-m+1}(\{u < v\}) \le H_u^{n-m+1}\left(\overline{\{u < v\}}\right)$.Applying this inequality to the functions $u + \varepsilon,\ v,\ \ \varepsilon > 0$, we have

$$H_v^{n-m+1}(\{u + \varepsilon < v\}) \le H_u^{n-m+1}\left(\overline{\{u + \varepsilon < v\}}\right)$$

and then tending $\varepsilon \to 0$, we obtain the proof of the Theorem. $\rhd$

Hessian $H^{n-m+1}(u)$ is closely related to maximal functions.

Definition 4. *A function $u(x) \in m - cv(D)$ is called maximal in the domain $D \subset \mathbb{R}^n$, if for it the maximum principle holds in the class of $m - cv$ functions, i.e. if $v(x) \in m - cv(D)$: $\varliminf\limits_{x \to \partial D} (u(x) - v(x)) \ge 0$, then $u(x) \ge v(x)$, $\forall\, x \in D$.*

In practice, it is very convenient to use the following criterion, that $u(x) \in m - cv(D)$ *is* maximal in the domain $D \subset \mathbb{R}^n$ if and only if, for any domain $G \subset\subset D$, the inequality $u(x) \ge v(x)$, $\forall\, x \in G$,is satisfied for all functions $v(x) \in m - psh(D)$: $u|_{\partial G} \ge v|_{\partial G}$.

Theorem 5. *A function $u(x) \in m - cv(D) \cap C(D)$ is maximal if and only if the Hessian $H^{n-m+1}(u) = 0$ in $D \subset \mathbb{R}^n$.*

P r o o f. Let $H^{n-m+1}(u) = 0$, but the function $u(x)$ is not maximal. Then there exists a domain $G \subset\subset D$ and a function $v(x) \in m - cv(D)$ such that $u(x) > v(x)\ \ \forall x \in \partial G$, but $v(x^0) - u(x^0) = \varepsilon > 0$ for some point $x^0 \in G$. Approximating v by infinitely smooth $m - cv$ functions, by Hartogs' lemma we find $j_0 \in \mathbb{N}$ such that $v_{j_0}|_{\partial G} < u|_{\partial G} + \frac{\varepsilon}{2}$. Compare the function $u(x)$ with the function $v_{j_0}(x) + \delta|x|^2$,where $\delta = \frac{\varepsilon}{3 \cdot \max\{\|x\|^2 : x \in \bar{G}\}}$. For such $\delta > 0$ the set $F = \left\{u(x) + \frac{\varepsilon}{2} < v_{j_0}(x) + \delta|x|^2\right\}$ is not empty and lies compactly in G. Then according to the comparison principle (Theorem 3)

$$\delta^{n-m+1} \int\limits_F H^{n-m+1}(|x|^2) \le \int\limits_F H^{n-m+1}(v + \delta|x|^2) \le \int\limits_F H^{n-m+1}(u) = 0.$$

It's a contradiction because $\int_F H^{n-m+1}(|x|^2) > 0$.

Let now $u(x)$ be maximum function. Then $u^c(z) = u(x)$ is also maximal function in the domain $\Omega = D \times i\mathbb{R}_x^n \subset \mathbb{C}^n$. Therefore, $H_{u^c}^{n-m+1} = 0$ in Ω . And this is equivalent to $H_u^{n-m+1} = 0$ in D . *The theorem is proven.*

Remark. We proved Theorems 3 and 4 for simplicity, in the class $m - cv(D) \cap C(D)$. In fact, they are also valid in the class $m - cv(D) \cap L_{loc}^\infty(D)$.

5. An estimations of Hessians in average. If $\Omega = \{\rho(z) < 0\} \subset \mathbb{C}^n$ is a strictly m −convex domain, where $\rho(z)$ is a doubly smooth, strictly $sh_m -$ function in some neighborhood $D^+ \supset D,\ \sigma = \min\limits_G \rho(\alpha)$ then for any $\sigma < r < 0$ and for any function $u \in sh_m(\Omega) \cap C^2(\bar{\Omega})$ the integral formula holds (see SA)

$$\int_\sigma^r dt \int_{\rho(z)\ \le\ t} (dd^c\rho)^{n-k} \wedge (dd^c u)^k \le (M - m) \int_{\rho(z) \le r} (dd^c\rho)^{n-k+1} \wedge (dd^c u)^{k-1}, \quad (10)$$

where $1 \le k \le n - m + 1,\ \ M = \sup\limits_\Omega u(z),\ \ m = \inf\limits_\Omega u(z)$.

Let us apply this integral formula to a function $u(z) = u(x) \in sh_m(B) \cap C^2(\bar{B})$, where $B = \{z \in \mathbb{C}^n:\ \ |z| < 1\}$ is the unit ball in the norm $|z|^2,\ \ \rho(z) = |z|^2 - 1$,

$$\int_{-1}^{r} dt \int_{|z|^2 \leq 1+t} \beta^{n-k} \wedge (dd^c u)^k \leq (M-m) \int_{|z|^2 \leq r} \beta^{n-k+1} \wedge (dd^c u)^{k-1}.$$

Let's move on to the Hessians here

$$\int_{-1}^{r} dt \int_{|z|^2 \leq 1+t} H_u^k \, dV \leq \frac{n-k+1}{k}(M-m) \int_{|z|^2 \leq r} H_u^{k-1}. \qquad (11)$$

Theorem 6. *If* $u(x) \in m - cv(B) \cap L^\infty(B)$, *then it is hold next uniform estimation of the Hessian* $H_u^k(x)$ *in terms of* $H_u^{k-1}(x)$

$$\int_{-1}^{r} dt \int_{|y|^2 \leq 1} dV(y) \int_{|x|^2 \leq 1+t-|y|^2} H_u^k(x) \, dV(x) \leq$$

$$\leq \frac{n-k+1}{k}(M-m) \int_{|y|^2 \leq 1} dV(y) \int_{|x|^2 \leq 1+r-|y|^2} H_u^{k-1}(x) \, dV(x).$$

where $r < 0, \ 1 \leq k \leq n-m+1.$

P r o o f. S t e p 1. Case $u(x) \in m - cv(B) \cap C^2(\bar{B})$. Applying formula (11) to the function $u^c(z) = u(x) \in sh_m(B \times i\mathbb{R}_y^n) \cap C^2(\bar{B} \times i\mathbb{R}_y^n)$, in the ball $B_z = \{z \in \mathbb{C}^n: \ |z|^2 < r\}$, $r < 0$, we obtain

$$\int_{-1}^{r} dt \int_{|z|^2 \leq 1+t} H_{u^c}^k(z) dV \leq \frac{n-k+1}{k}(M-m) \int_{|z|^2 \leq 1+r} H_{u^c}^{k-1}(z) dV.$$

Since $H_{u^c}^k(z) = \frac{1}{4^k} H_u^k(x), \ z = x + iy \in B \times i\mathbb{R}_y^n$ then

$$\int_{-1}^{r} dt \int_{|z|^2 \leq 1+t} H_u^k(x) \, dV(z) \leq$$

$$\leq \frac{n-k+1}{k}(M-m) \int_{|z|^2 \leq 1+r} H_u^{k-1}(x) dV(z), \ 1 \leq k \leq n-m+1. \qquad (12)$$

If we pass here from multiple integrals to repeated ones, we get the needed formula

$$\int_{-1}^{r} dt \int_{|y|^2 \leq 1} dV(y) \int_{|x|^2 \leq 1+t-|y|^2} H_u^k(x) \, dV(x) \leq$$

$$\leq \frac{n-k+1}{k}(M-m) \int_{|y|^2 \leq 1} dV(y) \int_{|x|^2 \leq 1+r+|y|^2} H_u^{k-1}(x) \, dV(x).$$

S t e p 2. General case $u(x) \in m - cv(B) \cap L^\infty(B)$. Assuming, without loss of generality, that the function $u(x)$ is m −convex in a certain neighborhood $B^+ \supset B$, we approximate $u(x)$ by smooth functions $u_j(x) \in m - cv(B^+) \cap C^\infty(B^+)$, $u_j(x) \downarrow u(x)$. Then the integral formula

$$\int_{-1}^{r} dt \int_{|y|^2 \leq 1} dV(y) \int_{|x|^2 \leq 1+t-|y|^2} H_u^k(x) \, dV(x) \leq$$

$$\leq \frac{n-k+1}{k}(M-m) \int_{|y|^2 \leq 1} dV(y) \int_{|x|^2 \leq 1+r+|y|^2} H_u^{k-1}(x) \, dV(x)$$

is easily obtained from the corresponding integral formulas written for $u_j(x)$ at $j \to \infty$. *The theorem is proven.*

Corollary. *In the class of locally uniformly bounded functions* $L = \{u(x) \in m - cv(D)\}$, *the family of integrals* $\int_K H_u^k(x) dV(x)$, $u \in L$, *is uniformly bounded for any compact set* $K \subset D, \ 1 \leq k \leq n-k+1.$

P r o o f. It is enough to prove the Corollary for

$$L = \{u(x) \in m - cv(B) \cap C^2(\bar{B})\}, \qquad |u(x)| \leq M, \ \forall x \in B, \qquad K \subset\subset B,$$

where $B = B(0,1) \subset\subset D$ is the unit ball. It is convenient for us to use formula (12). Applying it for $k, k-1, \ldots, 1$ we get

$$\int_{-1}^{0} dt_1 \ldots \int_{-1}^{t_{k-2}} dt_{k-1} \int_{-1}^{t_{k-1}} dt_k \int_{|z|^2 \le 1+t_k} H_u^k(x)\, dV(z) \le \frac{(n-k+1)!}{k!} (2M)^k Vol(B_z).$$

$1 \le k \le n-m+1.$

The left-hand side of the formula can be estimated from below for $\sigma < 0$

$$\int_{-1}^{0} dt_1 \ldots \int_{-1}^{t_{k-2}} dt_{k-1} \int_{-1}^{t_{k-1}} dt_k \int_{|z|^2 \le 1+t_k} H_u^k(x)\, dV(z) \ge$$

$$\ge \int_{\sigma}^{0} dt_1 \ldots \int_{\sigma}^{t_{k-2}} dt_{k-1} \int_{\sigma}^{t_{k-1}} dt_k \int_{|z|^2 \le 1+\sigma} H_u^k(x)\, dV(z) \ge$$

$$\ge \int_{|z|^2 \le 1+\sigma} H_u^k(x)\, dV(z) \int_{\sigma}^{0} dt_1 \ldots \int_{\sigma}^{t_{k-2}} dt_{k-1} \int_{\sigma}^{t_{k-1}} dt_k = \frac{|\sigma|^k}{k!} \int_{|z|^2 \le 1+\sigma} H_u^k(x)\, dV(z).$$

From here,

$$\int_{|z|^2 \le 1+\sigma} H_u^k(x)\, dV(z) \le \frac{(n-k+1)!}{|\sigma|^k} (2M)^k Vol(B_z),$$

which means that the family of integrals $\int_{|x|^2 \le r} H_u^k(x) dV(x)$, $u \in L$, $r < 1$, is uniformly bounded. *The corollary has been proven.*

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Simulation of Self-Heating Effect in FinFETs: Influence of Oxide Layer Thickness and Material

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Abstract

The self-heating effect on the fin field effect transistor (FinFET) is investigated. The dependence of the lattice temperature in the channel center of the transistor on the thickness of gate oxide as well as back oxide is simulated. Different types of the most used oxide materials (SiO_2, HfO_2, and Si_3N_4) and their combination SiO_2+Si_3N_4 are considered for gate and back oxides. 3D simulation is performed using Sentaurus TCAD.

We have studied an SOI FinFET with an equivalent thickness of HfO_2 gate oxide that varies between tox=1.0 nm and 1.5 nm. The length of the TiN gate between Lgate=10nm, and the thickness of the BOX varies between Tbox=10nm and 1000 nm. Other main values are that The source and drain of the transistor are phosphorus-doped with a concentration of 5×10^{18}cm-3, and the transistor's channel is boron-doped with a concentration of 1×10^{16} cm-3. At the base, it has the thickness Tsi =9 nm and a width of Wfin = 22 nm.

For simulation, TCAD Sentaurus is used. For estimation of the self-heating effect drift-diffusion transport model in conjugation with the thermodynamic transport model was used. To account for quantum effects the quantum correction Density gradient was also used. The doping-dependent mobility model and velocity saturation in the high field are taken into account. Coulomb and phonon scatterings are included in the simulation model to consider the mobility degradation at the interface as the high-k material HfO_2 is used as gated oxide. A simulation drift-diffusion transport model was calibrated by comparing the I-V characteristics of the simulated transistor with experimental results.

It is shown, that the lattice temperature slowly and monotonically decreased with the increasing the gate oxide thickness. However, the lattice temperature is monotonically increased with increasing the thickness of the back oxide. This behavior of the lattice temperature depends on the relation between heat generation and dissipation rates in the transistor channel. The obtained behavior of the lattice temperature is explained by a difference in the heat conductivity of the oxide materials. Also, the lattice temperature dependence on the gate oxide thickness is explained by the increase of the contact area between gate oxide and the gate with increasing the gate oxide thickness. Besides this, it is accounted that the Joule heat generation rate depends on drain current, which also depends on oxide materials.

Keywords: 3D simulation; FinFET, self-heating effect; thermal conductivity; lattice temperature.

Introduction

One of the main tasks of nanoelectronics is to reduce power consumption and increase the degree of integration of integrated circuits (ICs). This task is connected with the considerable reduction of the sizes of the transistors that make up the integrated circuit. Along with other elements, metal-oxide-semiconductor field-effect transistors (MOSFET) are one of the most important components of ICs. The reduction in the size of MOSFETs leads to the appearance of various degradation effects, including short-channel effects (SCE), self-heating effect (SHE), Negative Bias Temperature Instability (NBTI), etc.

SCEs are strongly manifested in MOSFETs based on planar technology, which is one of the initial technologies for manufacturing MOSFETs. In order to increase the resistance of MOSFETs to short-channel effects, a structure of a vertical (or Fin) MOSFET (FinFET) was proposed instead of planar MOSFETs, with gate lengths of the order of 20 nm and lower, which operate in the inversion mode[1]. FinFET has three gates and as a consequence has a high degree of electrostatic integrity that ensures high immunity against short-channel effects[2].

One of the features of the FinFETs is that they are based on silicon-on-insulator technology. This feature is characterized by the fact that the channel of the transistor borders an oxide layer, the so-called back oxide layer (BOX). This feature leads to the manifestation of a self-heating effect in FinFET due to the low thermal conductivity of the oxide layer, leading to a low dissipation rate of heat generated in the channel of the transistor[345]. As a result, Increasing the temperature in the transistor channel leads to changing the drain current and as a consequence degradation of the I-V characteristics of the FinFET.

Therefore finding the oxide materials that increase the immunity of the transistors against SHE and which are compatible with FinFET fabrication technology is a very important task. In this work, the impact of the BOX materials, as well as gate oxide materials on the temperature in the transistor channel is investigated[67]. SiO2, HfO2, Si3N4, and combination SiO2+Si3N4 mainly are used in FinFET fabrication technology, and therefore in this research, these materials are considered as BOX as well as gate oxide materials.

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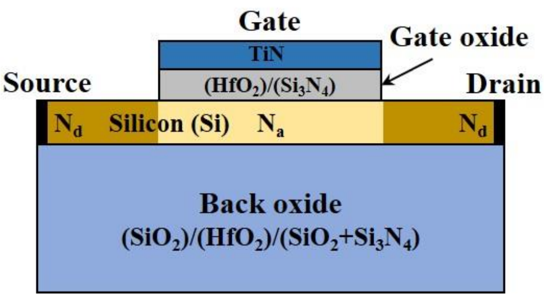
Transistor structure parameters and simulation models

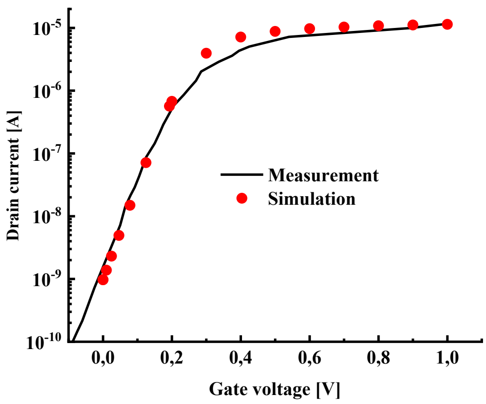
The cross-section along the channel of the 3D structure simulated in this work, silicon-based FinFET is shown in Fig.1. The silicon-based transistor's channel is n-type. TiN is used as gate material. Geometrical sizes of the different parts of the transistor and channel doping level are presented in Table I. In the simulation, the gate oxide thickness varies in the range between 1 and 1.5 nm, while back oxide thickness varies between 10 and 1000 nm.

Table 1. Geometrical and physical parameters of the considered transistor

Parameter	Designation	Value
Source and drain doping level	N_d	5×10^{18} cm^{-3} (n-type)
Channel doping level	N_a	1×10^{16} cm^{-3} (p-type)
Gate oxide (HfO_2, Si_3N_4) thickness	t_{ox}	t_{EOT} = 1.0-1.5 nm
Channel thickness	T_{si}	9 nm
Channel width	W_b	22 nm
Back oxide layer (SiO_2, HfO_2, SiO_2+Si_3N_4) thickness	T_{box}	10-1000 nm
Gate length	L_{gate}	10 nm

In the simulation, Sentaurus TCAD software is used. For estimation of the self-heating effect drift-diffusion transport model in conjugation with the thermodynamic transport model was used. To account for quantum effects the quantum correction Density gradient was also used. The doping-dependent mobility model and velocity saturation in the high field are taken into account. Coulomb and phonon scatterings are included in the simulation model to consider the mobility degradation at the interface as the high-k material HfO_2 is used as gated oxide. A simulation drift-diffusion transport model was calibrated by comparing the I-V characteristics of the simulated transistor with experimental results presented in[8]. The results of the comparison given in the Fig.2 show a good agreement.





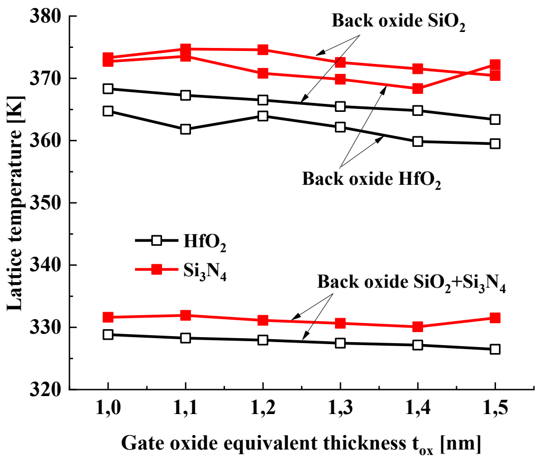
[8] V. S. Basker, T. Standaert, H. Kawasaki, C. C. Yeh, K. Maitra, T. Yamashita, J. Faltermeier, H. Adhikari, H. Jagannathan, J. Wang, H. Sunamura, S. Kanakasabapathy, S. Schmitz, J. Cummings, A. Inada, et al., in Proceedings of the Symposium on VLSI Technology (IEEE, 2010), p. 19.

Fig.1. Simulated FinFET cross-section structure.

Fig.2. Comparing the I-V characteristics of the simulated and experimental transistors for *Lgate* = 25 nm and *Vds* = 50 mV.

Simulation results and discussion

The dependence of the temperature in the channel center on the thickness of gate oxide in FinFET for HfO2 and Si3N4 as gate oxide and SiO2, HfO2, and SiO2+Si3N4 as back oxide is simulated. The results of the simulation are shown in Fig.3. Results show, that the lattice temperature very slowly decreased with increasing the gate oxide thickness for all considered oxide materials, while the contact area between the gate oxide and the channel increased monotonically with increasing the gate oxide thickness (Fig.4). Increasing the contact area lead to increasing heat dissipation rate, however drain current, and as consequence heat generation rate, practically is not changed with increasing the gate oxide thickness. Obviously, in this case, the heat generation rate has a considerable effect than the heat dissipation rate.



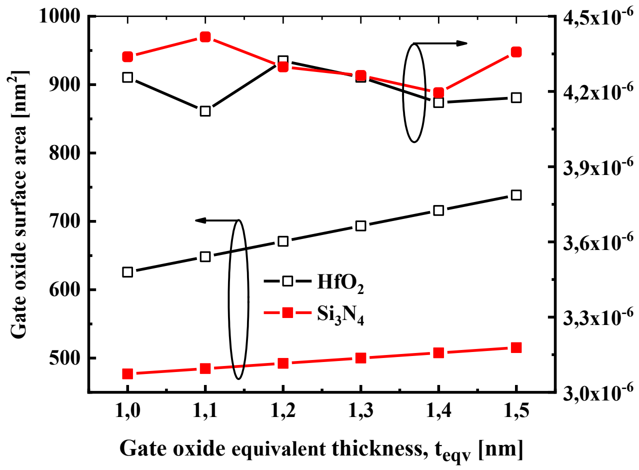


Fig.3. Dependence of the lattice temperature in the channel center on the gate oxide equivalent thickness for different gate oxide and back oxide materials.

Fig.4. Contact area and drain current dependence on the gate oxide equivalent thickness. BOX is HfO2

It is seen in Fig. 3, that in the case of using HfO2 as gate oxide the lattice temperature is lower than at using Si3N4, for all BOX materials. This dependence correlates with drain current dependence on the gate oxide thickness (Fig.4). The temperature dependence on the gate oxide thickness is the result of the combined effect of the dielectric constant, thermal conductivity, and the thickness of gate oxide

Lattice temperature dependence on the BOX thickness T_{box} is monotonous and the temperature growth with increasing the BOX thickness for all considered BOX materials. This dependence is in agreement with [9][10][11] and at higher thicknesses is expressed by the formula (1).

$$\Delta T = \frac{(P_t \cdot T_{box})}{K_b \cdot A} \tag{4}$$

where P_t stands for the heat power generated by the current in the channel, K_b is the heat conductivity of the oxide layer, and A represents the area of the contact surface between the oxide layer and the channel.

Increasing the temperature with increasing the BOX thickness is explained by the circumstance that with increasing the thickness the distance between the channel and metallic contact is increased. The lowest temperature for SiO_2+Si_3N_4 BOX material is connected with the highest thermal conductivity of this material (Table 3).

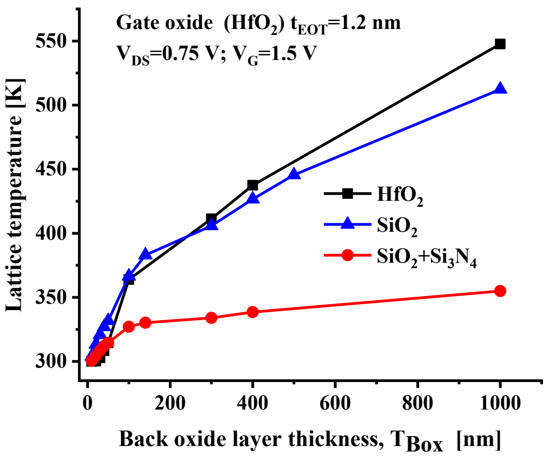


Fig.5. Lattice temperature dependence on the thickness of the BOX for different materials.

Table 2. Thermal Conductivity of the oxide materials

Oxide Material	Thermal Conductivity K_b (W m^{-1} K^{-1})
HfO_2	2.3
SiO_2	1.4
Si_3N_4 (SiO_2+Si_3N_4)	18.5

Conclusion

Results of the simulation show that lattice temperature in the center of the FinFET channel depends on gate oxide as well as back oxide material. More considerable dependence on the temperature of the materials is seen for back oxide, where the maximal difference in the temperatures lies in the range between 50 and 170 K for BOX thicknesses from 100 to 1000 nm. The maximal temperature difference in using different considered gate oxide materials is approximately 10K in all considered ranges of oxide thicknesses. The material of the oxide layers influences the drain current, but a substantial influence on the temperature is the thermal conductivity of the back oxide material.

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The highest immunity against the self-heating effect is achieved using HfO_2 as gate oxide material and SiO_2+Si_3N_4 as back oxide material in the considered range of oxide materials in this work.

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An analogue of the Sadullaev-Dinew theorem for the class of *m*-convex functions

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Abstract. In this work, we study m - convex functions, which are defined using operators in Hessians. In particular, we prove analogue of the Sadullaev-Dinew theorem for the class of m - convex functions. It is known that A.Sadullaev proved, that the classes $(A)sh_m(D)\cap C^2(D)$ and $(B)sh_m(D)\cap C^2(D)$ coincide for $m=2$, and raised the question of whether $(B)sh_m(D)\cap C^2(D)=(A)sh_m(D)\cap C^2(D)$ for arbitrary $1\le m\le n$. Later, S. Dinew showed that for $n\le 7$ these classes actually coincide, but for $n=11$, $m=3$ these classes are different. After that, the authors somewhat improved Dinew's result, proving for $n=10$, $m=3$ Sadullaev's hypothesis is not true, but for $n=9$, $n=8$ and $m=3$ the hypothesis is true.

Keywords: m - convex functions, $(A)m-cv$ functions, $(B)sh_m$ functions, $(A)sh_m$ functions, Hessians.

1.Introduction

m - convex functions are a generalization of convex functions in $\mathbb{R}^n$. Below we define m - convex functions using operators on Hessians. Let $D\subset\mathbb{R}^n$ and $u(x)\in C^2(D)$. The matrix $\left(\frac{\partial^2 u}{\partial x_k\partial x_t}\right)$ is symmetric, $\frac{\partial^2 u}{\partial x_k\partial x_t}=\frac{\partial^2 u}{\partial x_t\partial x_k}$. Therefore, after a suitable orthonormal transformation, it is transformed into a diagonal form,

$$\left(\frac{\partial^2 u}{\partial x_k\partial x_t}\right)\to\begin{pmatrix}\lambda_1 & 0 & \dots & 0\\ 0 & \lambda_2 & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & \lambda_n\end{pmatrix}$$

where $\lambda_j = \lambda_j(x) \in \mathbb{R}$ - are the eigenvalues of the matrix $\left(\frac{\partial^2 u}{\partial x_k \partial x_t}\right)$. Then $H^k(u) = H^k(\lambda) = \sum_{1 \le j_1 < \ldots < j_k \le n} \lambda_{j_1} \ldots \lambda_{j_k}$ - is called the Hessian of degree k of the eigenvalue vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Definition 1. (see [29]) A twice smooth function $u \in C^2(D)$ is called m - *convex* in $D \subset \mathbb{R}^n$, $u \in m - cv(D)$, if its eigenvalue vector $\lambda = \lambda(x) = (\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x))$ satisfies the conditions

$$H^k(\lambda(x)) \ge 0, \ \forall x \in D, \ k = 1, \ldots, n - m + 1.$$

It is clear that $1 - cv(D) \subset m - cv(D) \subset n - cv(D) = sh(D)$. The theory of $m - cv$ functions is a little-studied and new direction in the theory of real geometry. However, for $m = n$ the class $n - cv \cap C^2(D) = \{\lambda_1 + \lambda_2 + \ldots + \lambda_n \ge 0\}$ coincides with the class of subharmonic functions, and for $m = 1$ this class $1 - cv \cap C^2(D) = \{H^1(\lambda) \ge 0, \ldots, H^n(\lambda) \ge 0\} = \{\lambda_1 \ge 0, \ldots, \lambda_n \ge 0\}$ coincides with the well known class of convex functions in $\mathbb{R}^n$. The class of convex functions has been well studied in works A.Aleksandrov [3], I.Bakelman [5], A.Pogorelov [22], A.Artikbaev[4], and etc. When $m > 1$ this class has been studied in a series of works by N.Ivochkina, N. Trudinger, X.Wang and others (see [28]-[33]).

We study $m - cv(D)$ functions based on a new approaches. Namely, through their connection with the well-known $(B)sh_m$ functions: let us embed $\mathbb{R}^n_x$ into $\mathbb{C}^n_z$, $\mathbb{R}^n_x \subset \mathbb{C}^n_z = \mathbb{R}^n_x + i\mathbb{R}^n_y \ (z = x + iy)$, as a real n - dimensional subspace of the complex space $\mathbb{C}^n_z$.

Theorem 1. (see [26]) *A twice smooth function* $u(x) \in C^2(D)$, $D \subset \mathbb{R}^n_x$, *is* $m - cv$ *in* D *if and only if a function* $u^c(z) = u^c(x + iy) = u(x)$, *that does not depend on variables* $y \in \mathbb{R}^n_y$, *is* $(B)sh_m$ *in the domain* $D \times i\mathbb{R}^n_y$.

Remember, a twice smooth function $u(z) \in C^2(D)$, $D \subset \mathbb{C}^n$ is called $(B)sh_m$ $(1 \le m \le n)$ if at each point of the domain D the inequalities

$$(dd^c u)^k \wedge \beta^{n-k} \ge 0, \ \forall k = \overline{1, n - m + 1}$$

hold, where $\beta = dd^c \|z\|^2$ - is the standart volume form in $\mathbb{C}^n$.

Operators $(dd^c u)^k \wedge \beta^{n-k}$ closely related to the Hessians. For a twice smooth function $u \in C^2(D)$, the second-order differential $dd^c u = \frac{i}{2}\sum_{k,t}\frac{\partial^2 u}{\partial z_k \partial \bar{z}_t} dz_k \wedge d\bar{z}_t$ is Hermitian quadratic form. After a suitable unitary coordinate transformation, it is reduced to the diagonal form $dd^c u = \frac{i}{2}\left[\lambda_1 dz_1 \wedge d\bar{z}_1 + \ldots + \lambda_n dz_n \wedge d\bar{z}_n\right]$ where $\lambda_1,\ldots,\lambda_n$ - are the eigenvalues of the Hermitian matrix $\left(\frac{\partial^2 u}{\partial z_k \partial \bar{z}_t}\right)$, which are real: $\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{R}^n$. Therefore, it is easy to see that

$$(dd^c u)^k \wedge \beta^{n-k} = k!(n-k)!H^k(u)\beta^n,$$

where $H^k(u) = \sum_{1 \le j_1 < \ldots < j_k \le n} \lambda_{j_1} \ldots \lambda_{j_k}$ - is the Hessian of dimension k of the vector $\lambda = \lambda(u) \in \mathbb{R}^n$.

Hence, the twice smooth function $u(z) \in C^2(D)$, $D \subset \mathbb{C}^n$ is $(B)sh_m$, if at each point $o \in D$ it satisfies the following inequalities

$$H_o^k(u) \ge 0, \; k = 1,2,\ldots,n-m+1.$$

The class of $(B)sh_m$ functions was studied by S.Y.Li (see [19]), Z. Blocki (see [6]) and others, where a number of important properties of this class were proven. The potential theory in the class $(B)sh_m$ was constructed by Sadullaev-Abdullaev (see [25]). At this time, this theory has become a well-developed and relevant topic in the theory of functions. It is reflected in the works of a number of authors such as S. Dinew and S. Kolodziej (see [7], [8] and [9]), H.C.Lu (see [20]), N.S. Nguyen (see [21]), etc.

To study the geometric properties of $(B)sh_m$ functions, we need to consider the class of $m-sh$ functions.

Definition 2. An upper semi-continuous function $u(z)$ in the domain $D \subset \mathbb{C}^n$ is called $m-sh$ function if the restriction $u|_\Pi$, $\dim_{\mathbb{C}} \Pi = m$, is a subharmonic function in $D \cap \Pi$.

Note that a twice smooth function $u(z) \in C^2(D)$, $D \subset \mathbb{C}^n$, is $m-sh$ $(1 \le m \le n)$, if and only if at each point of the domain D the inequality

$$(dd^c u) \wedge \beta^{m-1} \ge 0$$

holds, i.e. if at each point $o \in D$ the inequalities

$$\lambda_{j_1} + \lambda_{j_2} + \ldots + \lambda_{j_m} \ge 0, \; 1 \le j_1 < \ldots < j_m \le n$$

are satisfied.

To construct a theory of potential in the class of $m-sh$ functions, Abdullaev introduced the class $(A)sh_m(D)$.

Definition 3. A twice smooth function $u(z)\in C^2(D),\ D\subset \mathbb{C}^n$ is called $(A)sh_m$ $(1\le m\le n)$, if at each point of the domain D the inequalities

$$(dd^c u)\wedge\beta^{m-1}\ge 0,\ (dd^c u)^{n-m+1}\wedge\beta^{m-1}\ge 0,$$

hold, i.e. if at each point $o\in D$ the inequalities

$$\lambda_{j_1}+\lambda_{j_2}+\ldots+\lambda_{j_m}\ge 0,\ 1\le j_1<\ldots<j_m\le n \text{ and } H_o^{n-m+1}(u)\ge 0$$

are satisfied.

In the work [24], A. Sadullaev proved that classes $(A)sh_m(D)\cap C^2(D)$ and $(B)sh_m(D)\cap C^2(D)$ coincide for $m=2$, and raised the question of whether $(B)sh_m(D)\cap C^2(D)=(A)sh_m(D)\cap C^2(D)$ for arbitrary $1\le m\le n$.

In the work [7], S. Dinew showed that for $n\le 7$ these classes actually coincide, but for $n=11,\ m=3$ these classes are different. In the work [2], we somewhat improved Dinew's result, proving that for $n=10,\ m=3$ Sadullaev's hypothesis is not true, but for $n=9,\ n=8$ and $m=3$ the hypothesis is true.

In this article we will consider these questions for the class of $m-$ convex functions.

2. $(A)m-cv$ functions and their relationship with $(A)sh_m$ functions

Definition 4. A twice smooth function $u\in C^2(D)$ is called $(A)m-cv$ in $D\subset\mathbb{R}^n$, if its eigenvalue vector $\lambda=\lambda(x)=(\lambda_1(x),\lambda_2(x),\ldots,\lambda_n(x))$ satisfies the conditions

$$\lambda_{j_1}(x)+\lambda_{j_2}(x)+\ldots+\lambda_{j_m}(x)\ge 0,\ 1\le j_1<\ldots<j_m\le n,\quad H^{n-m+1}(\lambda(x))\ge 0,\ \forall x\in D.$$

Theorem 2. *A twice smooth function $u(x)\in C^2(D),\ D\subset\mathbb{R}^n_x$, is $(A)m-cv$ in D if and only if a function $u^c(z)=u^c(x+iy)=u(x)$, that does not depend on variables $y\in\mathbb{R}^n_y$, is $(A)sh_m$ in the domain $D'\times\mathbb{R}^n_y$.*

Proof. Recall that $u^c(z)$ is $(A)sh_m$ if and only if the eigenvalues $\lambda_j = \lambda_j(z) \in \mathbb{R}$ - of the matrix $\left\| \dfrac{\partial^2 u^c(z)}{\partial z_k \partial \bar{z}_t} \right\|$ satisfies $\lambda_{j_1} + \lambda_{j_2} + \ldots + \lambda_{j_m} \geq 0,\ 1 \leq j_1 < \ldots < j_m \leq n$ and $H^{n-m+1}(\lambda) \geq 0$.

But

$$\frac{\partial^2 u^c(z)}{\partial z_k \partial \bar{z}_t} = \frac{\partial^2 u^c(x+iy)}{\partial z_k \partial \bar{z}_t} = \frac{\partial^2 u(x)}{\partial z_k \partial \bar{z}_t} = \frac{1}{4}\frac{\partial^2 u(x)}{\partial x_k \partial x_t}.$$

It follows that, the eigenvalues of the matrices $\left\| \dfrac{\partial^2 u^c(z)}{\partial z_k \partial \bar{z}_t} \right\|$ and $\left\| \dfrac{\partial^2 u(x)}{\partial x_k \partial x_t} \right\|$ differ by a fixed constant. Therefore, $u(x) \in (A)m-cv(D) \Leftrightarrow u^c(z) \in (A)sh_m(D \times i\mathbb{R}^n_y)$. *Theorem 2 is proved.*

Example 1. Let's check function $u(x) = \ln\left(x_1^2 + x_2^2 + \ldots + x_n^2\right)$ (1) on $(A)m-cv$. B.I.Abdullaev proved in article [1] that the function

$$u = \ln\left(x_1^2 + x_2^2 + x_3^2\right) = \ln[(z_1+\bar{z}_1)^2 + (z_2+\bar{z}_2)^2 + (z_3+\bar{z}_3)^2] - 2\ln 2$$

belongs to the classes $1-sh$ and $2-sh$. According to the above theorem $u(x) \in (A)m-cv(D) \Leftrightarrow u^c(z) \in (A)sh_m(D \times i\mathbb{R}^n_y)$. Therefore, we can check function

$$u^c(z) = \ln[(z_1+\bar{z}_1)^2 + (z_2+\bar{z}_2)^2 + \ldots + (z_n+\bar{z}_n)^2] \qquad (2)$$

on $(A)sh_m$. We find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the Hermitian matrix $\left\| \dfrac{\partial^2 u^c(z)}{\partial z_k \partial \bar{z}_t} \right\|$ of the function (2).

First, we carry out the following calculations:

$$\frac{\partial u}{\partial z_1} = \frac{1}{\left|z+\bar{z}\right|^2} 2\left(z_1+\bar{z}_1\right) = 2\frac{z_1+\bar{z}_1}{\left|z+\bar{z}\right|^2};$$

$$\frac{\partial u}{\partial z_2} = \frac{1}{\left|z+\bar{z}\right|^2} \cdot 2(z_2+\bar{z}_2) = 2\frac{z_2+\bar{z}_2}{\left|z+\bar{z}\right|^2};$$

$$\frac{\partial u}{\partial z_3} = \frac{1}{|z+\bar z|^2}\cdot 2(z_3+\bar z_3) = 2\frac{z_3+\bar z_3}{|z+\bar z|^2};$$

....

$$\frac{\partial u}{\partial z_n} = \frac{1}{|z+\bar z|^2}\cdot 2(z_n+\bar z_n) = 2\frac{z_n+\bar z_n}{|z+\bar z|^2};$$

1) $u_{1\bar 1} = \dfrac{\partial^2 u}{\partial z_1 \partial \bar z_1} = \dfrac{2|z+\bar z|^2}{|z+\bar z|^4} - \dfrac{2(z_1+\bar z_1)\cdot 2(z_1+\bar z_1)}{|z+\bar z|^4} = \dfrac{2}{|z+\bar z|^2} - \dfrac{4(z_1+\bar z_1)^2}{|z+\bar z|^4};$

2) $u_{1\bar 2} = \dfrac{\partial^2 u}{\partial z_1 \partial \bar z_2} = -\dfrac{2(z_1+\bar z_1)\cdot 2(z_2+\bar z_2)}{|z+\bar z|^4} = -4\dfrac{(z_1+\bar z_1)(z_2+\bar z_2)}{|z+\bar z|^4};$

3) $u_{1\bar 3} = \dfrac{\partial^2 u}{\partial z_1 \partial \bar z_3} = -4\dfrac{(z_1+\bar z_1)(z_3+\bar z_3)}{|z+\bar z|^4};$

4) $u_{2\bar 1} = \dfrac{\partial^2 u}{\partial z_2 \partial \bar z_1} = -4\dfrac{(z_1+\bar z_1)(z_2+\bar z_2)}{|z+\bar z|^4};$

5) $u_{2\bar 2} = \dfrac{\partial^2 u}{\partial z_2 \partial \bar z_2} = \dfrac{2}{|z+\bar z|^2} - \dfrac{4(z_2+\bar z_2)^2}{|z+\bar z|^4};$

6) $u_{2\bar 3} = \dfrac{\partial^2 u}{\partial z_2 \partial \bar z_3} = -4\dfrac{(z_2+\bar z_2)(z_3+\bar z_3)}{|z+\bar z|^4};$

7) $u_{3\bar 1} = \dfrac{\partial^2 u}{\partial z_3 \partial \bar z_1} = -4\dfrac{(z_1+\bar z_1)(z_3+\bar z_3)}{|z+\bar z|^4};$

8) $u_{3\bar 2} = \dfrac{\partial^2 u}{\partial z_3 \partial \bar z_2} = -4\dfrac{(z_2+\bar z_2)(z_3+\bar z_3)}{|z+\bar z|^4}$ $u_{3\bar 3} = \dfrac{\partial^2 u}{\partial z_3 \partial \bar z_3} = \dfrac{2}{|z+\bar z|^2} - \dfrac{4(z_3+\bar z_3)^2}{|z+\bar z|^4};$

$$\Rightarrow \quad u_{s\bar t} = \frac{\partial^2 u}{\partial z_s \partial \bar z_t} = -4\frac{(z_s+\bar z_s)(z_t+\bar z_t)}{|z+\bar z|^4}, \quad s,t = \overline{1,n};\ s \neq t$$

$$u_{s\bar s} = \frac{\partial^2 u}{\partial z_s \partial \bar z_s} = \frac{2}{|z+\bar z|^2} - \frac{4(z_s+\bar z_s)^2}{|z+\bar z|^4}, \quad s = \overline{1,n}\ .$$

Now we find the eigenvalues of the matrix $\left\| \dfrac{\partial^2 u^c(z)}{\partial z_k \partial \bar z_t} \right\|$:

$$\begin{vmatrix} u_{1\bar{1}}-\lambda & \cdots & u_{1\bar{n}} \\ \vdots & \ddots & \vdots \\ u_{n\bar{1}} & \cdots & u_{n\bar{n}}-\lambda \end{vmatrix} = \begin{vmatrix} \dfrac{2}{|z+\bar{z}|^2}-\dfrac{4(z_1+\bar{z}_1)^2}{|z+\bar{z}|^4}-\lambda & \cdots & -4\dfrac{(z_1+\bar{z}_1)(z_n+\bar{z}_n)}{|z+\bar{z}|^4} \\ \vdots & \ddots & \vdots \\ -4\dfrac{(z_1+\bar{z}_1)(z_n+\bar{z}_n)}{|z+\bar{z}|^4} & \cdots & \dfrac{2}{|z+\bar{z}|^2}-\dfrac{4(z_n+\bar{z}_n)^2}{|z+\bar{z}|^4}-\lambda \end{vmatrix} =$$

$$= \left(-4\frac{(z_1+\bar{z}_1)}{|z+\bar{z}|^4}\right)\cdots\left(-4\frac{(z_n+\bar{z}_n)}{|z+\bar{z}|^4}\right)\cdot$$

$$\begin{vmatrix} \dfrac{|z+\bar{z}|^4}{-4(z_1+\bar{z}_1)}\left(\dfrac{2}{|z+\bar{z}|^2}-\dfrac{4(z_1+\bar{z}_1)^2}{|z+\bar{z}|^4}-\lambda\right) & \cdots & (z_n+\bar{z}_n) \\ \vdots & \ddots & \vdots \\ (z_1+\bar{z}_1) & \cdots & \dfrac{|z+\bar{z}|^4}{-4(z_n+\bar{z}_n)}\left(\dfrac{2}{|z+\bar{z}|^2}-\dfrac{4(z_n+\bar{z}_n)^2}{|z+\bar{z}|^4}-\lambda\right) \end{vmatrix} = 0$$

To make it convenient to find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, we make the following substitutions:

$$x_k = \frac{2}{|z+\bar{z}|^2}-\frac{4(z_k+\bar{z}_k)^2}{|z+\bar{z}|^4}-\lambda,\quad k=\overline{1,n},\quad a_k = z_k+\bar{z}_k,\ k=\overline{1,n}.$$

Thus, we come to the calculation of the following determinant (see [23]):

$$\begin{vmatrix} x_1 & a_2 \cdots & a_n \\ a_1 & \ddots & \vdots \\ \vdots & & \\ a_1 & a_2 \cdots & x_n \end{vmatrix} = (x_1-a_1)(x_2-a_2)\cdots(x_n-a_n)\left(1+\frac{a_1}{x_1-a_1}+\frac{a_2}{x_2-a_2}+\ldots+\frac{a_n}{x_n-a_n}\right),$$

$$x_1-a_1 = \frac{|z+\bar{z}|^4}{-4(z_1+\bar{z}_1)}\left(\frac{2}{|z+\bar{z}|^2}-\frac{4(z_1+\bar{z}_1)^2}{|z+\bar{z}|^4}-\lambda\right)-(z_1+\bar{z}_1) = \frac{2|z+\bar{z}|^2-|z+\bar{z}|^4\lambda}{-4(z_1+\bar{z}_1)}$$

$$\frac{a_1}{x_1-a_1}=\frac{-4(z_1+\bar z_1)^2}{2\left|z+\bar z\right|^2-\left|z+\bar z\right|^4\lambda},\ \ldots,\ \frac{a_n}{x_n-a_n}=\frac{-4(z_n+\bar z_n)^2}{2\left|z+\bar z\right|^2-\left|z+\bar z\right|^4\lambda}$$

$$1+\frac{a_1}{x_1-a_1}+\frac{a_2}{x_2-a_2}+\ldots+\frac{a_n}{x_n-a_n}=1-\frac{4}{2-\left|z+\bar z\right|^2\lambda}=\frac{-2-\left|z+\bar z\right|^2\lambda}{2-\left|z+\bar z\right|^2\lambda}\Rightarrow$$

$$\begin{vmatrix} u_{1\bar 1}-\lambda & \cdots & u_{1\bar n} \\ \vdots & \ddots & \vdots \\ u_{n\bar 1} & \cdots & u_{n\bar n}-\lambda \end{vmatrix}=\frac{\left(2-\left|z+\bar z\right|^2\lambda\right)^{n-1}\left(-2-\left|z+\bar z\right|^2\lambda\right)}{\left|z+\bar z\right|^{2n}}=$$

$$=-\left(\frac{2}{\left|z+\bar z\right|^2}-\lambda\right)^{n-1}\left(\frac{2}{\left|z+\bar z\right|^2}-\lambda\right)=0\qquad\Rightarrow$$

$$\lambda_{1,2,3,\ldots,n-1}=\frac{2}{\left|z+\bar z\right|^2},\quad \lambda_n=-\frac{2}{\left|z+\bar z\right|^2}.$$

In the case $n=2$, the eigenvalues of the Hermitian matrix $\left\|\frac{\partial^2 u^c(z)}{\partial z_k \partial \bar z_t}\right\|$ of the function (2) are equal to $\lambda_{1,2}=\pm\frac{2}{\left|z+\bar z\right|^2}$, and $H^1(\lambda)=\lambda_1+\lambda_2=0,\ H^2(\lambda)=\lambda_1\cdot\lambda_2<0.$

In the case $n=3$, the eigenvalues of the Hermite matrix $\left\|\frac{\partial^2 u^c(z)}{\partial z_k \partial \bar z_t}\right\|$ of the function (2) are equal to $\lambda_{1,2}=\frac{2}{\left|z+\bar z\right|^2},\ \lambda_3=-\frac{2}{\left|z+\bar z\right|^2};$ $H^1(\lambda)=\lambda_1+\lambda_2+\lambda_3>0,\ H^2(\lambda)=\lambda_1\cdot\lambda_2+\lambda_1\cdot\lambda_3+\lambda_2\cdot\lambda_3<0,\ H^3(\lambda)=\lambda_1\cdot\lambda_2\cdot\lambda_3<0.$

If $n=4$, then $H^1(\lambda)>0,\ H^2(\lambda)=0,\ H^3(\lambda)<0, H^4(\lambda)<0.$

For $n=2k+1$ we know, $\lambda_{1,2,\ldots,2k-1,2k}=\frac{2}{\left|z+\bar z\right|^2},\quad \lambda_{2k+1}=-\frac{2}{\left|z+\bar z\right|^2}.$

The sum $H^s(\lambda)$ has C^s_{2k+1} terms and C^s_{2k} of them are positive, since these terms do not contain λ_{2k+1}. The number of participants in λ_{2k+1} is

$C^s_{2k+1} - C^s_{2k} = C^{s-1}_{2k}$, that is, in the sum $H^s(l)$ $\dfrac{(2k)!}{(s-1)!(2k-s+1)!}$ terms are negative, $\dfrac{(2k)!}{s!(2k-s)!}$ terms are positive and all terms have the same value. It is easy to see that the inequality $\dfrac{(2k)!}{(s-1)!(2k-s+1)!} \le \dfrac{(2k)!}{s!(2k-s)!}$ holds for $s \le k + \frac{1}{2}$. So, $H^s(l) \ge 0$ for $s = \overline{1,k}$ and $H^s(l) < 0$ for $s = \overline{(k+1),2k+1}$. For $n = 2k$ we can see that $H^1(l), H^2(l), \ldots, H^k(l) \ge 0,\ H^{k+1}(l) < 0, \ldots, H^{2k}(l) < 0$ based on the same considerations as above. In general, the inequalities $H^1(l), \ldots, H^m(l) \ge 0,\ H^{m+1}(l) < 0, \ldots, H^n(l) < 0$ hold for $1 \le m \le \left[\frac{n}{2}\right] n \ge 2$.

It follows that for $n - \left[\frac{n}{2}\right] + 1 \le m \le n,\ n \ge 2$ the inequalities $H^1(l), \ldots, H^{n-m+1}(l) \ge 0,\ H^{n-m+2}(l) < 0, \ldots, H^n(l) < 0$ are valid.

Thus, we come to the following conclusion:

Proposition1. The function $u^c(z) = \ln[(z_1 + \bar{z}_1)^2 + (z_2 + \bar{z}_2)^2 + \ldots + (z_n + \bar{z}_n)^2]$ is $m-sh$ function for $2 \le m \le n$.

Proposition2. The function $u^c(z) = \ln[(z_1 + \bar{z}_1)^2 + (z_2 + \bar{z}_2)^2 + \ldots + (z_n + \bar{z}_n)^2]$ is in $(B)sh_m$ and $(A)sh_m$ for $n - \left[\frac{n}{2}\right] + 1 \le m \le n, n \ge 2$. When $m < n - \left[\frac{n}{2}\right] + 1,\ n \ge 2$ this function is not in $(B)sh_m$ and $(A)sh_m$.

Proposition 3. The function $u(x) = \ln\left(x_1^2 + x_2^2 + \ldots + x_n^2\right)$ is $m-cv$ and $(A)m-cv$ function for $n - \left[\frac{n}{2}\right] + 1 \le m \le n, n \ge 2$. When $m < n - \left[\frac{n}{2}\right] + 1,\ n \ge 2$ this function is not $m-cv$ and $(A)m-cv$.

3. $m-psh$ functions

$m-psh$ functions are analogue of $m-sh$ functions in real Euclidian space $\mathbb{R}^n$. In the works of R. Harvey and B. Lawson, D. Joyce, B. Drnjvzek and F. Forstnerich [10-18] the class of functions that are subharmonic on m-dimensional real planes in Euclidean space $\mathbb{R}^n$ is considered. They called such functions m-plurisubharmonic $(m-psh)$functions, which were then successfully used in convex geometry, in descriptions of convex hulls of geometric bodies. Class of m- plurisubharmonic functions and its rich properties have found a number of applications in calibrated geometry, in the study of minimal surfaces on manifolds.

Definition 5. A function $u(x)\in L^1_{loc}(D)$, given in a domain $D\subset\mathbb{R}^n$ is to be $m-psh$ *function* (subharmonic function on real plans of dimension m), $1\le m\le n$, in D if:

1) it is upper semicontinuous in D, i.e.,

$$\overline{\lim_{x\to x^0}}\, u(x)=\lim_{\varepsilon\to 0}\sup_{B(x^0,\varepsilon)}u(x)\le u(x^0);$$

2) for any $\Pi\subset\mathbb{R}^n$, $\dim_{\mathbb{R}}\Pi=m$, the restriction

$$u\big|_{\Pi}\in sh(\Pi\cap D).$$

Let us present the following properties of functions that we need later.

1) (see [27]) $m-cv\subset m-psh$;

2) $m-cv(D)\cap C^2(D)\subset(A)m-cv(D)\cap C^2(D)\subset m-psh(D)\cap C^2(D)$;

3) a linear combination of $m-psh$ functions with non-negative coefficients is $m-psh$ function, i.e. if

$$u_j(z)\in m-psh(D),\quad a_j\in R_+\quad (j=1,2,\ldots,N)\ \Rightarrow$$

$$a_1u_1(z)+a_2u_2(z)+\ldots+a_Nu_N(z)\in m-psh(D);$$

4) the limit of a monotonically decreasing sequence of $m-psh$ functions is $m-psh$ function, i.e.

$$u_j(z)\in m-psh(D),\ u_j(z)\ge u_{j+1}(z),\ (j=1,2,\ldots)\Rightarrow \lim_{j\to\infty}u_j(z)\in m-psh(D);$$

5) a uniformly convergent sequence of $m-psh$ functions converges to $m-psh$ function , i.e. if

$u_j(z)\in m-psh(D),\ (j=1,2,\ldots),\ u_j(z)\rightrightarrows u(z)$, then $u(z)\in m-psh(D)$;

4) (maximum principle). Let $u(z)\in m-psh(D)$. If at some point $z^0\in D$

$u(z)$ reaches its maximum, i.e. $u(z^0)=\sup\limits_{z\in D}u(z)$ then $u(z)\equiv const$.

4. On equality of the functions $(A)m-cv$ and $m-cv$

Theorem 3. *(An analogue of A. Sadullaev's theorem) Let $D\subset\mathbb{R}^n$. Then the equality $(A)2-cv(D)\cap C^2(D)=2-cv(D)\cap C^2(D)$ is true.*

Proof. Let $u(x)\in(A)2-cv(D)\cap C^2(D)$. Then by Theorem 2, $u^c(z)\in(A)sh_2(D\times i\mathbb{R}^n_y)\cap C^2(D\times i\mathbb{R}^n_y)$. If $u^c(z)\in(A)sh_2(D\times i\mathbb{R}^n_y)\cap C^2(D\times i\mathbb{R}^n_y)$ then $u^c(z)\in(B)sh_2(D\times i\mathbb{R}^n_y)\cap C^2(D\times i\mathbb{R}^n_y)$ was proven in [24]. So, according to theorem 1, $u(x)\in 2-cv(D)\cap C^2(D)$. *Theorem 3 is proved.*

Theorem 4. *(An analogue of S. Dinew's theorem). Let $D\subset\mathbb{R}^n$, $1\le m\le n\le 7$. Then the equality $(A)m-cv(D)\cap C^2(D)=m-cv(D)\cap C^2(D)$ is true.*

Proof. Let $u(x)\in(A)m-cv(D)\cap C^2(D)$. By Theorem 2 $u^c(z)\in(A)sh_m(D\times i\mathbb{R}^n_y)\cap C^2(D\times i\mathbb{R}^n_y)$. If $u^c(z)\in(A)sh_m(D\times i\mathbb{R}^n_y)\cap C^2(D\times i\mathbb{R}^n_y)$ then $u^c(z)\in(B)sh_m(D\times i\mathbb{R}^n_y)\cap C^2(D\times i\mathbb{R}^n_y)$ for $1\le m\le n\le 7$ was proven in [7]. So, by Theorem 1 $u(x)\in m-cv(D)\cap C^2(D)$, $1\le m\le n\le 7$. *Theorem 4 is proved.*

Theorem 5. *(An analogue of the theorem of B. Abdullaev, D. Qalandarova) Let $D\subset\mathbb{R}^n$. Then for $n=9$ and $n=8$ the equality*

$$(A)3-cv(D)\cap C^2(D)=3-cv(D)\cap C^2(D)$$

is true.

Proof. Let $u(x)\in(A)3-cv(D)\cap C^2(D)$. Then by Theorem 2 $u^c(z)\in(A)sh_3(D\times i\mathbb{R}^n_y)\cap C^2(D\times i\mathbb{R}^n_y)$. If $u^c(z)\in(A)sh_3(D\times i\mathbb{R}^n_y)\cap C^2(D\times i\mathbb{R}^n_y)$, then

$u^c(z)\in(B)sh_3\left(D\times\mathbb{C}^n_y\right)\cap C^2\left(D\times\mathbb{C}^n_y\right)$ for $n=9$ and $n=8$ was proven in [2]. So, by Theorem 1 $u(x)\in 3-cv(D)\cap C^2(D)$ for $n=9$ and $n=8$. *Theorem 5 is proved.*

Now we will show in an example that the class of functions $(A)m-cv$ and $m-cv$ do not coincide when $n=10, m=3$.

Example 2. We take a function

$$u(x_1,x_2,\dots,x_{10})=-\sum_{s=1}^{2}x_s^2+2\sum_{t=3}^{10}x_t^2.$$

First, we carry out the following calculations:

$$\frac{\partial u}{\partial x_1}=-2x_1,\ \frac{\partial u}{\partial x_2}=-2x_2,\ \frac{\partial u}{\partial x_3}=4x_3,\dots,\ \frac{\partial u}{\partial x_{10}}=4x_{10};$$

$$u_{11}=\frac{\partial^2 u}{\partial x_1^2}=-2,\ u_{22}=\frac{\partial^2 u}{\partial x_2^2}=-2,\ u_{33}=\frac{\partial^2 u}{\partial x_3^2}=4,\dots,u_{10,10}=\frac{\partial^2 u}{\partial x_{10}^2}=4;$$

$$u_{st}=\frac{\partial^2 u}{\partial x_s\partial x_t}=0,\ t\neq s.$$

$$\left(\frac{\partial^2 u}{\partial x_s\partial x_t}\right)\to\begin{pmatrix}2&0&\dots&0\\0&-2&\dots&0\\\dots&\dots&\dots&\dots\\0&0&\dots&4\end{pmatrix},$$

We can see from this that $\lambda_1=\lambda_2=-2,\ \lambda_3=\lambda_4=\dots=\lambda_{10}=4$ and

$$H^8(\lambda)=\lambda_1\lambda_2\left(\sum_{3\le j_1<\dots<j_6\le 10}\lambda_{j_1}\lambda_{j_2}\dots\lambda_{j_6}\right)+\lambda_1\left(\sum_{3\le j_1<\dots<j_7\le 10}\lambda_{j_1}\lambda_{j_2}\dots\lambda_{j_7}\right)+$$
$$+\lambda_2\left(\sum_{3\le j_1<\dots<j_7\le 10}\lambda_{j_1}\lambda_{j_2}\dots\lambda_{j_7}\lambda\right)+\lambda_3\lambda_4\dots\lambda_{10}=0,$$

the following inequalities also hold $\lambda_{j_1}+\lambda_{j_2}+\lambda_{j_3}\ge 0,\quad 1\le j_1<j_2<j_3\le 10$.

So, $u(x)\in(A)3-cv\left(\mathbb{R}^{10}\right)$. Since

$$H^7(\lambda)=\lambda_1\lambda_2\left(\sum_{3\le j_1<\ldots<j_5\le 10}\lambda_{j_1}\lambda_{j_2}\ldots\lambda_{j_5}\right)+\lambda_1\left(\sum_{3\le j_1<\ldots<j_6\le 10}\lambda_{j_1}\lambda_{j_2}\ldots\lambda_{j_6}\right)+$$
$$+\lambda_2\left(\sum_{3\le j_1<\ldots<j_6\le 10}\lambda_{j_1}\lambda_{j_2}\ldots\lambda_{j_6}\right)+\sum_{3\le j_1<\ldots<j_7\le 10}\lambda_{j_1}\lambda_{j_2}\ldots\lambda_{j_7}=-6\cdot 4^7<0,$$

it follows that $u(x)\notin 3-cv(\mathbb{C}^{10})$.

So, in the case of $n=10, m=3,$ classes $(A)m-cv$ and $m-cv$ do not coincide, that is, when $n\ge 10,$ these classes are different.

In cases where $n=8, n=9$ and $4\le m\le n$ we do not know whether these classes coincide or not.

Conclusion. The theorems on the equality of $m-$ convex and $(A)m-$ convex functions proved in the paper are of theoretical importance, and these theorems can be used in potential theory and mathematical physics.

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Integration of the loaded modified Korteweg-de Vries equation with an integral source in the case of multiple eigenvalues in the class of rapidly decreasing functions

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Abstract

This work is devoted to integration of the loaded modified Korteweg-de Vries equation with an integral source using the method of inverse problems of scattering theory in the class of rapidly decreasing functions. In this paper, we consider the case when the Dirac operator included in the Lax pairs is not self-adjoint, therefore the eigenvalues of the Dirac operator can be multiples. The evolution of scattering data is obtained for the non-self-adjoint Dirac operator, the potential of which is a solution of the loaded modified Korteweg-de Vries equation with an integral source in the class of rapidly decreasing functions. An example is given to illustrate the application of the results obtained.

Introduction

Modified Korteweg-de Vries equation (mKdV)

$$u_t \pm 6u^2 u_x + u_{xxx} = 0,$$

describing the nonlinear propagation of waves in systems with polar symmetry, is one of the most well-known models that allow the existence of solitons in the solution. This equation can be applied in many areas, anharmonic lattices [1], Alfven waves [2] and lines transmission through the Schottky barrier [3]. To calculate the exact solutions of the mKdV equation, many methods have been created. For example, one of these methods is the inverse scattering method. This method greatly contributed to the development of the solitons theory. Solitons theory in mathematics is one of the important concepts connecting physics and mathematics. Issues of the theory of solitons and the integration of various nonlinear evolution equations are considered in a lot of monographs and articles, of which it should be noted [4-9].

In modern literature, if the value of the solution or its derivative at a point participates in the coefficients or on the right side of the equation, such equations are called the loaded equations. The study of integrable loaded nonlinear evolution equations is developing because these equations make it possible to model various complex nonlinear phenomena. Among the works devoted to loaded equations, works [10-19] and others should be especially noted.

In this article, the integration of the loaded modified Korteweg-de Vries equation with an integral source in the class of rapidly decreasing functions is studied using the inverse scattering method.

Let us consider the following system of equations

$$u_t + P\big(u(x_0,t)\big)\big(6u^2 u_x + u_{xxx}\big) + Q\big(u(x_1,t)\big)u_x = iF\big(u(x_2,t)\big)\int_{-\infty}^{\infty}\big(\phi_1^2 - \phi_2^2\big)\,d\eta, \qquad (1)$$

$$L(t)\phi = \eta\phi, \qquad x \in \mathbb{R},$$

where $L(t)=i\begin{pmatrix}\frac{d}{dx} & -u(x,t)\\ -u(x,t) & -\frac{d}{dx}\end{pmatrix}$ and $P\left(u(x_0,t)\right)$, $Q\left(u(x_1,t)\right)$, $F\left(u(x_2,t)\right)$ functions are polynomials of the following form

$$P\left(u(x_0,t)\right)=a_0(t)+a_1(t)u(x_0,t)+...+a_m(t)\left(u(x_0,t)\right)^m,$$

$$Q\left(u(x_1,t)\right)=b_0(t)+b_1(t)u(x_0,t)+...+b_s(t)\left(u(x_0,t)\right)^s,$$

$$F\left(u(x_2,t)\right)=c_0(t)+c_1(t)u(x_2,t)+...+c_l(t)\left(u(x_0,t)\right)^l$$

with degree not exceeding $n=\max\{m,s,l\}$. Where $a_1(t)\neq 0$, $b_1(t)\neq 0$, $c_1(t)\neq 0$, $a_j(t), j=\overline{0,m}$, $b_j(t), j=\overline{0,s}$, $c_j(t)\in C[0,+\infty), j=\overline{0,l}$.

Equation (1) is considered under the initial condition

$$u(x,0)=u_0(x). \tag{2}$$

In the problem under consideration, the initial function $u_0(x)$ $(-\infty<x<\infty)$ has the following properties:

1) $$\int_{-\infty}^{\infty}(1+|x|)|u_0(x)|\,dx<\infty, \tag{3}$$

2) operator $L(0)=i\begin{pmatrix}\frac{d}{dx} & -u_0(x)\\ -u_0(x) & -\frac{d}{dx}\end{pmatrix}$ in the upper half-plane of the complex plane has exactly N eigenvalues $\xi_1(0),\xi_2(0),...,\xi_N(0)$ with multiplicities $m_1(0),m_2(0),...,m_N(0)$ and has no spectral singularities.

Function $\phi=\left(\phi_1(x,\eta,t),\phi_2(x,\eta,t)\right)^T$ has the following asymptotics for $x\to\infty$

$$\phi\to\begin{pmatrix}h(\eta,t)e^{-i\eta x}\\ h(\eta,t)e^{i\eta x}\end{pmatrix}, \tag{4}$$

where $h(\eta,t)=h(-\eta,t)$ is a continuous function satisfying the condition:

$$\int_{-\infty}^{\infty}|h(\eta,t)|^2\,d\eta<\infty,\ \text{for}\ \ t\geq 0. \tag{5}$$

Let us assume that function $u(x,t)$ has the required smoothness and quickly tends to its limits under $x\to\pm\infty$, i.e.

$$\int_{-\infty}^{\infty}\left((1+|x|)|u(x,t)|+\sum_{k=1}^{3}\left|\frac{\partial^k u(x,t)}{\partial x^k}\right|\right)dx<\infty,\ \ k=1,2,3. \tag{6}$$

The main goal of this work is to obtain representations for the solution $\{u(x,t),\phi_1(x,\eta,t),\phi_2(x,\eta,t)\}$ of problem (1)–(6) within the framework of the inverse scattering method for the operator $L(t)$.

Preliminaries

Let us consider the following system of Dirac equations

$$\begin{cases} v_{1x}+i\xi\, v_1 = u_0(x)v_2\,, \\ v_{2x}-i\xi\, v_2 = -u_0(x)v_1, \end{cases} \tag{7}$$

on the entire axis ($-\infty < x < \infty$), with potential $u_0(x)$ satisfying condition (3). It can be seen that using the operator $L(0)$ and vector function $v=(v_1,v_2)$, system of equations (7) can be rewritten in the form

$$L(0)v=\xi v\,. \tag{8}$$

System of equations (7) has Jost solutions with the following asymptotics

$$\left.\begin{aligned} \varphi(x,\xi) &\sim \begin{pmatrix}1\\0\end{pmatrix}e^{-i\xi x} \\ \tilde\varphi(x,\xi) &\sim \begin{pmatrix}0\\-1\end{pmatrix}e^{i\xi x} \end{aligned}\right\},\ \operatorname{Im}\xi=0,\ \ x\to-\infty; \qquad \left.\begin{aligned} \psi(x,\xi) &\sim \begin{pmatrix}0\\1\end{pmatrix}e^{i\xi x} \\ \tilde\psi(x,\xi) &\sim \begin{pmatrix}1\\0\end{pmatrix}e^{-i\xi x} \end{aligned}\right\},\ \operatorname{Im}\xi=0,\ x\to\infty\,. \tag{9}$$

For real ξ, the pairs of vector functions $\{\varphi,\tilde\varphi\}$ and $\{\psi,\tilde\psi\}$ are pairs of linearly independent solutions for the system of equations (7). Therefore, the following relations hold

$$\left.\begin{aligned} \varphi &= a(\xi)\tilde\psi+b(\xi)\psi, \\ \tilde\varphi &= -\bar a(\xi)\psi+\bar b(\xi)\tilde\psi \end{aligned}\right\} \ \text{ and } \ \left.\begin{aligned} \psi &= -a(\xi)\tilde\varphi+\bar b(\xi)\varphi, \\ \tilde\psi &= \bar a(\xi)\varphi+b(\xi)\tilde\varphi \end{aligned}\right\}, \tag{10}$$

where $a(\xi)=W\{\varphi,\psi\}$, $b(\xi)=W\{\tilde\varphi,\psi\}$. The following equalities are true

$$|a(\xi)|^2+|b(\xi)|^2=1\,, \qquad \bar a(\xi)=a(-\xi)\,, \qquad \bar b(\xi)=b(-\xi)\,. \tag{11}$$

Coefficients $a(\xi)$ and $b(\xi)$ are continuous under $\xi\in R$ and satisfy the asymptotic equalities: $a(\xi)=1+O\left(|\xi|^{-1}\right), b(\xi)=O\left(|\xi|^{-1}\right),\ |\xi|\to\infty.$

The non-real zeros $\{\xi_k\}_{k=1}^N$ of the function $a(\xi)$ are the eigenvalues of the operator $L(0)$ in the upper half-plane $\operatorname{Im}\xi>0$. The eigenvalues of operator $L(0)$ in the lower half-plane $\operatorname{Im}\xi<0$ coincide with the zeros of function $\bar a(\xi)$. So, set $\{\xi_k,-\xi_k\}_{k=1}^N$ is the eigenvalues of operator $L(0)$, and this operator has no other eigenvalues. The requirement that operator $L(0)$ has no spectral singularities means that function $a(\xi)$ does not have real zeros, i.e. $a(\xi)\neq 0,\ \xi\in\mathbb{R}$.

There is such a chain of numbers $\left\{\chi_0^k,\chi_1^k,\ldots,\chi_{m_k-1}^k\right\}$, that the relations

$$\overset{(l)}{\varphi}(x,\xi_k)=\sum_{\nu=0}^{l}\chi_{l-\nu}^k\frac{l!}{\nu!}\overset{(\nu)}{\psi}(x,\xi_k),\ \ k=\overline{1,N}, \qquad l=\overline{0,m_k-1}\,, \tag{12}$$

hold. Sequence of numbers $\left\{\chi_0^k,\chi_1^k,\ldots,\chi_{m_k-1}^k\right\}$ are called the normalizing chain of the non-self-adjoint operator $L(0)$.

The following integral representation is valid for the function $\psi(x,\xi)$ [see 20, p. 33]

$$\psi(x,\xi)=\begin{pmatrix}0\\1\end{pmatrix}e^{i\xi x}+\int_x^{\infty}\mathbf{K}(x,s)e^{i\xi s}ds, \tag{13}$$

where $\mathbf{K}(x,s)=(K_1(x,s),K_2(x,s))^T$. In representation (13), the kernel $\mathbf{K}(x,s)$ does not depend on ξ and the equality

$$u(x)=-2K_1(x,x), \tag{14}$$

is satisfied. The components of the kernel $K(x,y)$ with $y>x$ are solutions to the system of integral equations of Gelfand-Levitan-Marchenko

$$K_2(x,y)+\int_x^{\infty}K_1(x,s)F(s+y)ds=0,$$

$$-K_1(x,y)+F(x+y)+\int_x^{\infty}K_2(x,s)F(s+y)ds=0,$$

here

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}r^+(\xi)e^{i\xi x}d\xi-i\sum_{k=1}^{N}\sum_{\nu=0}^{m_k-1}\chi^k_{m_k-\nu-1}\frac{1}{\nu!}\frac{d^\nu}{dz^\nu}\left[\frac{(z-\xi_k)^{m_k}}{a(z)}e^{izx}\right]\Bigg|_{z=\xi_k},$$

$r^+(\xi)\equiv\dfrac{b(\xi)}{a(\xi)}$, $a(z)$ – analytical continuation of function $a(\xi)$, $\operatorname{Im}\xi=0$ into the upper half-plane $\operatorname{Im}z>0$, which is determined by the formula

$$a(z)=\exp\left\{-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln\left(1+|r^+(\xi)|\right)}{\xi-z}d\xi\right\}\prod_{j=1}^{N}\left(\frac{\xi-\xi_j}{\xi-\bar{\xi}_j}\right)^{m_j}.$$

Now the potential $u(x)$ is determined from equality (14).

Definition 1. The set of quantities

$$\left\{r^+(\xi),\ \xi\in R^1;\ \xi_k,\operatorname{Im}\xi_k>0;\ \chi^k_j,\quad k=\overline{1,N};\ j=\overline{0,m_k-1}\right\}$$

is called the scattering data for system (7).

The following theorem is true [see 21, §6.2].

Theorem 1. The scattering data of operator L uniquely determine L.

In what follows we will often use the results of the following lemmas.

Lemma 1. If vector functions $Y=(y_1(x,\xi),y_2(x,\xi))^T$ and $Z=(z_1(x,\eta),z_2(x,\eta))^T$ are the solutions to equations $LY=\xi Y$ and $LZ=\eta Z$, then their components satisfy equalities

$$\frac{d}{dx}(y_1z_1+y_2z_2)=-i(\xi+\eta)(y_1z_1-y_2z_2),\quad \frac{d}{dx}(y_1z_2-y_2z_1)=-i(\xi-\eta)(y_1z_2+y_2z_1).$$

Lemma 2. Let vector functions $\varphi(x,\xi), f_k^s(x,t)$, $s=\overline{0,m_k-1}$ be solutions to the following equations

$$L(t)\varphi=\xi\varphi,\quad L(t)f_k^s=\xi_kf_k^s+sf_k^{s-1},\quad s=\overline{0,m_k-1},$$

then equalities

$$\frac{d}{dx}\left(f_{k1}^{s}\varphi_2 - f_{k2}^{s}\varphi_1\right) = i\left(\xi-\xi_k\right)\left(f_{k1}^{s}\varphi_2 + f_{k2}^{s}\varphi_1\right) - is\left(f_{k1}^{s-1}\varphi_2 + f_{k2}^{s-1}\varphi_1\right), \tag{15}$$

$$\frac{d}{dx}\left(f_{k_1}^{s}\varphi_1 + f_{k_2}^{s}\varphi_2\right) = -i\left(\xi+\xi_k\right)\left(f_{k_1}^{s}\varphi_1 - f_{k_2}^{s}\varphi_2\right) - is\left(f_{k_1}^{s-1}\varphi_1 - f_{k_2}^{s-1}\varphi_2\right),\ s=\overline{0,m_k-1}. \tag{16}$$

are valid.

These lemmas are proved by direct verification.

Corollary. If the conditions of Lemma 2 are met, the following equalities are valid

$$f_{k_1}^{s}\varphi_1 - f_{k_2}^{s}\varphi_2 = i\sum_{l=0}^{s}\frac{(-1)^l}{\left(\xi+\xi_k\right)^{l+1}}\frac{s\,!}{(s-l)\,!}\frac{d}{dx}V\left\{f_k^{s-l},\varphi\right\}, \tag{17}$$

for $\xi \neq \xi_k$

$$f_{k_1}^{s}\varphi_2 + f_{k_2}^{s}\varphi_1 = i\sum_{l=0}^{s}\frac{(-1)^l}{\left(\xi_k-\xi\right)^{l+1}}\frac{s\,!}{(s-l)\,!}\frac{d}{dx}W\left\{f_k^{s-l},\varphi\right\},\ s=\overline{0,m_k-1}, \tag{18}$$

where $V\{f,g\} \equiv f_1g_1 + f_2g_2$.

The following Lemma can also be proven by direct verification.

Lemma 3. If $\varphi_k = \left(\varphi_{k_1},\varphi_{k_2}\right)^T$ is an eigenvector function of the operator L with potential $u(x)$, corresponding to the eigenvalue ξ_k, then the equalities

$$\int_{-\infty}^{\infty} u\left(\varphi_{k1}^2 - \varphi_{k2}^2\right)dx = 0, \qquad \int_{-\infty}^{\infty} u_x\left(\varphi_{k1}^2 + \varphi_{k2}^2\right)dx = 0,$$

$$\int_{-\infty}^{\infty} u^3\left(\varphi_{k1}^2 - \varphi_{k2}^2\right)dx = 0, \qquad \int_{-\infty}^{\infty} u_{xx}\left(\varphi_{k1}^2 - \varphi_{k2}^2\right)dx = 0$$

are valid.

Evolution of the scattering data

Let the potential $u(x,t)$ in the system of equations $LY = \xi Y$ be a solution to the following equation

$$u_t + P\left(u(x_0,t)\right)(u_{xxx} + 6u^2u_x) = G(x,t), \tag{19}$$

where $G(x,t) = -Q\left(u(x_1,t)\right)u_x(x,t) + iF\left(u(x_2,t)\right)\int_{-\infty}^{\infty}\left(\phi_1^2 - \phi_2^2\right)d\eta$. Then the operator

$$A = P\left(u(x_0,t)\right)\begin{pmatrix} -4i\xi^3 + 2iu^2\xi & 4u\xi^2 + 2iu_x\xi - 2u^3 - u_{xx} \\ -4u\xi^2 + 2iu_x\xi + 2u^3 + u_{xx} & 4i\xi^3 - 2iu^2\xi \end{pmatrix}, \tag{20}$$

satisfies the following Lax relation

$$[L,A]\equiv LA-AL=i\,P\big(u(x_0,t)\big)\begin{pmatrix}0 & -6u^2u_x-u_{xxx}\\ -6u^2u_x-u_{xxx} & 0\end{pmatrix}. \tag{21}$$

Therefore, equation (19) can be rewritten as

$$L_t+[L,A]=iR, \tag{22}$$

where $R=\begin{pmatrix}0 & -G\\ -G & 0\end{pmatrix}$. Differentiating equality $L\varphi=\xi\varphi$ with respect to t, taking into account (22), we have

$$\big(L-\xi\big)\big(\varphi_t-A\varphi\big)=-iR\varphi. \tag{23}$$

Using the method of variation of constants, we can write

$$\varphi_t-A\varphi=B(x)\psi+D(x)\varphi. \tag{24}$$

Then to define $B(x)$ and $D(x)$ we obtain

$$\mathbf{M}B_x\psi+\mathbf{M}D_x\varphi=-R\varphi, \tag{25}$$

where $\mathbf{M}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. To solve the equation (25), it is convenient to introduce the following notation $\hat{\varphi}=\left(\varphi_2,\varphi_1\right)^T$, $\hat{\psi}=\left(\psi_2,\psi_1\right)^T$. According to (21) and the definition of Wronskian the following equalities are true

$$\hat{\psi}^T\mathbf{M}\varphi=-\hat{\varphi}^T\mathbf{M}\psi=a, \qquad \hat{\psi}^T\mathbf{M}\psi=\hat{\varphi}^T\mathbf{M}\varphi=0.$$

Multiplying (25) by $\hat{\varphi}^T$ and $\hat{\psi}^T$, we get

$$B_x=\frac{\hat{\varphi}^TR\varphi}{a},\quad D_x=-\frac{\hat{\psi}^TR\varphi}{a}. \tag{26}$$

According to (20), for $x\to-\infty$ we have

$$\varphi_t-A\varphi\to\begin{pmatrix}4i\xi^3P\big(u(x_0,t)\big)\\ 0\end{pmatrix}e^{-i\xi x}.$$

Therefore, based on (24) for $x\to-\infty$, we have $D(x)\to4i\xi^3P\big(u(x_0,t)\big)$, $B(x)\to0$. Therefore, from (25) we can determine

$$D(x)=-\frac{1}{a}\int_{-\infty}^{x}\hat{\psi}^TR\varphi\,dx+4i\xi^3P\big(u(x_0,t)\big),\quad B(x)=\frac{1}{a}\int_{-\infty}^{x}\hat{\varphi}^TR\varphi\,dx.$$

Thus, the equality (24) has the following form

$$\varphi_t-A\varphi=\frac{1}{a}\int_{-\infty}^{x}\hat{\varphi}^TR\varphi\,dx\psi+\left(-\frac{1}{a}\int_{-\infty}^{x}\hat{\psi}^TR\varphi\,dx+4i\xi^3P\big(u(x_0,t)\big)\right)\varphi. \tag{27}$$

According to (10), the equality (27) can be rewritten in the following form

$$a_t\tilde{\psi}+b_t\psi-A\big(a\tilde{\psi}+b\psi\big)=$$

$$=\frac{1}{a}\int_{-\infty}^{x}\hat{\varphi}^TR\varphi dx\psi+\left(-\frac{1}{a}\int_{-\infty}^{x}\hat{\psi}^TR\varphi\,dx+4i\xi^3P\big(u(x_0,t)\big)\right)\big(a\tilde{\psi}+b\psi\big).$$

Passing to the limit at $x \to +\infty$ in the last equality and taking into account (20), we can derive the following equalities

$$a_t = -\int_{-\infty}^{\infty} \hat{\psi}^T R\varphi dx, \qquad b_t = \frac{1}{a}\int_{-\infty}^{\infty} \hat{\varphi}^T R\varphi dx - \frac{b}{a}\int_{-\infty}^{\infty} \hat{\psi}^T R\varphi\, dx + 8i\xi^3 P\left(u(x_0,t)\right)b. \tag{28}$$

Consequently, for $\operatorname{Im}\xi = 0$ we have

$$\frac{dr^+}{dt} = 8i\xi^3\alpha(t)r^+ - \frac{1}{a^2}\int_{-\infty}^{\infty} G\left(\varphi_1^2+\varphi_2^2\right)dx. \tag{29}$$

Lemma 4. If the vector function $\varphi(x,\xi)$ is a solution to equation (7), then its components satisfy the following equality

$$\int_{-\infty}^{+\infty} G\left(\varphi_1^2+\varphi_2^2\right)dx = 2i\xi a(\xi)b(\xi)Q\left(u(x_1,t)\right) -$$

$$-2a(\xi)b(\xi)\left(\text{v.p.}\int_{-\infty}^{\infty}\frac{F\left(u(x_2,t)\right)h^2(\eta,t)}{\eta+\xi}d\eta - i\pi F\left(u(x_2,t)\right)h^2(\xi,t)\right), \tag{30}$$

where v.p. is integral in the sense of principal value.

Proof. Let us calculate the following integral using formulas (7), (9), (10):

$$-\int_{-\infty}^{\infty} Q\left(u(x_1,t)\right)u_x\left(\varphi_1^2+\varphi_2^2\right)dx = -Q\left(u(x_1,t)\right)\int_{-\infty}^{\infty}\left(\varphi_1^2+\varphi_2^2\right)du$$

$$= -Q\left(u(x_1,t)\right)u\left(\varphi_1^2+\varphi_2^2\right)\Big|_{-\infty}^{\infty} + Q\left(u(x_1,t)\right)\int_{-\infty}^{\infty} u\left(\varphi_1^2+\varphi_2^2\right)' dx = 2i\xi Q\left(u(x_1,t)\right)a(\xi)b(\xi).$$

Thus we get the following equality

$$\int_{-\infty}^{\infty} Q\left(u(x_1,t)\right)u_x\left(\varphi_1^2+\varphi_2^2\right)dx = -2i\xi Q\left(u(x_1,t)\right)a(\xi)b(\xi). \tag{31}$$

In the future, to simplify the notation, if this is not essential, we will not write the dependence of the functions on t. According to lemma 1 we have

$$\int_{-\infty}^{+\infty}\left(\phi_1^2(x,\eta)-\phi_2^2(x,\eta)\right)\left(\varphi_1^2(x,\xi)+\varphi_2^2(x,\xi)\right)dx =$$

$$= \frac{i}{2}\lim_{R\to\infty}\left(\frac{\left(\phi_1(x,\eta)\varphi_1(x,\xi)+\phi_2(x,\eta)\varphi_2(x,\xi)\right)^2}{\eta+\xi}\right)\Bigg|_{-R}^{R} +$$

$$+\frac{i}{2}\lim_{R\to\infty}\left(\frac{\left(\phi_1(x,\eta)\varphi_2(x,\xi)-\phi_2(x,\eta)\varphi_1(x,\xi)\right)^2}{\eta-\xi}\right)\Bigg|_{-R}^{R}.$$

By virtue of (4) $\phi(x,\eta,t) = h(\eta,t)\left(\tilde{\psi}(x,\eta,t)+\psi(x,\eta,t)\right)$, therefore, using the asymptotics for $\varphi(x,\xi)$ and the Sokhotski formula [22], we find

$$F\left(u(x_2,t)\right)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\phi_1^2(x,\eta)-\phi_2^2(x,\eta)\right)\left(\varphi_1^2(x,\xi)+\varphi_2^2(x,\xi)\right)dxd\eta =$$

$$=2a(\xi)b(\xi)F\left(u(x_2,t)\right)\left(i\pi h^2(\xi,t)-\text{v.p.}\int_{-\infty}^{\infty}\frac{h^2(\eta,t)}{\xi+\eta}d\eta\right). \tag{32}$$

According to equalities (31) and (32), we get formula (30). **The lemma is proven.**

Using equalities (29) and (30), we have

$$\begin{aligned}\frac{dr^+}{dt}&=\left(8i\xi^3P\left(u(x_0,t)\right)-2i\xi Q\left(u(x_1,t)\right)\right)r^+ +\\ &+\text{v.p.}\int_{-\infty}^{\infty}\frac{2F\left(u(x_2,t)\right)h^2(\eta,t)}{\eta+\xi}d\eta-2i\pi F\left(u(x_2,t)\right)h^2(\xi,t),\ \left(\operatorname{Im}\xi=0\right).\end{aligned} \tag{33}$$

Lemma 5. If the system of functions $\{u(x,t),\ \varphi_1(\eta,x,t),\ \varphi_2(\eta,x,t)\}$ is a solution to the problem (1) – (6), then the eigenvalues of the operator $L(t)$ with potential $u(x,t)$ do not depend on t.

Proof. Let ξ_k be the eigenvalue of the operator $L(t)$ of multiplicity m_k with the eigenvector function $\varphi_k=\left(\varphi_{k_1},\varphi_{k_2}\right)^T$.

Let us assume that the eigenvalue ξ_k is simple, i.e. $m_k=1$. Differentiating with respect to t equality

$$L\varphi_k=\xi_k\varphi_k,$$

we have

$$\frac{\partial L}{\partial t}\varphi_k+L\frac{\partial\varphi_k}{\partial t}=\frac{d\xi_k}{dt}\varphi_k+\xi_k\frac{\partial\varphi_k}{\partial t}. \tag{34}$$

Substituting L_t from (22) into (34) we get

$$\left(L-\xi_k\right)\left(\frac{\partial\varphi_k}{\partial t}-A\varphi_k\right)=\frac{d\xi_k}{dt}\varphi_k-iR\varphi_k. \tag{35}$$

Multiplying both sides of equality (35) on the left by $\hat{\varphi}_k^T\equiv\left(\varphi_{k_2},\varphi_{k_1}\right)$ we integrate over x from $-\infty$ to ∞, noticing that

$$\int_{-\infty}^{\infty}\hat{\varphi}_k^T\left(L-\xi_k\right)\left(\frac{\partial\varphi_k}{\partial t}-A\varphi_k\right)dx=0,$$

we get

$$\frac{d\xi_k}{dt}=\frac{\int_{-\infty}^{\infty}\left(G\varphi_{k_2}^2+G\varphi_{k_1}^2\right)dx}{2i\int_{-\infty}^{\infty}\varphi_{k_1}\varphi_{k_2}dx},\quad k=\overline{1,N}. \tag{36}$$

Now using lemma 1 we have the following equality

$$iF\left(u(x_2,t)\right)\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty}\left(\phi_1^2-\phi_2^2\right)\left(\varphi_{k_1}^2+\varphi_{k_2}^2\right)d\eta dx=0\,. \tag{37}$$

Let us calculate the integral,

$$\int_{-\infty}^{\infty}Q\left(u(x_1,t)\right)u_x\left(\varphi_{k_1}^2+\varphi_{k_2}^2\right)dx=$$

$$=Q\left(u(x_1,t)\right)u(x,t)\left(\varphi_{k_1}^2+\varphi_{k_2}^2\right)\Big|_{-\infty}^{+\infty}+2i\xi_k Q\left(u(x_1,t)\right)\varphi_{k_1}\varphi_{k_2}\Big|_{-\infty}^{+\infty}=0\,.$$

According to equality (37) and the last relation, we obtain the equality

$$\int_{-\infty}^{\infty}G\left(\varphi_{k1}^2+\varphi_{k2}^2\right)dx=0. \tag{38}$$

Consequently $\dfrac{d\xi_k}{dt}=0$.

When $m_k\neq 1$, this proof does not work, since equality

$$\left.\frac{da(\xi)}{d\xi}\right|_{\xi=\xi_k}=-\frac{2i}{\chi_0^k}\int_{-\infty}^{\infty}\varphi_{k_1}\varphi_{k_2}dx\,,$$

implies

$$\int_{-\infty}^{\infty}\varphi_{k_1}\varphi_{k_2}dx=0\,. \tag{39}$$

In this case, differentiating the equality

$$L\overset{(1)}{\varphi}_k=\xi_k\overset{(1)}{\varphi}_k+\varphi_k\,,$$

with respect to t relatively, we get

$$\left(L-\xi_k\right)\left(\frac{\partial\overset{(1)}{\varphi}_k}{\partial t}-A\overset{(1)}{\varphi}_k\right)-\frac{\partial\varphi_k}{\partial t}+A\varphi_k=\frac{d\xi_k}{dt}\overset{(1)}{\varphi}_k-iR\overset{(1)}{\varphi}_k\,. \tag{40}$$

Let us multiply both sides of equality (40) on the left by $\hat{\varphi}_k^T$ and integrate over x from $-\infty$ to ∞:

$$\int_{-\infty}^{\infty}\hat{\varphi}_k^T\left(\left(L-\xi_k\right)\left(\frac{\partial\overset{(1)}{\varphi}_k}{\partial t}-A\overset{(1)}{\varphi}_k\right)-\frac{\partial\varphi_k}{\partial t}+A\varphi_k\right)dx=\int_{-\infty}^{\infty}\hat{\varphi}_k^T\left(\frac{d\xi_k}{dt}\overset{(1)}{\varphi}_k-iR\overset{(1)}{\varphi}_k\right)dx\,. \tag{41}$$

Differentiating equality (39) with respect to t, we obtain

$$\int_{-\infty}^{\infty}\hat{\varphi}_k^T\frac{\partial\varphi_k}{\partial t}dx=0\,.$$

According to the definition of operators A we have

$$\hat{\varphi}_k^T A\varphi_k = P\big(u(x_0,t)\big)\big(\varphi_{k2},\varphi_{k1}\big)\begin{pmatrix} -4i\xi_k^3+2u^2 i\xi_k & 4u\xi_k^2+2iu_x\xi_k-2u^3-u_{xx} \\ -4u\xi_k^2+2iu_x\xi_k+2u^3+u_{xx} & 4i\xi_k^3-2iu^2\xi_k \end{pmatrix}\times$$

$$\times\begin{pmatrix}\varphi_{k1}\\ \varphi_{k2}\end{pmatrix} = P\big(u(x_0,t)\big)\big(\varphi_{k1}^2-\varphi_{k2}^2\big)\big(2u^3+u_{xx}-4u\xi_k^2\big)+2iP\big(u(x_0,t)\big)u_x\xi_k\big(\varphi_{k1}^2+\varphi_{k2}^2\big).$$

By this equality and Lemma 3 we have

$$\int_{-\infty}^{\infty}\hat{\varphi}_k^T A\varphi_k dx = 0\,.$$

Thus, the left side of equality (41) is equal to zero, therefore

$$\frac{d\xi_k}{dt} = \frac{i\int_{-\infty}^{\infty} G\left(\overset{(1)}{\varphi}_{k_1}\varphi_{k_1}+\overset{(1)}{\varphi}_{k_2}\varphi_{k_2}\right)dx}{2\int_{-\infty}^{\infty}\left(\overset{(1)}{\varphi}_{k_1}\varphi_{k_2}+\overset{(1)}{\varphi}_{k_1}\varphi_{k_2}\right)dx}. \tag{42}$$

According to Lemma 1 and the corollary of Lemma 2, we find

$$i\int_{-\infty}^{\infty}\big(\phi_1^2(x,\eta)-\phi_2^2(x,\eta)\big)\left(\overset{(1)}{\varphi}_{k1}\varphi_{k1}+\overset{(1)}{\varphi}_{k2}\varphi_{k2}\right)dx =$$

$$=\frac{-1}{2}\lim_{R\to\infty}\left(\frac{V\left\{\phi,\overset{(1)}{\varphi}_k\right\}V\{\phi,\varphi_k\}}{\eta+\xi_k}-\frac{\big(V\{\phi,\varphi_k\}\big)^2}{2\big(\eta+\xi_k\big)^2}+\frac{W\left\{\phi,\overset{(1)}{\varphi}_k\right\}W\{\phi,\varphi_k\}}{\eta-\xi_k}-\frac{\big(W\{\phi,\varphi_k\}\big)^2}{2\big(\eta-\xi_k\big)^2}\right)\Bigg|_{-R}^{R}=0.$$

Now let's calculate the following integral

$$Q\big(u(x_1,t)\big)\int_{-\infty}^{+\infty}u_x\left(\overset{(1)}{\varphi}_{k_1}\varphi_{k_1}+\overset{(1)}{\varphi}_{k_2}\varphi_{k_2}\right)dx = Q\big(u(x_1,t)\big)\int_{-\infty}^{+\infty}\left(\overset{(1)}{\varphi}_{k_1}\varphi_{k_1}+\overset{(1)}{\varphi}_{k_2}\varphi_{k_2}\right)du =$$

$$= i\xi_k Q\big(u(x_1,t)\big)\int_{-\infty}^{+\infty}\left(\overset{(1)}{\varphi}_{k_1}\varphi_{k_2}+\overset{(1)}{\varphi}_{k_2}\varphi_{k_1}\right)'dx = i\xi_k Q\big(u(x_1,t)\big)\lim_{R\to+\infty}\left(\overset{(1)}{\varphi}_{k_1}\varphi_{k_2}+\overset{(1)}{\varphi}_{k_2}\varphi_{k_1}\right)\Bigg|_{-R}^{R}=0.$$

Therefore, $\dfrac{d\xi_k}{dt}=0$. **The lemma is proven.**

Thus, based on lemma 5, we have the following relation

$$m_k(t)=m_k(0),\ \xi_k(t)=\xi_k(0),\ k=1,..,N\,. \tag{43}$$

Let us rewrite equality (27) in the form

$$\frac{\partial\varphi}{\partial t}-A\varphi=\frac{1}{a(\xi)}\left[\int_{-\infty}^{x}G(\psi_1(x,\xi)\varphi_1(x,\xi)+\psi_2(x,\xi)\varphi_2(x,\xi))dx\right]\varphi(x,\xi)$$
$$-\frac{1}{a(\xi)}\left[\int_{-\infty}^{x}G(\varphi_1^2(x,\xi)+\varphi_2^2(x,\xi))dx\right]\psi(x,\xi)+4i\xi^3P\left(u(x_0,t)\right)\varphi(x,\xi). \tag{44}$$

Using Lemma 1 and asymptotics for φ, ϕ, we have

$$iF\left(u(x_2,t)\right)\int_{-\infty}^{\infty}\int_{-\infty}^{x}\left(\phi_1^2(x,\eta)-\phi_2^2(x,\eta)\right)\left(\varphi_1(x,\xi)\psi_1(x,\xi)+\varphi_2(x,\xi)\psi_2(x,\xi)\right)dxd\eta=$$
$$=-\frac{F\left(u(x_2,t)\right)}{2}\int_{-\infty}^{+\infty}\frac{\left(\phi_1(x,\eta)\psi_1(x,\xi)+\phi_2(x,\eta)\psi_2(x,\xi)\right)\left(\phi_1(x,\eta)\varphi_1(x,\xi)+\phi_2(x,\eta)\varphi_2(x,\xi)\right)}{\eta+\xi}d\eta-$$
$$-\frac{F\left(u(x_2,t)\right)}{2}\int_{-\infty}^{+\infty}\frac{\left(\phi_1(x,\eta)\psi_2(x,\xi)-\phi_2(x,\eta)\psi_1(x,\xi)\right)\left(\phi_1(x,\eta)\varphi_2(x,\xi)-\phi_2(x,\eta)\varphi_1(x,\xi)\right)}{\eta-\xi}d\eta+$$
$$+F\left(u(x_2,t)\right)a(\xi)\int_{-\infty}^{+\infty}\frac{h^2(\eta,t)\left(a(\eta)\bar{a}(\eta)-b(\eta)\bar{b}(\eta)\right)}{\eta+\xi}d\eta.$$

Hence,

$$iF\left(u(x_2,t)\right)\int_{-\infty}^{+\infty}\int_{-\infty}^{x}\left(\phi_1^2(x,\eta)-\phi_2^2(x,\eta)\right)\left(\varphi_1^2(x,\xi)+\varphi_2^2(x,\xi)\right)dxd\eta=$$
$$=-\frac{\gamma(t)}{2}\int_{-\infty}^{\infty}\frac{\left(\phi_1(x,\eta)\varphi_1(x,\xi)+\phi_2(x,\eta)\varphi_2(x,\xi)\right)^2}{\eta+\xi}d\eta-$$
$$-\frac{\gamma(t)}{2}\int_{-\infty}^{\infty}\frac{\left(\phi_1(x,\eta)\varphi_2(x,\xi)-\phi_2(x,\eta)\varphi_1(x,\xi)\right)^2}{\eta-\xi}d\eta.$$

Based on the last two equalities and Lemma 1, we have the following equality

$$iF\left(u(x_2,t)\right)\int_{-\infty}^{\infty}\int_{-\infty}^{x}\left(\phi_1^2(x,\eta)-\phi_2^2(x,\eta)\right)\left(\varphi_1(x,\xi)\psi_1(x,\xi)+\varphi_2(x,\xi)\psi_2(x,\xi)\right)dxd\eta\,\varphi(x,\xi)-$$
$$-iF\left(u(x_2,t)\right)\int_{-\infty}^{\infty}\int_{-\infty}^{x}\left(\phi_1^2(x,\eta)-\phi_2^2(x,\eta)\right)\left(\varphi_1^2(x,\xi)+\varphi_2^2(x,\xi)\right)dxd\eta\,\psi(x,\xi)=$$
$$=-\frac{F\left(u(x_2,t)\right)}{2}\int_{-\infty}^{\infty}\frac{\mathbf{C}(x,\eta,t)}{\eta+\xi}d\eta\,a(\xi)\varphi(x,\xi)+$$
$$+F\left(u(x_2,t)\right)\int_{-\infty}^{\infty}\frac{h^2(\eta,t)\left(a(\eta)\bar{a}(\eta)-b(\eta)\bar{b}(\eta)\right)}{\eta+\xi}d\eta\,a(\xi)\varphi(x,\xi) \tag{45}$$

where $\mathbf{C}(x,\eta,t)$ the following matrix

$$\mathbf{C}=\begin{pmatrix}\phi_1(x,\eta)\phi_2(x,\eta)+\phi_1(x,-\eta)\phi_2(x,-\eta) & \phi_2^2(x,\eta)-\phi_1^2(x,-\eta)\\ -\phi_1^2(x,\eta)+\phi_2^2(x,-\eta) & -\phi_1(x,\eta)\phi_2(x,\eta)-\phi_1(x,-\eta)\phi_2(x,-\eta)\end{pmatrix}.$$

From (4) it follows that

$$\lim_{x\to\infty}\mathbf{C}(x,\eta,t)=2h^2(\eta,t)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{46}$$

Using formulas (6), (7), (9) and (10), we calculate the following integral:

$$\int_{-\infty}^{x} u_x(\varphi_1\psi_1+\varphi_2\psi_2)dx = u(x,t)(\varphi_1\psi_1+\varphi_2\psi_2) - \int_{-\infty}^{x}(\varphi_1\psi_2+\varphi_2\psi_1)'dx =$$
$$= u(x,t)(\varphi_1\psi_1+\varphi_2\psi_2) + 2i\xi\varphi_2\psi_1 \,. \tag{47}$$

The validity of the following equality is shown in the same way

$$\int_{-\infty}^{x} u_x(\varphi_1^2(x,\xi)+\varphi_2^2(x,\xi))dx = u(x,t)(\varphi_1^2(x,\xi)+\varphi_2^2(x,\xi)) - 2i\xi\varphi_1(x,\xi)\varphi_2(x,\xi)\,. \tag{48}$$

Thus, according to equalities (45), (47) and (48), equality (44) can be rewritten in the following form

$$\frac{\partial\varphi(x,\xi)}{\partial t} - A\varphi(x,\xi) = F\left(u(x_2,t)\right)\int_{-\infty}^{\infty}\frac{h^2(\eta,t)\left(a(\eta)\bar{a}(\eta)-b(\eta)\bar{b}(\eta)\right)}{\eta+\xi}d\eta\begin{pmatrix}\varphi_1(x,\xi)\\ \varphi_2(x,\xi)\end{pmatrix} -$$
$$-\frac{F\left(u(x_2,t)\right)}{2}\int_{-\infty}^{\infty}\frac{\mathbf{C}(x,\eta,t)}{\eta+\xi}d\eta\begin{pmatrix}\varphi_1(x,\xi)\\ \varphi_2(x,\xi)\end{pmatrix} + Q\left(u(x_1,t)\right)u(x,t)\begin{pmatrix}-\varphi_2(x,\xi)\\ \varphi_1(x,\xi)\end{pmatrix} -$$
$$-2i\xi Q\left(u(x_1,t)\right)\begin{pmatrix}0\\ \varphi_2(x,\xi)\end{pmatrix} + 4i\xi^3 P\left(u(x_0,t)\right)\begin{pmatrix}\varphi_1(x,\xi)\\ \varphi_2(x,\xi)\end{pmatrix}. \tag{49}$$

Differentiating equality (49) m_n-1 times with respect to ξ and assuming $\xi=\xi_n$ we obtain the following expression

$$\frac{\partial \overset{(m_n-1)}{\varphi_n}}{\partial t} - P\left(u(x_0,t)\right)\begin{pmatrix}-4i\xi_n^3+2iu\xi_n & 4u\xi_n^2+2iu_x\xi_n-2u^3-u_{xx}\\ -4i\xi_n^2+2iu_x\xi_n+2iu^3+u_{xx} & -4i\xi_n^3-2iu^2\xi_n\end{pmatrix}\overset{(m_n-1)}{\varphi_n} -$$
$$-C_{m_n-1}^1 P\left(u(x_0,t)\right)\begin{pmatrix}-12i\xi_n^2+2iu^2 & 8u\xi_n+2iu_x\\ -8i\xi_n+2iu_x & 12i\xi_n^2-2iu^2\end{pmatrix}\overset{(m_n-2)}{\varphi_n} - C_{m_n-1}^2 P\left(u(x_0,t)\right)\begin{pmatrix}-24i\xi_n & 8u\\ -8u & 24i\xi_n\end{pmatrix}\overset{(m_n-3)}{\varphi_n} -$$
$$-C_{m_n-1}^3 P\left(u(x_0,t)\right)\begin{pmatrix}-24i & 0\\ 0 & 24i\end{pmatrix}\overset{(m_n-4)}{\varphi_n} =$$
$$= F\left(u(x_2,t)\right)\sum_{s=0}^{m_n-1}C_{m_n-1}^s\int_{-\infty}^{\infty}\frac{(-1)^{(m_n-1-s)}h^2(\eta,t)(m_n-1-s)!}{(\eta+\xi_n)^{(m_n-s)}}(a(\eta)\bar{a}(\eta)-$$
$$-(b(\eta)\bar{b}(\eta))d\eta\,\overset{(s)}{\varphi_n} - \frac{F\left(u(x_2,t)\right)}{2}\sum_{s=0}^{m_n-1}C_{m_n-1}^3\int_{-\infty}^{\infty}\frac{(-1)^{(m_n-1-s)}(m_n-1-s)!}{(\eta+\xi_n)^{(m_n-s)}}\mathbf{C}(x,\eta,t)\,d\eta\,\overset{(s)}{\varphi_n} +$$
$$+4i\xi_n^3 P\left(u(x_0,t)\right)\overset{(m_n-1)}{\varphi_n} + C_{m_n-1}^1 12i\xi_n^2 P\left(u(x_0,t)\right)\overset{(m_n-2)}{\varphi_n} +$$
$$+C_{m_n-1}^2 12i\xi_n P\left(u(x_0,t)\right)\overset{(m_n-3)}{\varphi_n} + C_{m_n-1}^3 4iP\left(u(x_0,t)\right)\overset{(m_n-4)}{\varphi_n} +$$
$$+Q\left(u(x_1,t)\right)\begin{pmatrix}-\varphi_{2n}^{(m_n-1)}\\ \varphi_{1n}^{(m_n-1)}\end{pmatrix} - 2i\xi_n Q\left(u(x_1,t)\right)\begin{pmatrix}0\\ \varphi_{2n}^{(m_n-1)}\end{pmatrix} - 2i(m_n-1)Q\left(u(x_1,t)\right)\begin{pmatrix}0\\ \varphi_{2n}^{(m_n-2)}\end{pmatrix}. \tag{50}$$

Using (7), (12) and (44) we pass in equality (50) to the limit at $x\to\infty$, equating the coefficients of $\begin{pmatrix}0\\ 1\end{pmatrix}(ix)^s e^{i\xi_n x}$, $s=m_n-1, m_n-2,...,0$, and using equality $a(\xi)\bar{a}(\xi)=\dfrac{1}{1+r^+(\xi)r^+(-\xi)}$, we obtain the following equalities

$$\frac{d\chi_0^n}{dt}=\left(8i\xi_n^3P\left(u(x_0,t)\right)-2i\xi_nQ\left(u(x_1,t)\right)+2F\left(u(x_2,t)\right)\int\limits_{-\infty}^{\infty}\frac{h^2(\eta,t)}{(\eta+\xi_n)(1+r^+(\eta)r^+(-\eta))}d\eta\right)\chi_0^n,$$

$$\frac{d\chi_1^n}{dt}=\left(8i\xi_n^3P\left(u(x_0,t)\right)-2i\xi_nQ\left(u(x_1,t)\right)+2F\left(u(x_2,t)\right)\int\limits_{-\infty}^{\infty}\frac{h^2(\eta,t)}{(\eta+\xi_n)(1+r^+(\eta)r^+(-\eta))}d\eta\right)\chi_1^n+$$
$$+\left(24i\xi_n^2P\left(u(x_0,t)\right)-2iQ\left(u(x_1,t)\right)-2F\left(u(x_2,t)\right)\int\limits_{-\infty}^{\infty}\frac{h^2(\eta,t)}{(\eta+\xi_n)^2(1+r^+(\eta)r^+(-\eta))}d\eta\right)\chi_0^n,$$

$$\frac{d\chi_2^n}{dt}=\left(8i\xi_n^3P\left(u(x_0,t)\right)-2i\xi_nQ\left(u(x_1,t)\right)+2F\left(u(x_2,t)\right)\int\limits_{-\infty}^{\infty}\frac{h^2(\eta,t)}{(\eta+\xi_n)(1+r^+(\eta)r^+(-\eta))}d\eta\right)\chi_2^n+$$
$$+\left(24i\xi_n^2P\left(u(x_0,t)\right)-2iQ\left(u(x_1,t)\right)-2F\left(u(x_2,t)\right)\int\limits_{-\infty}^{\infty}\frac{h^2(\eta,t)}{(\eta+\xi_n)^2(1+r^+(\eta)r^+(-\eta))}d\eta\right)\chi_1^n+$$
$$+\left(24i\xi_nP\left(u(x_0,t)\right)+2F\left(u(x_2,t)\right)\int\limits_{-\infty}^{\infty}\frac{h^2(\eta,t)}{(\eta+\xi_n)^3(1+r^+(\eta)r^+(-\eta))}d\eta\right)\chi_0^n,$$

$$\frac{d\chi_3^n}{dt}=\left(8i\xi_n^3P\left(u(x_0,t)\right)-2i\xi_nQ\left(u(x_1,t)\right)+2F\left(u(x_2,t)\right)\int\limits_{-\infty}^{\infty}\frac{h^2(\eta,t)}{(\eta+\xi_n)(1+r^+(\eta)r^+(-\eta))}d\eta\right)\chi_3^n+$$
$$+\left(24i\xi_n^2P\left(u(x_0,t)\right)-2iQ\left(u(x_1,t)\right)-2F\left(u(x_2,t)\right)\int\limits_{-\infty}^{\infty}\frac{h^2(\eta,t)}{(\eta+\xi_n)^2(1+r^+(\eta)r^+(-\eta))}d\eta\right)\chi_2^n+$$
$$+\left(24i\xi_nP\left(u(x_0,t)\right)+2F\left(u(x_2,t)\right)\int\limits_{-\infty}^{\infty}\frac{h^2(\eta,t)}{(\eta+\xi_n)^3(1+r^+(\eta)r^+(-\eta))}d\eta\right)\chi_1^n+$$
$$+\left(8iP\left(u(x_0,t)\right)-2F\left(u(x_2,t)\right)\int\limits_{-\infty}^{\infty}\frac{h^2(\eta,t)}{(\eta+\xi_n)^4(1+r^+(\eta)r^+(-\eta))}d\eta\right)\chi_0^n,$$

$$\frac{d\chi_l^n}{dt}=\left(8i\xi_n^3P\left(u(x_0,t)\right)-2i\xi_nQ\left(u(x_1,t)\right)+2F\left(u(x_2,t)\right)\int\limits_{-\infty}^{\infty}\frac{h^2(\eta,t)}{(\eta+\xi_n)(1+r^+(\eta)r^+(-\eta))}d\eta\right)\chi_l^n+$$
$$+\left(24i\xi_nP\left(u(x_0,t)\right)+2F\left(u(x_2,t)\right)\int\limits_{-\infty}^{\infty}\frac{h^2(\eta,t)}{(\eta+\xi_n)^3(1+r^+(\eta)r^+(-\eta))}d\eta\right)\chi_{l-2}^n+$$
$$+\left(8iP\left(u(x_0,t)\right)-2F\left(u(x_2,t)\right)\int\limits_{-\infty}^{\infty}\frac{h^2(\eta,t)}{(\eta+\xi_n)^4(1+r^+(\eta)r^+(-\eta))}d\eta\right)\chi_{l-3}^n+$$
$$+2F\left(u(x_2,t)\right)\sum_{s=0}^{l-4}\left(\int\limits_{-\infty}^{\infty}\frac{(-1)^{l-s}h^2(\eta,t)}{(\eta+\xi_n)^{l-s+1}(1+r^+(\eta)r^+(-\eta))}d\eta\right)\chi_s^n,\ n=\overline{1,N},\ l=\overline{4,m_n-1}. \tag{45}$$

Thus, we were able to prove the following theorem.

Theorem 2. If the system of functions $\{u(x,t),\ \varphi_1(\eta,x,t),\ \varphi_2(\eta,x,t)\}$ is a solution to problem (1)-(6), then the scattering data of non-self-adjoint operator $L(t)$ with potential $u(x,t)$ satisfy the differential equations (33), (43), (45).

Let the function $u_0(x)$ be given that satisfies condition (3). Then the solution to problem (1)-(6) is found by the following algorithm.

- We solve the direct scattering problem with the initial function $u_0(x)$ and obtain scattering data

$$\left\{r^+(\xi,0),\ \xi \in R;\ \xi_k(0), \operatorname{Im}\xi_k > 0;\quad \chi_j^k(0),\quad k=\overline{1,N};\ j=\overline{0,m_k-1}\right\}$$

for the operator $L(0)$.

- Using the results of Theorem 2, we find the scattering data for $t>0$:

$$\left\{r^+(\xi,t),\ \xi \in R;\ \xi_k(t), \operatorname{Im}\xi_k > 0;\quad \chi_j^k(t),\quad k=\overline{1,N};\ j=\overline{0,m_k-1}\right\}.$$

- Using the method based on the Gelfand-Levitan-Marchenko integral equation, we solve the inverse scattering problem, i.e. We find the unique (according to theorem) $u(x,t)$ from the scattering data for $t>0$ obtained at the previous step.
- Solve the direct problem for the operator $L(t)$ with potential $u(x,t)$ and find functions $\phi_1(x,\eta,t), \phi_2(x,\eta,t)$.

The obtained equalities completely determine the evolution of the scattering data, which allows us to apply the inverse scattering problem method to solve problem (1)-(6).

Example. Let us consider the following Cauchy problem

$$u_t + \beta(t)u(1,t)\left(6u^2u_x + u_{xxx}\right) + \gamma(t)u(\ln 3,t)u_x = i\alpha(t)u(2,t)\int_{-\infty}^{\infty}\left(\phi_1^2 - \phi_2^2\right)d\eta \ , \quad (46)$$

$$L(t)\phi = \eta\phi\,,\quad x \in R\,,$$

$$u(x,0) = -\frac{2}{\operatorname{ch} 2x}\,. \quad (47)$$

where $h(\eta,t) = \dfrac{1}{\sqrt{\eta^2+1}}$, $\beta(t) = \dfrac{e^{4-\frac{2t}{t+1}}+1}{-32(t+1)^2 e^{2-\frac{t}{t+1}}}$, $\alpha(t) = \dfrac{2(t^2+1)e^{4-\frac{t}{t+1}}}{e^{8-\frac{2t}{t+1}}+1}$,

$$\gamma(t) = \frac{\left(1+2\pi i(t^2+1)(t+1)^2\right)\left(81+e^{\frac{2t}{t+1}}\right)}{-72(t+1)^2 e^{\frac{t}{t+1}}}.$$

The solution to the Cauchy problem (46)-(47) has the following form

$$u(x,t) = \frac{-2}{\operatorname{ch}\left(2x - \dfrac{t}{t+1}\right)},$$

$$\phi_1(x,\eta,t)=\frac{e^{i\eta x}}{(1-i\eta)\sqrt{1+\eta^2}\,\mathrm{ch}\left(2x-\frac{t}{t+1}\right)}+\frac{e^{-i\eta x}}{\sqrt{1+\eta^2}}\left(1-\frac{e^{-2x+\frac{t}{t+1}}}{(1+i\eta)\,\mathrm{ch}\left(2x-\frac{t}{t+1}\right)}\right),$$

$$\phi_2(x,\eta,t)=\frac{e^{i\eta x}}{\sqrt{1+\eta^2}}\left(1-\frac{e^{-2x+\frac{t}{t+1}}}{(1-i\eta)\,\mathrm{ch}\left(2x-\frac{t}{t+1}\right)}\right)-\frac{e^{-i\eta x}}{\sqrt{1+\eta^2}(1+i\eta)\,\mathrm{ch}\left(2x-\frac{t}{t+1}\right)}.$$

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Local automorphisms on naturally graded filiform Leibniz algebra

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Abstract. *In this paper, we study automorphisms and local automorphisms on naturally graded filiform Leibniz algebra. We describe the local automorphisms of naturally graded filiform Leibniz algebra.*

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1. Introduction

In last decades a series of papers have been devoted to study of mappings which are close to automorphism and derivation of associative algebras (especially of operator algebras and C*-algebras). Namely, the problems of describing so-called local automorphisms (respectively, local derivations) and 2-local automorphisms (respectively, 2-local derivations) have been considered [23, 27, 28, 5, 21, 6, 26, 8, 29, 1]. Later similar problems were extended for non associative algebras, in particular, for the case of Lie algebras.

Let $\mathcal{A}$ be an associative algebra. Recall that a linear mapping Φ of $\mathcal{A}$ into itself is called a local automorphism (respectively, a local derivation) if for every $x \in \mathcal{A}$ there exists an automorphism (respectively, a derivation) Φ_x of $\mathcal{A}$, depending on x, such that $\Phi_x(x) = \Phi(x)$. These notions were introduced and investigated independently by Kadison [23] and Larson and Sourour [27]. Later, in 1997, P. Šemrl [28] introduced the concepts of 2-local automorphisms and 2-local derivations. A map $\Phi : \mathcal{A} \to \mathcal{A}$ (not linear in general) is called a 2-local automorphism (respectively, a 2-local derivation) if for every $x, y \in \mathcal{A}$, there exists an automorphism (respectively, a derivation) $\Phi_{x,y} : \mathcal{A} \to \mathcal{A}$ (depending on x, y) such that $\Phi_{x.y}(x) = \Phi(x)$, $\Phi_{x,y}(y) = \Phi(y)$. In [28], P. Šemrl described 2-local derivations and 2-local automorphisms on the algebra $B(H)$ of all bounded linear operators on the infinite-dimensional separable Hilbert space H by proving that every 2-local automorphism (respectively, 2-local derivation) on $B(H)$ is an automorphism (respectively, a derivation). A similar result for finite-dimensional case appeared later in [25]. Further, in [5], a new technique was introduced to prove the same result for an arbitrary Hilbert space H (no separability is assumed.)

Afterwards, the considerations above gave rise to similar questions in von Neumann algebras framework. First positive results have been obtained in [6] and [7] for finite and semi-finite von Neumann algebras respectively, by showing that all 2-local derivations on these algebras are derivations. Finally, in [8], the same result was obtained for purely infinite von Neumann algebras. This completed the solution of the above problem for arbitrary von Neumann algebras.

It is natural to study the corresponding analogues of these problems for automorphisms or derivations of non-associative algebras.

Let $\mathcal{L}$ be a Lie algebra. A derivation (respectively, an automorphism) Φ of $\mathcal{L}$ is a linear (respectively, an invertible linear) map $\Phi : \mathcal{L} \to \mathcal{L}$ which satisfies the condition $\Phi([x, y]) = [\Phi(x), y] + [x, \Phi(y)]$ (respectively, $\Phi([x, y]) = [\Phi(x), \Phi(y)]$) for all $x, y \in \mathcal{L}$. The set of all automorphisms of a Lie algebra $\mathcal{L}$ is denoted by $\mathrm{Aut}\mathcal{L}$.

The notions of a local derivation (respectively, a local automorphism) and a 2-local derivation (respectively, a 2-local automorphism) for Lie algebras are defined as above, similar to the associative case. Every derivation (respectively, automorphism) of a Lie algebra $\mathcal{L}$ is a local derivation (respectively, local automorphism) and a 2-local derivation (respectively, 2-local automorphism). For a given Lie algebra $\mathcal{L}$, the main problem concerning these notions is to prove that they automatically become a derivation (respectively, an automorphism) or to give examples of local and 2-local derivations or automorphisms of $\mathcal{L}$, which are not derivations or automorphisms, respectively. For a finite-dimensional semi-simple Lie algebra $\mathcal{L}$ over an algebraically closed field of characteristic zero, the derivations and automorphisms of $\mathcal{L}$ are completely described in [22].

Recently in [10] we have proved that every local derivation on semi-simple Lie algebras is a derivation and gave examples of nilpotent finite-dimensional Lie algebras with local derivations which are not derivations. In [13], Sh.A. Ayupov, A.Kh. Khudoyberdiyev and B.B. Yusupov proved similar results concerning local derivations on solvable Leibniz algebras. The results of the paper [14] shows that p-filiform Leibniz algebras as a rule admit local derivations which are not derivations.

Earlier in [12] the authors have proved that every 2-local derivation on a semi-simple Lie algebra $\mathcal{L}$ is a derivation, and showed that each finite-dimension nilpotent Lie algebra, with dimension larger than two, admits a 2-local derivation which is not a derivation. In [15], Sh.A. Ayupov, K.K. Kudaybergenov and B.B. Yusupov proved similar results concerning 2-local derivations on Locally Simple Lie Algebras.

In [18], Chen and Wang initiated study of 2-local automorphisms of finite-dimensional Lie algebras. They proved that if $\mathcal{L}$ is a simple Lie algebra of type $A_l\,(l \geq 1), D_l\,(l \geq 4)$, or $E_k\,(k = 6,7,8)$ over an algebraically closed field of characteristic zero, then every 2-local automorphism of $\mathcal{L}$, is an automorphism. Finally, in [9] Sh. Ayupov and K. Kudaybergenov generalized this result of [18] and proved that every 2-local automorphism of a finite-dimensional semi-simple Lie algebra over an algebraically closed field of characteristic zero is an automorphism. Moreover, they also showed that every nilpotent Lie algebra with finite dimension larger than two admits 2-local automorphisms which are not automorphisms.

Local automorphisms of certain finite-dimensional simple Lie and Leibniz algebras are investigated in [11]. Concerning local automorphism, T. Becker, J. Escobar, C. Salas and R. Turdibaev in [17] established that the set of local automorphisms $LAut(sl_2)$ coincides with the group $Aut^{\pm}(sl_2)$ of all automorphisms and anti-automorphisms. Later in [20] M. Costantini proved that a linear map on a simple Lie algebra is a local automorphism if and only if it is either an automorphism or an anti-automorphism. The first example of a simple (ternary) algebra with nontrivial local derivations is constructed by B. Ferreira, I. Kaygorodov and K. Kudaybergenov in [19]. After that, the first example of a simple (binary) algebra with non-trivial local derivations(automorphisms) was constructed by Sh. Ayupov, A. Elduque and K. Kudaybergenov in [4]. J. Adashev and B. Yusupov proved that direct sum null-filiform nilpotent Leibniz algebras as a rule admit local automorphisms which are not automorphisms [2]. J. Adashev, B. Yusupov and etc. proved that quasi-filiform Leibniz algebras of type I and type II as a rule admit local automorphisms which are not automorphisms [3, 31]. B. Yusupov proved that p-filiform Leibniz algebras as a rule admit local automorphisms which are not automorphisms [30].

In the paper [24], I. Karimjanov, S. Umrzaqov, and B. Yusupov describe automorphisms, local and 2-local automorphisms of solvable Leibniz algebras with a model or abelian null-radicals. They show that any local automorphism on solvable Leibniz algebras with a model nilradical, the dimension of the complementary space of which is maximal, is an automorphism. But solvable Leibniz algebras with an abelian nilradical with a 1-dimensional complementary space admit local automorphisms which are not automorphisms.

In the present paper we study automorphisms and local automorphisms on naturally graded filiform Leibniz algebra. We describe the local automorphisms of naturally graded filiform Leibniz algebra.

2. Preliminaries

In this section we give some necessary definitions and preliminary results.

Definition 2.1. *A vector space with bilinear bracket $(\mathcal{L}, [\cdot,\cdot])$ is called a Leibniz algebra if for any $x, y, z \in \mathcal{L}$ the so-called Leibniz identity*

$$\big[x,[y,z]\big] = \big[[x,y],z\big] - \big[[x,z],y\big],$$

holds.

Let $\mathcal{L}$ be a Leibniz algebra. For a Leibniz algebra $\mathcal{L}$ consider the following central lower and derived sequences:

$$\mathcal{L}^1 = \mathcal{L}, \quad \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}^1], \quad k \geq 1,$$
$$\mathcal{L}^{[1]} = \mathcal{L}, \quad \mathcal{L}^{[s+1]} = [\mathcal{L}^{[s]}, \mathcal{L}^{[s]}], \quad s \geq 1.$$

Definition 2.2. *A Leibniz algebra $\mathcal{L}$ is called nilpotent (respectively, solvable), if there exists $p \in \mathbb{N}$ $(q \in \mathbb{N})$ such that $\mathcal{L}^p = 0$ (respectively, $\mathcal{L}^{[q]} = 0$).The minimal number p (respectively, q) with such property is said to be the index of nilpotency (respectively, of solvability) of the algebra $\mathcal{L}$.*

Definition 2.3. *An n-dimensional Leibniz algebra $\mathcal{L}$ is said to be filiform if* $\dim \mathcal{L}^i = n - i$, *for* $2 \leq i \leq n$.

Now let us define a natural graduation for a nilpotent Leibniz algebra.

Definition 2.4. *Given a nilpotent Leibniz algebra $\mathcal{L}$, put $\mathcal{L}_i = \mathcal{L}^i/\mathcal{L}^{i+1}$, $1 \leq i \leq n-1$, and $gr(\mathcal{L}) = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_{n-1}$. Then $[\mathcal{L}_i, \mathcal{L}_j] \subseteq \mathcal{L}_{i+j}$ and we obtain the graded algebra $gr(\mathcal{L})$. If $gr(\mathcal{L})$ and L are isomorphic, then we say that an algebra $\mathcal{L}$ is naturally graded.*

In the following theorem were call the classification of the naturally graded filiform non-Lie Leibniz algebras given in [16].

Theorem 2.5. *Any complex n-dimensional naturally graded filiform non-Lie Leibniz algebra is isomorphic to one of the following non isomorphic algebras*
- $F_n^1 :\ [e_1, e_1] = e_3, \quad [e_i, e_1] = e_{i+1}, \quad 2 \leq i \leq n-1;$
- $F_n^2 :\ [e_1, e_1] = e_3, \quad [e_i, e_1] = e_{i+1}, \quad 3 \leq i \leq n-1.$

Now we give the definitions of automorphisms and local automorphisms.

Definition 2.6. *A linear bijective map $\varphi : \mathcal{L} \to \mathcal{L}$ is called an automorphism (resp. an anti- automorphism), if it satisfies $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ (resp. $\varphi([x, y]) = [\varphi(y), \varphi(x)]$) for all $x, y \in \mathcal{L}$.*

The set of all automorphisms on $\mathcal{L}$ we denote by $Aut(\mathcal{L})$.

Definition 2.7. *Let $\mathcal{L}$ be an algebra. A linear map $\Delta : \mathcal{L} \to \mathcal{L}$ is called a local automorphism, if for any element $x \in \mathcal{L}$ there exists an automorphism $\varphi_x : \mathcal{L} \to \mathcal{L}$ such that $\Delta(x) = \varphi_x(x)$.*

We denote the set of all local automorphisms on $\mathcal{L}$ by $LocAut(\mathcal{L})$.

3. Local automorphisms of filiform Leibniz algebras

In the following proposition we describe automorphisms of the algebras F_n^1 and F_n^2.

Proposition 3.1. *Any automorphism φ of the algebras F_n^1 or F_n^2 has the following form:*
for tha algebras F_n^1

$$\varphi(e_1) = \sum_{j=1}^{n} \alpha_j e_j, \quad \varphi(e_2) = (\alpha_1 + \alpha_2)e_2 + \sum_{j=3}^{n-1} \alpha_j e_j + \beta e_n,$$

$$\varphi(e_i) = \alpha_1^{i-2}(\alpha_1 + \alpha_2)e_i + \alpha_1^{i-2} \sum_{j=i+1}^{n} \alpha_{j-i+2} e_j, \quad 3 \leq i \leq n,$$

where $\alpha_1(\alpha_1 + \alpha_2) \neq 0$;
for tha algebras F_n^2

$$\varphi(e_1) = \sum_{j=1}^{n} \alpha_j e_j, \quad \varphi(e_2) = \beta e_2 + \gamma e_n,$$

$$\varphi(e_i) = \alpha_1^{i-1} e_i + \alpha_1^{i-2} \sum_{j=i+1}^{n} \alpha_{j-i+2} e_j, \quad 3 \leq i \leq n,$$

where $\alpha_1 \beta_2 \neq 0$;

Proof. Since the proof repeats the same arguments that were presented earlier for each case, a detailed proof will be given only for the algebra F_n^1, the rest of the case is similar.

Let φ be an automorphism of the algebra. We set

$$\varphi(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \quad \varphi(e_2) = \sum_{i=1}^{n} \beta_i e_i.$$

From the equality

$$0 = \varphi([e_1, e_2]) = [\varphi(e_1), \varphi(e_2)] = \left[\sum_{i=1}^{n} \alpha_i e_i, \sum_{i=1}^{n} \beta_i e_i\right] =$$

$$= \beta_1(\alpha_1 + \alpha_2)e_3 + \beta_1 \sum_{i=4}^{n} \alpha_{n-1} e_i,$$

we get $\beta_1 = 0$ and $\alpha_1 + \alpha_2 \neq 0$.

Further, we have

$$\varphi(e_3) = \varphi([e_1, e_1]) = [\varphi(e_1), \varphi(e_1)] = \left[\sum_{i=1}^{n} \alpha_i e_i, \sum_{i=1}^{n} \alpha_i e_i\right] =$$
$$= \alpha_1(\alpha_1 + \alpha_2)e_3 + \alpha_1 \sum_{i=4}^{n} \alpha_{i-1} e_i.$$

On the other hand,

$$\varphi(e_3) = \varphi([e_2, e_1]) = [\varphi(e_2), \varphi(e_1)] = \left[\sum_{i=2}^{n} \beta_i e_i, \sum_{i=1}^{n} \alpha_i e_i\right] =$$
$$= \alpha_1 \beta_2 e_3 + \alpha_1 \sum_{i=4}^{n} \beta_{n-1} e_i.$$

Therefore, $\beta_2 = \alpha_1 + \alpha_2$ and $\beta_i = \alpha_i$, where $4 \leq i \leq n-1$.

With similar arguments applied on the products $[e_i, e_1] = e_{i+1}$ and with an induction on i, it is easy to check that the following identities hold for $3 \leq i \leq n$:

$$\varphi(e_i) = \alpha_1^{i-2}(\alpha_1 + \alpha_2)e_i + \alpha_1^{i-2} \sum_{j=i+1}^{n} \alpha_{j-i+2} e_j.$$

Now, we shall prove the sufficiency condition of the theorem.

Let us take $u, v \in F_1^n$ and denote them by

$$u = \sum_{i=1}^{n} \lambda_i e_i, \quad v = \sum_{i=1}^{n} \mu_i e_i.$$

Consider

$$\varphi([u, v]) = \varphi\left(\left[\sum_{i=1}^{n} \lambda_i e_i, \sum_{i=1}^{n} \mu_i e_i\right]\right) = \varphi\left((\lambda_1\mu_1 + \lambda_2\mu_1)e_3 + \mu_1 \sum_{i=4}^{n} \lambda_i e_i\right) =$$
$$= (\lambda_1\mu_1 + \lambda_2\mu_1)\varphi(e_3) + \mu_1 \sum_{i=4}^{n} \lambda_i \varphi(e_i) = (\lambda_1\mu_1 + \lambda_2\mu_1)\alpha_1(\alpha_1 + \alpha_2)e_3 +$$
$$+ (\lambda_1\mu_1 + \lambda_2\mu_1)\alpha_1 \sum_{j=4}^{n} \alpha_{j-1} e_j + \mu_1 \sum_{i=4}^{n} \lambda_i(\alpha_1^{i-2}(\alpha_1 + \alpha_2)e_i) +$$
$$+ \mu_1 \sum_{i=4}^{n} \lambda_i \alpha_1^{i-2} \sum_{j=i+1}^{n} \alpha_{j-i+1} e_j.$$

On the other hand,

$$
\begin{aligned}
[\varphi(u), \varphi(v)] &= \left[\varphi\left(\sum_{i=1}^{n}\lambda_i e_i\right), \varphi\left(\sum_{i=1}^{n}\mu_i e_i\right)\right] = \\
&= [\lambda_1 \sum_{j=1}^{n}\alpha_j e_j + \lambda_2(\alpha_1+\alpha_2)e_2 + \lambda_2\sum_{j=3}^{n-1}\alpha_j e_j + \lambda_2\beta e_n + \\
&+ \sum_{i=3}^{n}\lambda_i\left(\alpha_1^{i-2}(\alpha_1+\alpha_2)e_i + \alpha_1^{i-2}\sum_{j=i+1}^{n}\alpha_{j-i+2}e_j\right), \\
&\mu_1\sum_{j=1}^{n}\alpha_j e_j + \mu_2(\alpha_1+\alpha_2)e_2 + \mu_2\sum_{j=3}^{n-1}\alpha_j e_j + \mu_2\beta e_n + \\
&+ \sum_{i=3}^{n}\mu_i\left(\alpha_1^{i-2}(\alpha_1+\alpha_2)e_i + \alpha_1^{i-2}\sum_{j=i+1}^{n}\alpha_{j-i+2}e_j\right)] = \\
&= (\lambda_1\mu_1 + \lambda_2\mu_1)\alpha_1(\alpha_1+\alpha_2)e_3 + \\
&+ (\lambda_1\mu_1+\lambda_2\mu_1)\alpha_1\sum_{j=4}^{n}\alpha_{j-1}e_j + \mu_1\sum_{i=4}^{n}\lambda_i(\alpha_1^{i-2}(\alpha_1+\alpha_2)e_i) + \\
&+ \mu_1\sum_{i=4}^{n}\lambda_i\alpha_1^{i-2}\sum_{j=i+1}^{n}\alpha_{j-i+1}e_j.
\end{aligned}
$$

Comparing coefficients at the basis elements we obtain

$$\varphi([u,v]) = [\varphi(u), \varphi(v)]$$

and we complete the proof of proposition. □

In the following theorem we give the description of local automorphism of the algebras F_1^n and F_2^n.

Theorem 3.2. *Let Δ be a linear mapping on a complex n-dimensional naturally graded filiform non-Lie Leibniz algebra. Any local automorphism Δ of the algebras F_n^1 or F_n^2 has the following form: for tha algebras F_n^1*

$$
\begin{cases}
\Delta(e_1) &= \sum\limits_{j=1}^{n} c_{j,1}e_j, \\
\Delta(e_2) &= (c_{1,1}+c_{2,1})e_2 + \sum\limits_{j=3}^{n-1} c_{j,1}e_j + c_{n,2}e_n, \\
\Delta(e_i) &= \sum\limits_{j=i}^{n} c_{j,i}e_j, \quad 3 \le i \le n,
\end{cases}
\tag{3.1}
$$

where $c_{1,1}^2 + (c_{1,1}+c_{2,1})^2 + c_{3,3}^2 + \cdots + c_{n,n} \neq 0$;
for tha algebras F_n^2

$$
\begin{cases}
\Delta(e_1) &= \sum\limits_{j=1}^{n} c_{j,1}e_j, \\
\Delta(e_2) &= c_{2,2}e_2 + c_{n,2}e_n, \\
\Delta(e_i) &= \sum\limits_{j=i}^{n} c_{j,i}e_j, \quad 3 \le i \le n,
\end{cases}
\tag{3.2}
$$

where $\prod\limits_{i=1}^{n} c_{i,i} \neq 0$;

Proof. Since the proof repeats the same arguments that were presented earlier for each case, a detailed proof will be given only for the algebra F_n^1, the rest of the case is similar.

Let Δ be a local automorphisms of the algebras F_n^1 and let

$$\Delta(e_i) = \sum_{j=1}^{n} c_{j,i}e_j, \quad 1 \le i \le n.$$

Step 1. Take an automorphism φ_{e_2} such that $\Delta(e_2) = \varphi_{e_2}(e_2)$. Then

$$
\begin{aligned}
\Delta(e_2) &= \sum_{j=1}^{n} c_{j,2} e_j, \\
\varphi_{e_2}(e_2) &= (\alpha_1 + \alpha_2)e_2 + \sum_{j=3}^{n-1} \alpha_j e_j + \beta e_n.
\end{aligned}
$$

Comparing the coefficients at the basis elements, we get $c_{1,2} = 0$.

Step 2. Take an automorphism φ_{e_i} such that $\Delta(e_i) = \varphi_{e_i}(e_i)$, where $3 \leq i \leq n$. Then

$$
\begin{aligned}
\Delta(e_i) &= \sum_{j=1}^{n} c_{j,i} e_j, \\
\varphi_{e_i}(e_i) &= \alpha_1^{i-2}(\alpha_1 + \alpha_2)e_i + \alpha_1^{i-2} \sum_{j=i+1}^{n} \alpha_{j-i+2} e_j.
\end{aligned}
$$

Comparing the coefficients at the basis elements for $\Delta(e_i)$ and $\varphi_{e_i}(e_i)$, we obtain $c_{1,i} = c_{2,i} = \cdots = c_{i-1,i} = 0$.

Taking the point $x_0(1, -1, 0, \cdots, 0) \in F_n^1$, consider the value of automorphisms on this point which means $\Delta(x_0) = \varphi_{x_0}(x_0)$:

$$
\begin{array}{ll}
\Delta(x_0)_1 & = c_{1,1}, \\
\Delta(x_0)_2 & = c_{2,1} - c_{2,2}, \\
\Delta(x_0)_i & = c_{i,1} - c_{i,2}, \quad 3 \leq i \leq n,
\end{array} .
$$

and

$$
\begin{array}{ll}
\varphi(x_0)_1 & = \alpha_1, \\
\varphi(x_0)_2 & = -\alpha_2, \\
\varphi(x_0)_i & = 0, \quad 3 \leq i \leq n-1 \\
\varphi(x_0)_2 & = \alpha_n - \beta.
\end{array} .
$$

Due to the equality $\Delta(x_0) = \varphi(x_0)$ we get the following relations:

$$
\begin{array}{l}
c_{2,2} = c_{1,1} + c_{2,1} \\
c_{i,1} = c_{i,2}, \quad 3 \leq i \leq n-1.
\end{array} .
$$

Thus, the operator Δ has the form (3.1).

Assume that the operator Δ has the form (3.1). Take an arbitrary element $x = \sum\limits_{j=1}^{n} x_j e_j$.

The coordinates of $\Delta(x)$ on the basis $\{e_1, e_2, \ldots, e_n\}$ are

$$
\begin{aligned}
\Delta(x)_1 &= c_{1,1} x_1, \\
\Delta(x)_2 &= c_{2,1} x_1 + (c_{1,1} + c_{2,1}) x_2, \\
\Delta(x)_i &= c_{i,1}(x_1 + x_2) + \sum_{j=1}^{i} c_{i+1,j} x_j, \quad 3 \leq i \leq n-1, \\
\Delta(x)_n &= \sum_{j=1}^{n} c_{n,j} x_j.
\end{aligned}
$$

The coordinates of $\varphi(x)$ are

$$\begin{array}{rcl} \varphi(x)_1 & = & \alpha_1 x_1, \\ \varphi(x)_2 & = & \alpha_2 x_1 + (\alpha_1+\alpha_2)x_2, \\ \varphi(x)_i & = & \alpha_i(x_1+x_2) + \sum\limits_{j=3}^{i-1} \alpha_1^{j-2}\alpha_{i+2-j}x_j + \alpha_1^{i-2}(\alpha_1+\alpha_2)x_i, \quad 3 \leq i \leq n-1, \\ \varphi(x)_n & = & \alpha_n x_1 + \beta x_2 + \sum\limits_{j=3}^{n-1} \alpha_1^{j-2}\alpha_{i+2-j}x_j + \alpha_1^{n-2}(\alpha_1+\alpha_2)x_n. \end{array}$$

Comparing the coordinates of $\Delta(x)$ and $\varphi(x)$, we obtain

$$\begin{cases} \alpha_1 x_1 & = c_{1,1}x_1, \\ \alpha_2 x_1 + (\alpha_1+\alpha_2)x_2 & = c_{2,1}x_1 + (c_{1,1}+c_{2,1})x_2, \\ \dots\ \dots\ \dots\dots\dots & \dots\dots\dots\dots \\ \alpha_i(x_1+x_2) + \sum\limits_{j=3}^{i-1} \alpha_1^{j-2}\alpha_{i+2-j}x_j + & \\ +\alpha_1^{i-2}(\alpha_1+\alpha_2)x_i & = c_{i,1}(x_1+x_2) + \sum\limits_{j=1}^{i} c_{i+1,j}x_j, \quad 3 \leq i \leq n-1, \\ \alpha_n x_1 + \beta x_2 + \sum\limits_{j=3}^{n-1} \alpha_1^{j-2}\alpha_{i+2-j}x_j + & \\ +\alpha_1^{n-2}(\alpha_1+\alpha_2)x_n & = \sum\limits_{j=1}^{n} c_{n,j}x_j. \end{cases} \tag{3.3}$$

Now we consider the following possible cases.

Case 1. Let $x_1 + x_2 \neq 0$.

Subcase 1.1. Let $x_1 \neq 0$ and $x_2 = 0$. In this case, putting $\beta = 0$, from (3.3) we uniquely determine $a_i,\ 1 \leq i \leq n$.

Subcase 1.2. Let $x_1 = 0$ and $x_2 \neq 0$. Then $\alpha_i, \beta, 1 \leq i \leq n-1$ can be found in infinitely many ways from (3.3).

Subcase 1.3. Let $x_1 \neq 0$ and $x_2 \neq 0$. Then $\alpha_i, 1 \leq i \leq n-1$ can be found in a unique way, $a_n,\ \beta$ can be done in infinitely many ways (3.3).

Case 2. Let $x_1 + x_2 = x_3 = \cdots = x_{k-1} = 0,\ x_k \neq 0,\ 3 \leq k \leq n$.

Subcase 2.1. Let $x_1 = 0$ and $x_2 = 0$. Then $a_i,\ \beta,\ 1 \leq i \leq n$ in the system (3.3) have infinitely many solution.

Subcase 2.2. Let $x_1 = -x_2$. Then putting $\beta = 0$, from (3.3) we uniquely determine $a_i,\ \beta,\ 1 \leq i \leq n$.

Theorem 3.2 is proved. □

Remark 3.3. *Note that the general form of the matrix of a local automorphism on an algebra includes the general form of the matrix of an automorphism on this algebra. The coincidence of these general forms denotes that every local automorphism of the considering algebra is an automorphism. But the general form of the matrix of an automorphism on filiform Leibniz algebras F_n^1 and F_n^2 does not coincide with the general form of the matrix of a local automorphism on these algebras by theorem 3.3. Therefore, the filiform Leibniz algebras F_n^1 and F_n^2 have local automorphisms that are not automorphisms*

Corollary 3.4. *The algebras F_n^1 and F_n^2 admit local automorphisms which are not automorphisms.*

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TO THE QUALITATIVE PROPERTIES OF SELF-SIMILAR SOLUTIONS OF A CROSS-DIFFUSION PARABOLIC SYSTEM NOT IN DIVERGENCE FORM WITH A SOURCE AND CONVECTIVE TRANSFER

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Abstract

The study is devoted to the analysis of the qualitative properties of asymptotic solutions of a system of non-divergent parabolic cross-diffusion equations. These equations describe a complex filtration process involving sources and convective effects. Our research aims to understand the complex dynamics exhibited by self-similar solutions in this system. Using a rigorous analytical approach, we aim to uncover the fundamental mechanisms governing their evolution and stability. The results of this study are expected to improve our understanding of system behavior and provide valuable information applicable to various domains.

Keywords: Nonlinear system of equations, not in divergence form, global solutions, self-similar solutions, asymptotic representation of solution.

Introduction. The primary strength of this study lies in its comprehensive analysis based on self-similar solutions of a system of non-divergent parabolic equations. Addressing such a system represents a significant mathematical challenge due to the precision and robust analytical methods required. This research introduces an effective algorithm designed to tackle these complexities, enabling a detailed examination of the system's asymptotic behavior. Furthermore, the study rigorously establishes the feasibility of determining upper and lower bounds for the solutions. Unlike existing methodologies, the proposed algorithm demonstrates superior accuracy in predicting the asymptotic behavior of non-divergent parabolic systems. Additionally, novel mathematical techniques for estimating solutions within these bounds are presented.

The central aim of this research was to investigate solutions derived from self-similar formulations and to analyze the mathematical characteristics of non-divergent parabolic systems in detail. The study developed an algorithm to identify weak solutions and provided rigorous proofs of relevant theorems, ensuring the reliability of the results. A key focus was to obtain an initial approximation for the numerical solution of these problems, a feature that distinguishes this approach due to its rapid convergence to exact solutions. The study successfully addresses this issue, marking an important step in advancing mathematical modeling for non-divergent systems. Numerical solutions, along with their graphical representations in both one-dimensional and multidimensional contexts, underscore the practical implications of the findings, particularly in modeling physical and biological phenomena. Consequently, this work provides a robust theoretical foundation for addressing more complex mathematical challenges in future research.

Methods and models. We consider convective transport defined in the domain $Q=\{(t,x):t>0,x\in R\}$ and the nonlinear nondivergent cross-diffusion parabolic equation system with source of Koshi problem

$$\frac{\partial u_i}{\partial t}=u_i^{\alpha_i}\nabla\left(u_{2-i}^{m_i-1}\left|\nabla u_i^k\right|^{p-2}\nabla u_i\right)-div\left(c(t)u_i\right)+u_i^{\beta_i}$$
(1)

$$u_i(0,x)=u_{0i}(x),\ x\in R \tag{2}$$

Here $k>0,p\geq 2,m_i\geq 1(i=1,2)$ numeric parameters, $\nabla(\cdot)=grad_x(\cdot)$, $u_i=u_i(t,x)\geq 0$ sought negative solutions, $u_{0,i}(x)$ bounded, continuous, nonnegative functions. It is known that the system of equations (1) is degenerate, $u_i(t,x)=0$ or $|\nabla u_i|=0$ may not have a solution in the classical sense. Therefore, solutions of (1)-(2) $u_{3-i}^{m_i-1}\left|\nabla u_i^k\right|^{p-2}\nabla u_i\in C\left(R_+\times(0,+\infty)\right)$ $u_i\geq 0(i=1,2)$ s considered as a generalized solution in the class, and this solution satisfies the system of equations (1) in the integral sense. In many works, special cases of problem (1)-(2) have been considered and it is stated that they describe many physical and biological processes. [1].

(1)- (2) task has been studied by many scientists in private cases. The first is the non-divergent parabolic type in the following form

$$\frac{\partial u}{\partial t}=u^p\Delta u+u^q \tag{3}$$

$$\begin{aligned} &u=0 \quad (x\in\partial\Omega,\ t>0),\\ &u(x,0)=\phi(x)\geq 0\ \left(x\in\Omega,\ \Omega\in R^N\right)\end{aligned} \tag{4}$$

The study of equations was studied by Friedman A., Mcleod J. in 1986 and by Dal Passo and Luckhaus in 1987. In particular, Friedman and McLeod (3)-(4) of the problem started by studying the global and generalized solutions for the case $p=2,\ q=3$. This problem (3)-(4) reflects the model of diffusion of weak magnetic fields in the plasma located between two walls [3].

In this work [4], representing the nondivergent diffusion process under the influence of varying density and source, the following

$$|x|^{-l}\frac{\partial u}{\partial t}=u^q\frac{\partial}{\partial x}\left(|x|^n u^{m-1}\left|\frac{\partial u^k}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right)+|x|^{-l}u^\beta$$

$$u(0,x)=u_0(x)\geq 0,\ x\in R$$

qualitative properties of the solutions of the Cauchy problem placed in the parabolic equation were studied. In this case, the conditions for the existence of the global solution were found and the asymptotic behavior of the solution was studied.

In [5]

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(v^{m_1-1}\frac{\partial u}{\partial x}\right),\frac{\partial v}{\partial t}=\frac{\partial}{\partial x}\left(u^{m_2-1}\frac{\partial v}{\partial x}\right),\ x\in R_+,t>0,$$

system of cross-diffusion parabolic equations is considered with nonlocal boundary conditions. Theorems about the global solution and the asymptotics of the solution were proved and numerical results were obtained. Based on the obtained results, graphs are depicted at different values of the parameters for slow and fast diffusion cases.

In this [6] work, it is as follows

$$\frac{\partial(\rho(x)u)}{\partial t}=div\left(|x|^n v^{m_1-1}|\nabla u|^{p-2}\nabla u\right)+\rho(x)\gamma(t)u^{\beta_1}$$
$$\frac{\partial(\rho(x)v)}{\partial t}=div\left(|x|^n u^{m_2-1}|\nabla v|^{p-2}\nabla v\right)+\rho(x)\gamma(t)v^{\beta_2}$$

$$u(0,x)=u_0(x)\geq 0,v(0,x)=v_0(x)\geq 0,x\in R^N$$

qualitative properties of the system of parabolic equations with variable density of divergent form were studied. In particular, a system of self-similar equations was constructed, solutions of the self-similar equation system were found, and asymptotics were analyzed. Theorems about the existence of global solution for the case of slow diffusion have been proved and numerical results have been obtained.

In this [7] work, the nonlinear divergent has the following form

$$|x|^n\frac{\partial u}{\partial t}=div\left(|x|^k v^{m_1-1}|\nabla u|^{p-2}\nabla u\right)+|x|^n u^{\beta_1}$$
$$|x|^n\frac{\partial v}{\partial t}=div\left(|x|^n u^{m_2-1}|\nabla v|^{p-2}\nabla v\right)+|x|^n v^{\beta_2}$$

the Cauchy problem applied to the system of parabolic equations is considered. Self-similar solution properties were studied and solutions of Zeldovich-Barenblatt type were obtained. The asymptotic behavior of the self-similar solution was analyzed, and global conditions were obtained for the case of slow and fast diffusion. Numerical results are used to plot the graphs for slow and fast diffusion cases.

In this work [8], in critical cases, two-component media with a source and variable density studied Koshi problem

$$|x|^{-l}\frac{\partial u_i}{\partial t}=u_i^{\alpha_i}\nabla\left(|x|^n u_{3-i}^{m_i-1}\left|\Delta u_i^k\right|^{p-2}\Delta u_i\right)+|x|^{-l}u_i^{\beta_i} \tag{5}$$

$$u_i(0,x)=u_{0,i}(x),x\in R^N\ (i=1,2) \tag{6}$$

Theorems about the global solution of the problem and the asymptotics of the solution are proved. Weak solutions of the system are considered. Theorems about global and asymptotic solutions are proved and the results derived from them are given. Special cases of the issue (5)-(6) are considered in cases [9-12]. In this, theorems about global and asymptotic solutions are proved and the results derived from them are given, and numerical results are obtained using the driving method in the Python programming language. Based on them, graphs for fast and slow diffusion processes are depicted.

Building a system of self-similar equations.

To do this, we perform a form substitution in the system of equations as follows.

$$u_i(t,x)=z_i(t,\zeta),\ \text{here}\ \zeta=\int_0^t c(y)dy-x \tag{7}$$

As a result, the system of equations (1) will have the following form.

$$\frac{\partial z_i}{\partial t}=z_i^{\alpha_i}\nabla\left(z_{2-i}^{m_i-1}\left|\nabla z_i^k\right|^{p-2}\nabla z_i\right)+z_i^{\beta_i} \tag{8}$$

To switch from the system of 8)-equations to the system of self-similar equations, we use the nonlinear separation algorithm presented in the literature [2]. That is, we look for the solution of the system of equations (8) in the following form.

$$z_i(t,\zeta)=\bar{u}_i(t)\cdot w_i(\tau(t),\zeta) \tag{9}$$

after replacing (9), the system of 8 equations looks like this

$$\frac{\partial w_i}{\partial \tau}=w_i^{\alpha_i}\frac{\partial}{\partial \zeta}\left(w_{3-i}^{m_i-1}\left|\frac{\partial w_i^k}{\partial \zeta}\right|^{p-2}\frac{\partial w_i}{\partial \zeta}\right)+\frac{\psi_i}{\tau}\left(w_i^{\beta_i}-\frac{1}{1-\beta_i}w_i\right) \tag{10}$$

Here the helper functions and parameters are defined as follows.

$$\tau(t)=\begin{cases}\dfrac{(T+t)^{\sigma}}{\sigma} & at \quad \sigma\neq 0\\ \ln(T+t) & at \quad \sigma=0\end{cases}$$

$$T>0,\sigma=\frac{k(p-2)+\alpha_1}{1-\beta_1}+\frac{m_1-1}{1-\beta_2}+1=\frac{k(p-2)+\alpha_2}{1-\beta_2}+\frac{m_2-1}{1-\beta_1}+1$$

$$\psi_i=\left(\frac{k(p-2)+\alpha_i}{1-\beta_i}+\frac{m_i-1}{1-\beta_{3-i}}+1\right)^{-1}$$

In order to switch from the system of equations (1) to the system of self-similar equations

$$w_i(\zeta,\tau)=f_i(\xi)\ ,\xi=\frac{|\zeta|}{\tau^{\frac{1}{p}}}$$

as a result, we get the following system of self-similar equations.

$$f_i^{\alpha_i}\frac{d}{d\xi}\left(f_{3-i}{}^{m_i-1}\left|\frac{df_i^k}{d\xi}\right|^{p-2}\frac{df_i}{d\xi}\right)+\frac{\xi}{p}\frac{df_i}{d\xi}+\psi_i\left(f_i^{\beta_i}-\frac{1}{1-\beta_i}f_i\right)=0 \qquad (11)$$

We are engaged in finding non-trivial, non-negative solutions of the system of self-similar equations (12) that satisfy the following conditions.

$$\begin{aligned}&f_1(0)=M_1,f_2(0)=M_2,M_1\in R,M_2\in R\\ &f_1(d_1)=f_2(d_2)=0,0<d_1<\infty,\ 0<d_2<\infty\end{aligned} \qquad (12)$$

Evaluation of the solution of problem (1)-(2).

The case of slow diffusion. Globality conditions of the solution.

We implement weak solutions of system (1) by using the comparison principle of global solution. For this, we will look for our solution in the following form.

$$u_{i+}(t,x)=(T+t)^{n_i}\,\bar{f}_i(\xi) \qquad (13)$$

$$\bar{f}_i(\xi)=A_i\left(a-\xi^{\gamma}\right)_+^{\gamma_i} \qquad (14)$$

Here, the numerical parameters are defined as follows.

$$a>0,\gamma_i=\frac{(p-1)\left(k(p-2)+\alpha_{3-i}-m_i+1\right)}{\left(k(p-2)+\alpha_i\right)\left(k(p-2)+\alpha_{3-i}\right)-(m_i-1)(m_{3-i}-1)}$$

$$\gamma=\frac{p}{p-1},\ A_i,n_i=\frac{1}{1-\beta_i}\ (i=1,2)$$

$$A_i^{\alpha_i+k(p-2)}A_{3-i}^{m_i-1}=\frac{\frac{\gamma\gamma_i}{p}+\frac{\psi_i}{1-\beta_i}}{\gamma\gamma_i|\gamma\gamma_i|^{p-2}(\gamma\gamma_{i+2}-1)},\ \gamma_{i+2}=(\gamma_i k-1)(p-2)+\gamma_{3-i}(m_i-1)+\gamma_i-1 \quad (13)$$

using our found function, we present theorems related to the topic and the results derived from them.

Theorem 1. (Terms of globality). Let's assume $\gamma_i>0$,

$$\gamma\gamma_{i+2}A_i^{\alpha_i+k(p-2)}\cdot A_{3-i}^{m_i-1}k^{p-2}|\gamma\gamma_i|^{p-2}=\frac{1}{p},$$

$$\psi_i\left(A_i^{\beta_i-1}a^{\gamma_i\beta_i-\gamma_i}-\frac{1}{1-\beta_i}\right)-\frac{\gamma_i}{p\gamma_{i+2}}\le 0,,$$

$$u_i(t,0)\le u_{i+}(t,0),x\in R\ (i=1,2),$$

t the relationship be appropriate. Then problem (1)-(2) has a global solution in the field and the following estimates hold for it.

$$u_i(t,x)\le u_{i+}(t,x)=(T+t)_i^{n_i}\bar{f}_i(\xi).$$

Proof. We use the comparison theorem to prove Theorem 1 [1]. We get (13) as comparison functions. We carry (13) into (1) and evaluate the following inequalities.

$$f_i^{\alpha_i}\frac{d}{d\xi}\left(f_{3-i}^{m_i-1}\left|\frac{df_i^k}{d\xi}\right|^{p-2}\frac{df_i}{d\xi}\right)+\frac{\xi}{p}\frac{df_i}{d\xi}+\psi_i\left(f_i^{\beta_i}-\frac{1}{1-\beta_i}f_i\right)\le 0 \quad (15)$$

If, as stated in the condition of the theorem $\gamma\gamma_{i+2}A_i^{\alpha_i+k(p-2)}\cdot A_{3-i}^{m_i-1}k^{p-2}|\gamma\gamma_i|^{p-2}=\frac{1}{p}$ let's say equalities are appropriate

$$\psi_i\left(A_i^{\beta_i-1}(a-\xi^{\gamma})^{\gamma_i\beta_i-\gamma_i}-\frac{1}{1-\beta_i}\right)-\frac{\gamma_i}{\gamma_{i+2}p}\le 0$$

we will have inequality. Here considering that, $A_i^{\beta_i-1}a^{\gamma_i\beta_i-\gamma_i}\ge A_i^{\beta_i-1}(a-\xi^{\gamma})^{\gamma_i\beta_i-\gamma_i}$ then

$$\psi_i\left(A_i^{\beta_i-1}a^{\gamma_i\beta_i-\gamma_i}-\frac{1}{1-\beta_i}\right)-\frac{\gamma_i}{p\gamma_{i+2}}\le 0,(i=1,2)$$

it follows that the inequality is appropriate. Moreover, according to the hypotheses of Theorem 1 and the principle of comparison, the following relations are valid. $u_{i+}(t,x)\ge u_i(t,x), x\in R$, in the domain Q if $u_{i+}(t,0)\ge u_i(t,0),(i=1,2),x\in R$ relationship is appropriate. The theorem is proved.

Asymptotic behavior of the solution of the self-similar system of equations.

Before presenting the theorems and results based on the study of the asymptotic behavior of the solution of the Self-similar system of equations, we introduce the following notations.

$$a_{1i}(\eta)=-\gamma_{i+2}+\frac{e^{-\eta}}{\gamma\left(a-e^{-\eta}\right)},\qquad a_{2i}(\eta)=\frac{1}{p\cdot\gamma^{p-1}}$$
$$a_{3i}(\eta)=\frac{\psi_i e^{(\gamma_i-\gamma_i\beta_i-1)\eta}}{\gamma^p\left(a-e^{-\eta}\right)},\ a_{4i}(\eta)=\frac{\psi_i e^{-\eta}}{(1-\beta_i)\gamma^p\left(a-e^{-\eta}\right)}. \tag{16}$$

Here $\gamma_{i+2}=\gamma_{3-i}(m_i-1)+(\gamma_i k-1)(p-2)+\gamma_i-1\ (i=1,2)$

Let us assume that the following equality holds between the coefficients of system (1).

$$\gamma_1\left(k(p-2)+\alpha_1\right)+\gamma_2(m_1-1)=\gamma_2\left(k(p-2)+\alpha_2\right)+\gamma_1(m_2-1) \tag{17}$$

In that case, we will give the following theorems as a result of them.

Theorem 2. Let's assume that $\gamma_i>0$. Then (11), (12) are solutions of the problem with a compact operator the following asymptotics at $\xi\to a^{\frac{p-1}{p}}$

$$f_i(\xi)=c_i\left(a-\xi^{\gamma}\right)^{\gamma_i}\left(1+o(1)\right),(i=1,2) \tag{18}$$

to be eligible, one of the following conditions must be met:

1. 1. $\gamma_1-\gamma_1\beta_1-1=0,\gamma_2-\gamma_2\beta_2-1=0$ and $c_i\ (i=1,2)$ corresponding solutions of any of the systems of nonlinear algebraic equations shown below

$$a_{11}\cdot c_2^{m_1-1}c_1^{k(p-2)+1}\gamma_1^{p-1}k^{p-2}+a_{12}\cdot\gamma_1\cdot c_1^{1-\alpha_1}-a_{13}\cdot c_1^{\beta_1-\alpha_1}=0$$
$$a_{21}\cdot c_1^{m_2-1}c_2^{k(p-2)+1}\gamma_2^{p-1}k^{p-2}+a_{22}\cdot\gamma_2\cdot c_2^{1-\alpha_2}-a_{23}\cdot c_2^{\beta_2-\alpha_2}=0.$$

2. $\gamma_1-\gamma_1\beta_1-1<0,\gamma_2-\gamma_2\beta_2-1=0$ and $c_i\ (i=1,2)$ corresponding solutions of any of the systems of nonlinear algebraic equations shown below

$$a_{11}\cdot c_2^{m_1-1}c_1^{k(p-2)+1}\gamma_1^{p-1}k^{p-2}+a_{12}\cdot\gamma_1\cdot c_1^{1-\alpha_1}=0$$
$$a_{21}\cdot c_1^{m_2-1}c_2^{k(p-2)+1}\gamma_2^{p-1}k^{p-2}+a_{22}\cdot\gamma_2\cdot c_2^{1-\alpha_2}-a_{23}\cdot c_2^{\beta_2-\alpha_2}=0.$$

3. $\gamma_1-\gamma_1\beta_1-1=0,\gamma_2-\gamma_2\beta_2-1<0$ and $c_i\ (i=1,2)$ corresponding solutions of any of the systems of nonlinear algebraic equations shown below

$$a_{11}\cdot c_2^{m_1-1}c_1^{k(p-2)+1}\gamma_1^{p-1}k^{p-2}+a_{12}\cdot\gamma_1\cdot c_1^{1-\alpha_1}-a_{13}\cdot c_1^{\beta_1-\alpha_1}=0$$
$$a_{21}\cdot c_1^{m_2-1}c_2^{k(p-2)+1}\gamma_2^{p-1}k^{p-2}+a_{22}\cdot\gamma_2\cdot c_2^{1-\alpha_2}=0.$$

4. $\gamma_1-\gamma_1\beta_1-1<0,\gamma_2-\gamma_2\beta_2-1<0$ and $c_i\ (i=1,2)$ corresponding solutions of any of the systems of nonlinear algebraic equations shown below

$$a_{11}\cdot c_2^{m_1-1}c_1^{k(p-2)+1}\gamma_1^{p-1}k^{p-2}+a_{12}\cdot\gamma_1\cdot c_1^{1-\alpha_1}=0$$
$$a_{21}\cdot c_1^{m_2-1}c_2^{k(p-2)+1}\gamma_2^{p-1}k^{p-2}+a_{22}\cdot\gamma_2\cdot c_2^{1-\alpha_2}=0.$$

Proof: In order to study the asymptotics of the solutions of problem (11)-(12), we will look for the solution of the system of automodel equations (11) in the following form.

$$f_i(\xi)=\bar{f}_i(\xi)\cdot y_i(\eta),\ \ \eta=-\ln\left(a-\xi^{\gamma}\right),\left(i=1,2\right) \tag{19}$$

Here $\bar{f}_i(\xi)=\left(a-\xi^{\gamma}\right)^{\gamma_i}, a>0$, $y_i(\eta)$ $(i=1,2)$- sought functions

We transfer (19) to system (11), as a result we have the following system with respect to $y_i(\eta)$.

$$\begin{gathered}\frac{dL_i y}{d\eta}+a_{i1}(\eta)\cdot L_i y+y^{-\alpha_i}(\eta)\cdot\left(\frac{dy(\eta)}{d\eta}-\gamma_i\cdot y(\eta)\right)\cdot a_{i2}(\eta)+y^{\beta_i-\alpha_i}(\eta)\cdot a_{i3}(\eta)-\\-y^{1-\alpha_i}(\eta)\cdot a_{i4}(\eta)=0\end{gathered} \tag{20}$$

Here defined as $L_i\left(y\right)=y_{3-i}^{m_i-1}\left|\frac{dy_i^k}{d\eta}-k\gamma_i y_i^k\right|^{p-2}\left(\frac{dy_i}{d\eta}-\gamma_i y_i\right),\left(i=1,2\right)$. Let's assume defined in

$\xi\in\left[\xi_0,\xi_1\right), 0<\xi_0<\xi_1$, $\xi_1=a^{\frac{p}{p-1}}$. In this case $h(x)$ function will have the following properties.

$$\xi\in\left(\xi_0,\xi_1\right]\text{da } \eta'(\xi)>0, \eta_0=\eta_0(\xi), \lim_{\xi\to\xi_0}\eta(\xi)=+\infty$$

In the sytem (20)

$$v_i(\eta)=L_i\left(y\right)=y_{3-i}^{m_i-1}\left|\frac{dy_i^k}{d\eta}-k\gamma_i y_i^k\right|^{p-2}\left(\frac{dy_i}{d\eta}-\gamma_i y_i\right) \tag{21}$$

we enter the definition. As a result, system (20) is expressed in the following form.

$$\begin{gathered}v'(\eta)=-a_{i1}(\eta)\cdot v_i(\eta)-y^{-\alpha_i}(\eta)\cdot\left(\frac{dy(\eta)}{d\eta}-\gamma_i\cdot y(\eta)\right)\cdot a_{i2}(\eta)-y^{\beta_i-\alpha_i}(\eta)\cdot a_{i3}(\eta)+\\+y^{1-\alpha_i}(\eta)\cdot a_{i4}(\eta)=0.\end{gathered} \tag{22}$$

Now let's consider at the following function.

$$\begin{gathered}g_i(\lambda_i,\eta)\equiv-a_{i1}(\eta)\cdot\lambda_i-y^{-\alpha_i}(\eta)\cdot\left(\frac{dy(\eta)}{d\eta}-\gamma_i\cdot y(\eta)\right)\cdot a_{i2}(\eta)-\\-y^{\beta_i-\alpha_i}(\eta)\cdot a_{i3}(\eta)+y^{1-\alpha_i}(\eta)\cdot a_{i4}(\eta)=0\end{gathered} \tag{23}$$

Here $\lambda_i\in R,\left(i=1,2\right)$

$\gamma_i>0$ let the inequality be fulfilled. Both at $\eta\to\infty$ take into account the following limits

$$\lim_{\eta\to\infty} a_{i1}(\eta) = -\gamma_{2+i}, \lim_{\eta\to\infty} a_{i2}(\eta) = \frac{1}{p\cdot\gamma^{p-1}}, \lim_{\eta\to\infty} a_{i3}(\eta) = \begin{cases} 0 & 1+\gamma_1\beta_1-\gamma_1>0 \\ \dfrac{\psi_i}{\gamma^p a} & 1+\gamma_1\beta_1-\gamma_1=0 \end{cases},$$

$\lim_{\eta\to\infty} a_{i4}(\eta) = 0\ (i=1,2)$ then $\lambda_i, (i=1,2)$ at each fixed value of the parameters for the functions $g_i(\lambda_i,\eta)$ respectively $[\eta_i,+\infty)\subset[\eta_0,+\infty)\ (i=1,2)$, here are intervals such that in these intervals $g_i(\lambda_i,\eta)$ will keep its sign. That is

$$g_i(\lambda_i,\eta)>0 \text{ yoki } g_i(\lambda_i,\eta)<0, (i=1,2) \tag{24}$$

Let's assume $v_i(\eta)$ function $\eta\to+\infty$ do not have a finite limit. If any of the inequalities (24) is valid, we consider the following cases. In line $v_i(h)(i=1,2)$ functions $\bar{v}_i = l_i\ (i=1,2)$ that oscillate around straight lines, its graphs accordingly crosses infinitely many times at $[\eta_i;\infty)$ $(i=1,2)$. However, this cannot happen because in the interval $[\eta_i;\infty)\ (i=1,2)$ only one of the inequalities (24) holds, and therefore according to the equalities (23) $v_i(h)$, $(i=1,2)$ graphic of the functions $\bar{v}_i = l_i\ (i=1,2)$ straight lines intersects at only one point in the interval $[\eta_i;\infty)\ (i=1,2)$. Therefore for the functions $v_i(h)(i=1,2)$ at $\eta\to+\infty$ has the limit. $v_i(\eta)\ (i=1,2)$ taking into account that the functions are represented by equation (21) $\eta\to+\infty$ If we say that there is a limit at , then $y'(\eta)$ has a limit and can be considered zero.

So, $\eta\to+\infty$

$$v_i(\eta) = y_{3-i}^{m_1-1}\left|\frac{dy_i^k}{d\eta}-k\gamma_i y_i^k\right|^{p-2}\left(\frac{dy_i}{d\eta}-\gamma_i y_i\right) = -y_{3-i}^{m_1-1}k^{p-2}\gamma_i^{p-1}y_i^{k(p-2)+1}+o(1), (i=1,2)$$

there is a limit. According to equation (22). $v_i(\eta)\ (i=1,2)$ of the derivative of functions $\eta\to+\infty$ has a limit in and is zero. As a result

$$\lim_{\eta\to\infty}\left(a_{i1}(\eta)\cdot v_i - y_i^{-\alpha_i}(\eta)\cdot\gamma_i\cdot a_{i2}(\eta) + y_i^{\beta_i-\alpha_i}(\eta)\cdot a_{i3}(\eta) - y_i^{1-\alpha_i}(\eta)\cdot a_{i4}(\eta)\right) = 0\ (i=1,2)$$

System at $\eta\to+\infty$ has nonzero limit at and these limits are conditional on the theorem determined by the indicated systems. The theorem is proved.

Result 1. If $\frac{(p-1)(k(p-2)+\alpha_{3-i}-m_i+1)}{k(p-2)(k(p-2)+\alpha_i+\alpha_{3-i})+\alpha_i\alpha_{3-i}-(m_i-1)(m_{3-i}-1)}>0$ if the inequality holds, then (1)-(2) is a generalized solution $|x|\to a^{\frac{p-1}{p}}\tau^{\frac{1}{p}}$ has asymptotics

$$u_{iA}(x,t)\approx c_i(T+t)^{n_i}\left(a-\left(|x|\tau^{-\frac{1}{p}}\right)^{\frac{p}{p-1}}\right)^{\frac{(p-1)(k(p-2)+\alpha_{3-i}-m_i+1)}{k(p-2)(k(p-2)+\alpha_i+\alpha_{3-i})+\alpha_i\alpha_{3-i}-(m_i-1)(m_{3-i}-1)}}(1+o(1))$$

Here $c_i\,(i=1,2)$ defined constants.

Fast diffusion($\gamma_i<0$).Globality conditions of the solution.

Suppose that the system (11) satisfies the following conditions.

$$f_i'(0)=0, f_i(\infty)=0 \tag{25}$$

Here is equal to $f_i(\xi)=A_i\left(a+\xi^{\gamma}\right)_+^{\gamma_i}$.

Theorem 3. (Globality conditions). Let's assume $\gamma_i<0$,

$$\gamma\gamma_{i+2}A_i^{\alpha_i+k(p-2)}\cdot A_{3-i}^{m_i-1}k^{p-2}|\gamma\gamma_i|^{p-2}=-\frac{1}{p},$$

$$\psi_i\left(A_i^{\beta_i-1}a^{\gamma_i\beta_i-\gamma_i}-\frac{1}{1-\beta_i}\right)-\frac{\gamma_i}{p\gamma_{i+2}}\ge 0,$$

$$u_i(t,0)\ge u_{i-}(t,0)\ ,x\in R\ \ (i=1,2),$$

let the relationship be appropriate. Then problem (1)-(2) has a global solution in the field and the following estimates hold for it.

$$u_i(t,x)\ge u_{i-}(t,x)=(T+t)^{n_i}\,\bar{f}_i(\xi),(i=1,2)$$

The proof of Theorem 3 is proved in the same way as Theorem 1.

Theorem 4. Assume that $\gamma_i<0$. Then for solutions (11), (20) of the problem vanishing at infinity $\xi\to\infty$ the following asymptotics are relevant

$$f_i(\xi)=c_i\left(a+\xi^{\gamma}\right)^{\gamma_i}(1+o(1)),\ i=1,2 \tag{26}$$

here $c_i\,(i=1,2)$ solution of the system of nonlinear algebraic equations shown below

$$a_{i1}c_i^{k(p-2)+1}c_{3-i}^{m_i-1}k^{p-2}\gamma_i^{p-1}+a_{i2}\gamma_i c_i^{1-\alpha_i}+a_{i4}c_i^{1-\alpha_i}=0$$

Proof: **I**n order to study the asymptotics of the solutions of problem (11), (20), we look for solutions of the system of self-similar equations (11) vanishing at infinity in the following form.

$$f_i(\xi) = \overline{f}_i(\xi)\cdot y_i(\eta),\ \ \eta = \ln\left(a+\xi^{\gamma}\right), (i=1,2) \tag{27}$$

Here $\overline{f}_i(\xi) = \left(a+\xi^{\gamma}\right)^{\gamma_i}, a>0$, $y_i(\eta)(i=1,2)$ sought functions.

We transfer (18) to system (11), as a result we have the following system with respect to $y_i(\eta)$.

$$\frac{dL_i y}{d\eta} + a_{i1}(\eta)\cdot L_i y + y^{-\alpha_i}(\eta)\cdot\left(\frac{dy(\eta)}{d\eta} + \gamma_i\cdot y(\eta)\right)\cdot a_{i2}(\eta) + y^{\beta_i-\alpha_i}(\eta)\cdot a_{i3}(\eta) -$$
$$-y^{1-\alpha_i}(\eta)\cdot a_{i4}(\eta) = 0$$

Here, $a_{i1}(\eta) = \gamma_{2+i} - \dfrac{e^{\eta}}{\gamma\left(a-e^{\eta}\right)}$, $a_{i2}(\eta) = \dfrac{1}{p\cdot\gamma^{p-1}}$, $a_{i3}(\eta) = \dfrac{\psi_i e^{(1+\gamma_i\beta_i-\gamma_i)\eta}}{\gamma^p\left(a-e^{\eta}\right)}$,

$a_{i4}(\eta) = \dfrac{\psi_i e^{\eta}}{\gamma^p(1-\beta_i)\left(a-e^{\eta}\right)}$, $L_i(y) = y_{3-i}^{m_i-1}\left|\dfrac{dy_i^k}{d\eta} + k\gamma_i y_i^k\right|^{p-2}\left(\dfrac{dy_i}{d\eta} + \gamma_i y_i\right), (i=1,2)$ Based on these, the proof of this theorem is proved in the same way as Theorem 2.

Result 2. If $\dfrac{(p-1)\left(k(p-2)+\alpha_{3-i}-m_i+1\right)}{k(p-2)\left(k(p-2)+\alpha_i+\alpha_{3-i}\right)+\alpha_i\alpha_{3-i}+(m_i-1)(m_{3-i}-1)} < 0$ the inequality is valid, then generalized solution of the problem (1)-(2) at $\xi\to\infty$ has an asymptotic

$$u_{iA}(x,t) \approx c_{i+2}(T+t)^{n_i}\left(a+\xi^{\frac{p}{p-1}}\right)^{\frac{(p-1)\left(k(p-2)+\alpha_{3-i}-m_i+1\right)}{k(p-2)\left(k(p-2)+\alpha_i+\alpha_{3-i}\right)+\alpha_i\alpha_{3-i}+(m_i-1)(m_{3-i}-1)}}(1+o(1))$$

here $c_{2+i}(i=1,2)$ defined constants.

Conclusion.

In this study, a detailed mathematical analysis was conducted to investigate the convergence and existence of solutions for a system of non-divergent parabolic equations. The research employed several advanced methodologies to address the complexity of these systems.

One key approach was the utilization of weak solutions, which are particularly valuable in cases where classical solutions are not available. This method ensures that the system of equations is satisfied in the sense of an integral identity, providing a robust framework for proving the existence of solutions.

The existence of solutions was further substantiated using the comparison principle. This principle allowed for precise estimation of solutions, both from above and below, thereby strengthening the theoretical foundation of the study.

To examine convergence, numerical algorithms based on the Peaceman-Rachford scheme were developed. These algorithms exhibited rapid convergence to exact solutions, demonstrating their efficiency and reliability for solving such complex systems.

Additionally, the asymptotic behavior of the solutions was thoroughly analyzed using self-similar solutions. This analysis provided critical insights into the stability of the solutions at infinity and allowed for further refinement of the upper and lower bound estimates.

Overall, the methodologies and results presented in this study contribute significantly to the theoretical understanding of non-divergent parabolic systems. They also offer practical tools for addressing real-world problems in fields such as physics and biology, where such equations are often applied.

Investigating self-similar solutions in a cross-diffusion parabolic system, which does not follow the divergence form and includes a source and convective transfer, contributes to our comprehension of these intricate systems. By establishing the presence, singularity, stability, and long-term behavior, we establish a strong mathematical framework for various scientific and engineering challenges. Subsequent research can expand upon these discoveries to investigate more complex situations, such as interactions between multiple species and source terms that vary over time. This will enhance the usefulness of these mathematical models in a wider range of scenarios. This study highlights the significance of interdisciplinary approaches combining advanced mathematical methods with practical applications to tackle intricate real-world problems effectively.

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Obtaining and electrophysical study of solid solution $(Ge_2)_{1-x}(GaAs)_x$

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Abstract
Solid solutions $(Ge_2)_{1-x}(GaAs)_x$ were obtained from lead and bismuth solution-melt at crystallization onset temperatures in the range, respectively, 860-760 ^{0}C (Pb), 650-600 ^{0}C (Bi). The cooling rate of the melt solution was 1-1.5 K/minute. Single-crystal GaAs substrates with (100) orientation, n-type conductivity, and deviation from the plane not exceeding 15' - 30' were used as substrates. The concentration of charge carriers and the concentration of acceptor centers were determined. The mechanism has been established to influence the relaxation time of charge carriers on impurity and dislocation centers in the film.

Currently, on the basis of elementary, binary semiconductors, obtaining solid solutions with specific photoelectric and electrophysical properties with expanded physical capabilities is an urgent task in the field of crystal growth. By technologically controlling the chemical composition of the solid solution, it is possible to grow crystalline perfect continuous epitaxial layers and obtain heterostructures by smoothing out differences in crystal lattice parameters and the coefficient of thermal expansion between the film and the substrate[1,2]. The resulting graded-gap solid solutions cover a relatively wider range of photosensitivity of incident quanta than homogeneous semiconductors. Therefore, we were tasked with selecting more suitable pairs of types of semiconductors forming a functional material with similar physical parameters. From this point of view, Ge and GaAs are promising materials[3,4]. Using literature data and carrying out experimental measurements on the solubility of Ge, GaAs in Pb, Bi solvents, the growth temperature range of the solid solution and the composition of the solution-melt were determined[5,6]. It is also very important to study the electrical properties of graded-gap solid solutions depending on their chemical composition. Impurity and dislocation centers in the film affect the mobility of charge carriers, including the relaxation time[7,8].

Solid solutions $(Ge_2)_{1-x}(GaAs)_x$ were grown from a lead, bismuth solution-melt at the crystallization onset temperature in the range, respectively, 860-760 ^{0}C (Pb), 650-600 ^{0}C (Bi), and the cooling rate of the melt solution is 1-1.5 K/min. The gap between the substrates was 0.5-1.2 mm. Single-crystal GaAs substrates with (100) orientation, n-type conductivity, and deviation from the plane not exceeding 15' - 30' were used as substrates.

The composition and structure of the resulting solid solutions were studied using XRD and a scanning electron microscope, which showed that the films were single-crystalline and contained Ge, Ga, and As components.

Quantitative analysis of the chemical composition of the film surface obtained using SEM indirectly showed that films $(Ge_2)_{1-x}(GaAs)_x$ with solid solution structure c with different final compositions on the surface. In this case, the final chemical composition of the film surface depended on the growth temperature range.

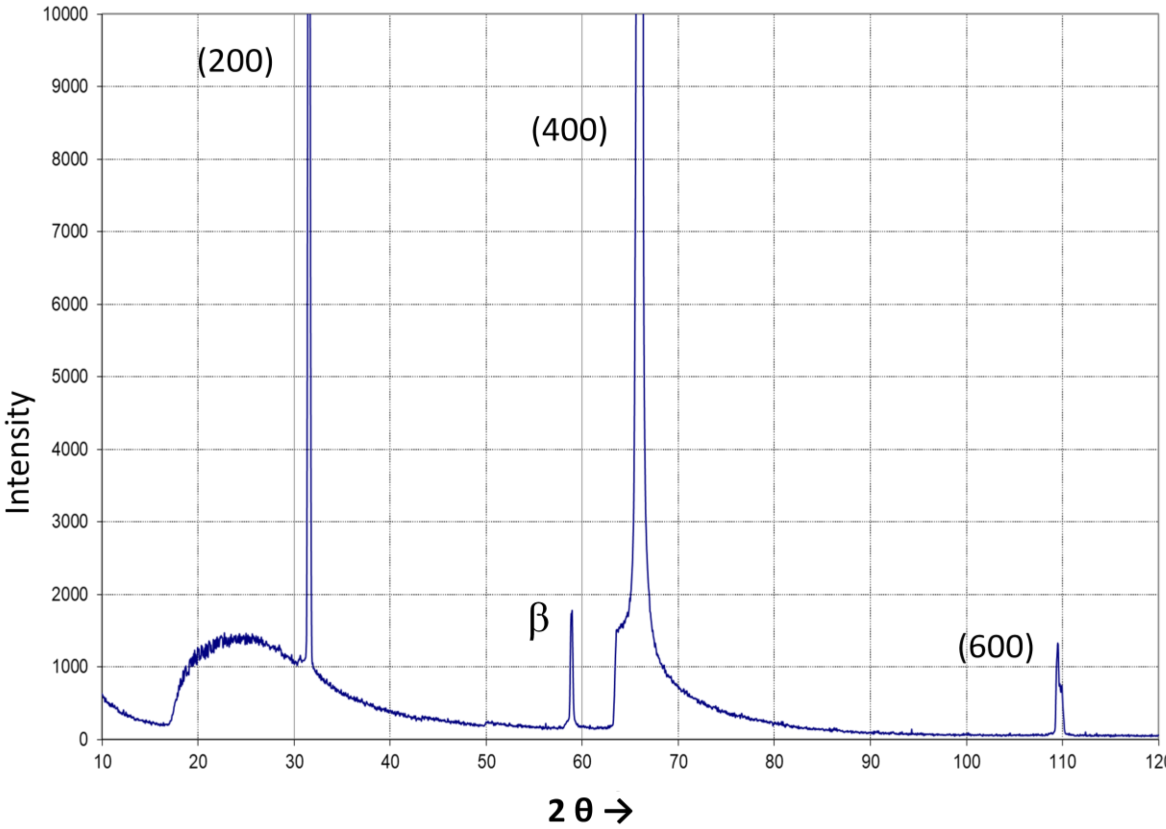

Figure.1. XRD patterns from the surface of the $(Ge_2)_{1-x}(GaAs)_x$epitaxial layer (Cu-K$_\alpha$ radiation)

XRD studies showed that the grown $(Ge_2)_{1-x}(GaAs)_x$ solid solutions are single crystals; therefore, reflections from the (200), (400) and (600) planes were observed in the diffraction patterns of the surface layers of the grown film (Fig. 1). It is necessary to note a slight shift of the diffraction maxima to the region of larger Bragg angles compared to reflections from the GaAs substrate lattice. Precision measurement of the lattice parameter of the resulting solid solution indicated its decrease $\left(a_{(Ge_2)_{1-x}(GaAs)_x} = 0{,}56521 \pm 0{,}00005\,nm\right)$ compared to the lattice parameters of germanium $(a_{Ge} = 0{,}56581 \pm 0{,}0005\,nm)$, and gallium arsenide $(a_{GaAs} = 0{,}56532 \pm 0{,}0005\,nm)$.

And also, based on experimental data, Poisson's ratio $v_{(Ge_2)_{0{,}05}(GaAs)_{0{,}5}} = 0.31$ and generalized lattice constant parameters $(a^{v}_{(Ge_2)_{0{,}5}(GaAs)_{0{,}5}} = 0.56523 \pm 0{,}00005\,nm)$were determined) solid solution $(Ge_2)_{1-x}(GaAs)_x$ using formulas (1) and (2)[9]. These parameters determine the movement of charge carriers in the crystal; accordingly, they characterize the electrophysical properties of the film.

$$a^{v}_{(Ge_2)_{1-x}(GaAs)_x} = a^{\perp}_{(Ge_2)_{1-x}(GaAs)_x} \frac{1-v_{(Ge_2)_{1-x}(GaAs)_x}}{1+v_{(Ge_2)_{1-x}(GaAs)_x}} + a^{v}_{GaAs} \frac{2v_{(Ge_2)_{1-x}(GaAs)_x}}{1+v_{(Ge_2)_{1-x}(GaAs)_x}} \quad (1)$$

$$a^{v}_{(Ge_2)_{1-x}(GaAs)_x} = a^{\perp}_{(Ge_2)_{1-x}(GaAs)_x} \frac{1-(xv_{Ge}-(1-x)v_{GaAs})}{1+(xv_{Ge}+(1-x)v_{GaAs})} + a^{v}_{GaAs} \frac{2(xv_{Ge}+(1-x)v_{GaAs})}{1+(xv_{Ge}+(1-x)v_{GaAs}} \quad (2)$$

The above show that the $(Ge_2)_{1-x}(GaAs)_x$ solid solution with a specific structure has different electrical properties than elementary and binary semiconductors. In a graded-gap $(Ge_2)_{1-x}(GaAs)_x$ film, determining the participating current carriers associated with impurity and other centers in the crystal lattice is quite difficult than in elementary semiconductor materials[10]. Therefore, in this work we proposed a new approach to determining the concentration of the main charge carriers and impurity centers in the solid solution $(Ge_2)_{1-x}(GaAs)_x$ taking into account the variable parameters of the solid solution depending on the composition. In fig.2 the band diagram of $GaAs$ $(Ge_2)_{1-x}(GaAs)_x$ structures is shown.

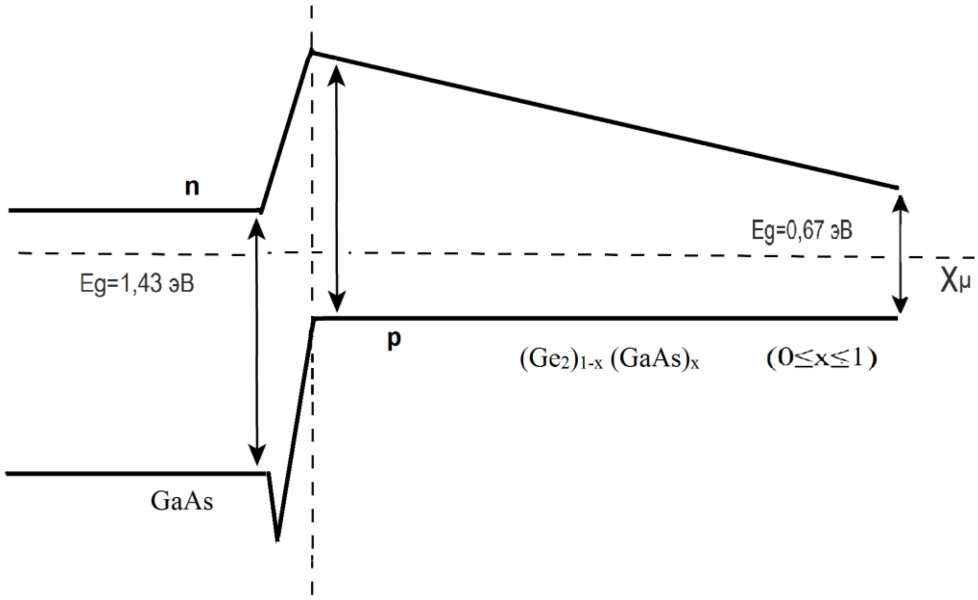

Figure.2. Band diagram of GaAs − $(Ge_2)_{1-x}(GaAs)_x$ structures.

The concentration and mobility of charge carriers in the solid solution $(Ge_2)_{1-x}(GaAs)_x$. $n_p = 1{,}5 \cdot 10^{18} cm^{-3}$ $\mu = \frac{80 cm^2}{B \cdot c}$ $(T = 300\ K)$.

According to the statistical theory of impurity levels in semiconductors, the probability of a given level being occupied by a charge carrier, we can determine the total number of acceptor centers ($n_з$) per unit volume with vacancies for electrons at T>0 K[11]:

$$n_з = \frac{n_p^2}{2 \cdot g_p \left(\frac{2 \cdot \pi \cdot m_p \cdot k \cdot T}{h^2}\right)^{\frac{3}{2}}} \cdot e^{\frac{\Delta E_A}{kT}} = \frac{n_p^2 \cdot e^{\frac{\Delta E_A}{kT}} \cdot h^3}{2 \cdot g_p (2 \cdot \pi \cdot m_p \cdot k \cdot T)^{\frac{3}{2}}} \tag{3}$$

where n_p - the concentration of holes, $g_p -$ the statistical weight, $m_p -$ the effective mass of holes, k - Boltzmann's constant, T - the absolute temperature, ΔE_A -the activation energy of acceptor impurities, h - Planck's constant.

The total number of acceptor centers per unit volume is described by the expression:

$$n_{0p} = n_p + n_з \tag{4}$$

where $n_p, n_{0p} -$ the concentration of holes and impurity acceptor centers in the volume, $n_з -$ the total number of vacant acceptor centers per unit volume at T>0K.

The activation energy of the acceptor level of the solid solution $(Ge_2)_{1-x}(GaAs)_x$ is determined using the expression:

$$\Delta E_A = \frac{m_p \cdot e^4}{2\hbar^2 \cdot \varepsilon_r{}^2} \tag{5}$$

where $\hbar = \frac{h}{2\pi} - Planck's\ constant,\ \varepsilon_r -$the dielectric constant of the solid solution.

In this way, the concentration of charge carriers, the total concentration of acceptor centers, the concentration of acceptor centers participating in current passage at T=300K, the possible participating holes of the impurity level at T>300K of the graded-gap solid solution are determined: $(Ge_2)_{1-x}(GaAs)_x$ $n_{0p} = 2{,}187 \cdot 10^{18} cm^{-3}$, $n_p = 1{,}5 \cdot 10^{18} cm^{-3}$, $n_з = 6{,}87 \cdot 10^{17} cm^{-3}$. Then the values of the coefficient of holes participating in the current passage from the entire acceptor center and the coefficient of acceptor centers ($n_з$) of vacancies for electrons, respectively, will be:

$$\mathrm{K_p} = \frac{\mathrm{n_p}}{n_{0p}} = \frac{1{,}5 \cdot 10^{18}}{2{,}187 \cdot 10^{18}} = 0{,}686; \qquad \mathrm{K_з} = \frac{\mathrm{n_з}}{n_{0p}} = \frac{0{,}685 \cdot 10^{18}}{2{,}187 \cdot 10^{18}} = 0{,}314 \tag{6}$$

Using the value of the total concentration of acceptor centers (including the concentration of ionized atoms from them), one can determine the average relaxation time of charge carriers that are associated with ionized, neutral impurity atoms and dislocation centers in the crystal lattice[12,13]. Depending on the proportion of their influence on the "survival" time of charge carriers, it is possible to determine the dominant factor in the mechanism involved in the current passage of the solid solution. Therefore, to explain the electrical properties of the $(\mathrm{Ge_2})_{1-\mathrm{x}}(\mathrm{GaAs})_\mathrm{x}$ solid solution, the relaxation time of charge carriers was studied.

The values of the relaxation time of charge carriers associated with ionized (τ_I), neutral impurity atoms(τ_N), dislocation centers(τ_D), with thermal vibrations of the lattice(τ_l), respectively, are determined using the following formulas (7)-(10):

$$\tau_I = \tau_o \cdot E^{\frac{3}{2}} = \frac{3}{4} \cdot \frac{\sigma m_p^* (\pi k T)^{\frac{1}{2}}}{e^2 \cdot n_p} \cdot \left(\frac{Z \cdot e^2 \cdot n_p^{\frac{1}{3}}}{\varepsilon_r} \sqrt{e^{\frac{(2m_p^*)^{\frac{1}{2}} \cdot \varepsilon_r^2}{\pi \cdot Z^2 \tau_{0I} \cdot e^4 \cdot n_p}} - 1} \right)^{\frac{3}{2}} \tag{7}$$

where, E - the energy of charge carriers.

$$\tau_N = \frac{e^2 m_p{}^{*2}}{20\varepsilon_r \hbar^3} \cdot \frac{1}{n_з} \quad (8)$$

$$\tau_D = \frac{3}{8R\vartheta} \cdot \frac{1}{N_D} \quad (9)$$

where, N_D - the dislocation density, R - the dislocation radius.

$$\tau_l = \frac{\tau_0}{m_p{}^{*\frac{3}{2}}} \cdot T^{-1} \cdot E^{-\frac{1}{2}} \quad (10)$$

The following results were obtained for the relaxation time of charge carriers:

$\tau_l = 1{,}98 \cdot 10^{-14} s$
$\tau_N = 1{,}795 \cdot 10^{-23} s$
$\tau_D = 4{,}29 \cdot 10^{-8} s$
$\tau_I = 8{,}3 \cdot 10^{-22} s$

Comparing the values of the results obtained, the following conclusions were drawn: a relatively long relaxation time of charge carriers is observed in the dislocation center, which is related to the dislocation density and does not strongly depend on the temperature of the crystal; at room temperature, vibrational scattering centers of atoms manifest themselves; neutral impurity atoms depend on the concentration of impurities; therefore, this scattering mechanism manifests itself at relatively low temperatures; The scattering of charge carriers in ionized impurity centers depends on the speed of charge carriers and its effective mass.

This work demonstrates the possibility of obtaining crystalline-perfect graded-gap epitaxial layers $(\mathrm{Ge_2})_{1-x}(\mathrm{GaAs})_x$ from a limited solution-melt by varying the chemical composition of the film. The electrical properties of the film and the band diagram of the GaAs − Ge21−x(GaAs)x structures are presented. A mechanism has been established for the relaxation time of charge carriers in the film, which are associated with ionized, neutral impurity atoms, dislocation centers and thermal vibrations in the crystal lattice. The $(\mathrm{Ge_2})_{1-x}(\mathrm{GaAs})_x$ solid solution is a promising semiconductor material in semiconductor device engineering.

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Molecular dynamic Simulation of Emission of Cationized Water Clusters from a Water Film under Fast Ion Bombardment

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Abstract
This paper presents the results of a molecular dynamics computer simulation of the process of sputtering an ice film containing a positive ion on the Au (111) surface. It is shown that the energy of the bombarding particle is transferred to the target and forms a large ion cluster consisting of water molecules and an embedded ion, and also creates deformation in the region of incidence. It is shown that the introduced ion is located in the center of the ion cluster and this cluster is more stable. In the mass spectrum, a peak is formed that relates to the ion cluster. The results obtained are of interest in understanding the mechanisms of ion cluster formation and purification of metal surfaces from water molecules.

Introductions

The interaction of fast ions with a solid lead to the knocking out of atoms and molecules of the material, both in a neutral and in a charged state. The SIMS method is based on this phenomenon of comparative effective formation of charged particles (secondary ions) and on the principle of highly sensitive mass spectrometric measurements [1-3].

Ion beam sputtering is a destructive process. But if it is required that the surface remain virtually unchanged, then SIMS analysis can be carried out at very low sample sputtering rates (less than 10^{-4} monolayers per second). In order to ensure sufficient sensitivity of the method ($\approx 10^{-4}$ monolayers), as can be seen from p and c .1.14, a primary ion beam with a current of 10^{-10} Å with a diameter of 1 mm is required. At such a low current density of primary ions (10^{-5} mA/cm ^{2}), the rate of arrival of atoms or molecules of residual gases onto the sample surface can exceed the rate of their sputtering by the primary beam[4-7]. Therefore, SIMS measurements under such conditions should be carried out in ultra-high or pure (cryogenic) vacuum. Also interesting is the spraying of a film of ice covering metal surfaces out of technological interest, since water molecules turn a clean surface into a dirty one. Therefore, removing ice films is more relevant. Sputtering of ionized molecular clusters of water was studied experimentally and theoretically in works [8-11]. In this chapter we will discuss the results of sputtering ionized water clusters using the molecular dynamics method.

Research method and results

We investigated the process of sputtering a film of water, which contains an adsorbed Na^+ ion. This process is a very interesting mechanism of sputtering molecules and atoms in the form of large clusters. The sodium ion located inside the film forms ionic bonds with its molecules. These bonds primarily depend on the charge of the ion. i.e., if an ion is positive, then negative components of molecules accumulate around it, or vice versa.

In our calculations, we considered the case when the Na^+ ion is located at the top of a film consisting of four layers of ice on the Au (111) surface, which consists of 1980 atoms located on eight layers (Fig. 1). The system was under observation for 25 picoseconds, and external influences were not taken into account in this case. This time is sufficient for the ion to approach the film surface and form ionic bonds with oxygen atoms. During the above time, the sodium ion is completely located on the first top layer of the film.

Structural changes in the film occur mainly only around the location of the ion. The rest of the film completely retains its original structure.

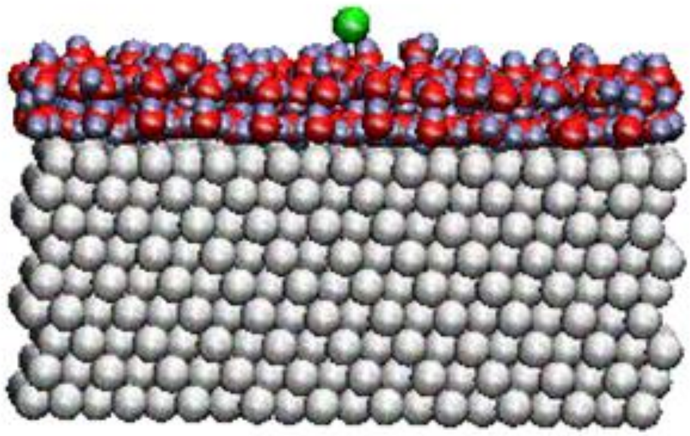

Figure 1. Target consisting of four layers of ice on the Au (111) surface

To describe the interaction of water-water, water- Au , Au - A and Na^+ - water, the interaction potentials Dipole - dipole were used potential , Modified Spohr potential MD / MC - CEM potential and $U = U_q + U_{LJ}$. Here Uq is the Coulomb potential, U_{LJ} is the Lenhard-Jones potential[12].

Figure 2 shows sputtering of an Au (111) surface covered with a water film when bombarded with Ar^{+} ions with an initial energy E_o =400 eV. The process of interaction of the target with the falling particle continued for a time t = 4 picoseconds. The case under consideration differs from the previous one in that we use boundary conditions and the sodium ion is located on the upper layers of the film.

Figure 2a shows the state of the target before bombing. Here it is clearly visible that the sodium ion, located on the upper layer, already has ionic bonds with oxygen atoms. This explains the changes in the film structure associated with the sodium ion.

Figure 2b shows the state of the target at a bombardment time equal to t = 0.4 picoseconds. The figure clearly shows that the falling particle, after interacting with the film, is repelled from its surface. This is due to the fact that the initial energy of the incident particle is not sufficient to penetrate deeper into the target. During the specified time of interaction of the ion with the target, significant changes mainly occur in the upper two layers of the film. It should be noted that at this time, due to the energy of the falling particle, sputtering of film molecules is observed on the surface.

When the interaction time reaches t = 1.6 picoseconds (Fig. 2c), the sodium ion, together with water molecules, begins to move away from the surface of the film, while the molecules around it form a large spherical cluster due to attractive forces. At the same time, around this cluster there is a sputtering of small molecular clusters consisting of 2-7 molecules. There is also an accumulation of water molecules not involved in sputtering on the surface of the film itself, which is explained by the superiority of the binding energy between surface molecules compared to the kinetic energy of these molecules.

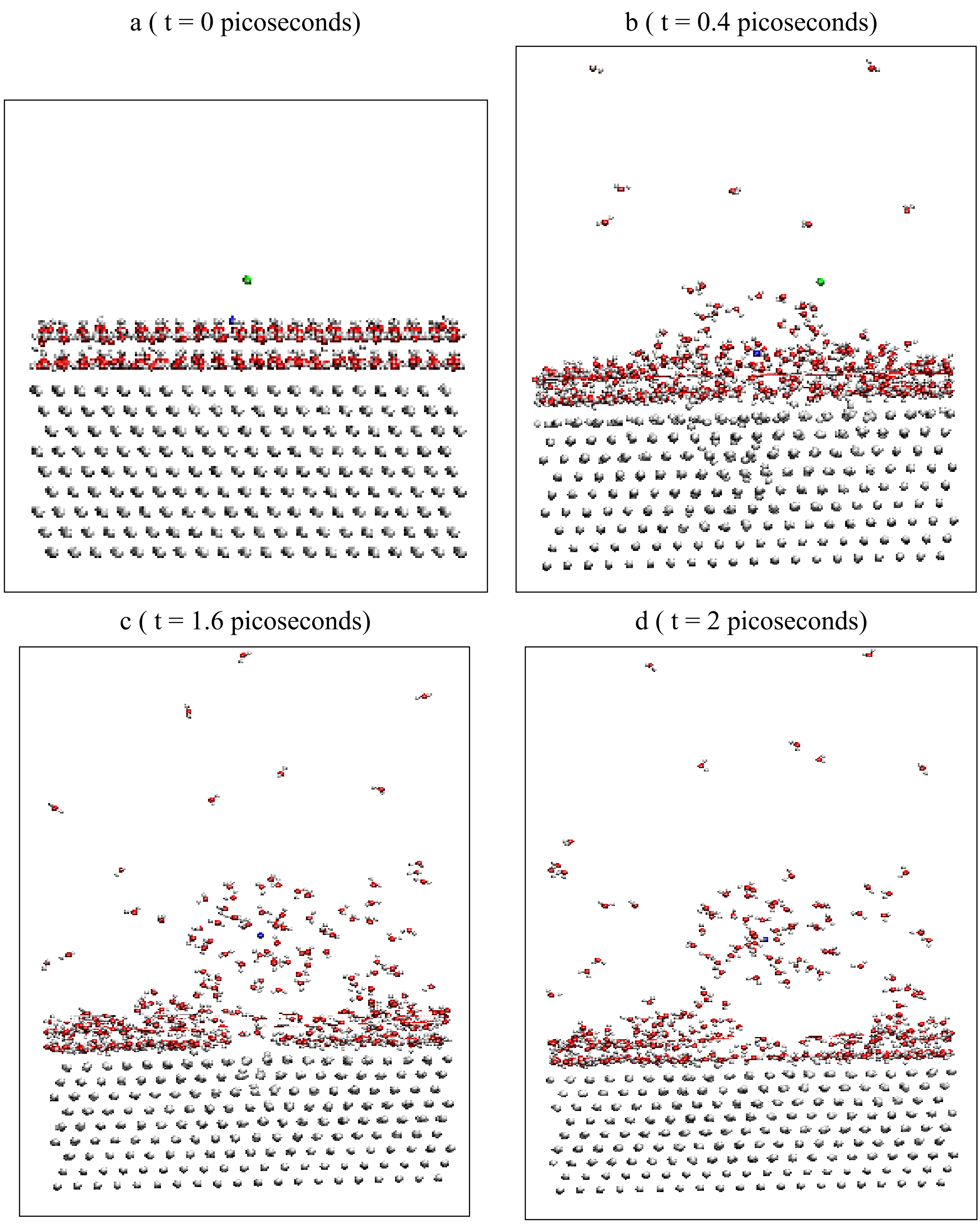

Figure 2. Snapshots of the sputtering process from 0 ps to 2.0 ps during bombardment with Ar^{+} ions with $E_0 = 400$ eV of an ice film with an implanted Na^{+} ion (ion at the top of the film), coated on the gold surface

By this time of bombardment, vibrations of the substrate atoms continue, but with the lowest amplitude values. Our calculations showed that these oscillations lasted for 4 picoseconds. Consequently, when describing the following states of the target, we will not dwell on these minor vibrations of gold atoms.

Figure 2e shows the state of the target at an interaction time equal to t = 2 picoseconds. It can be seen that by this time the formation of a molecular ion cluster is observed, which has managed to separate from the surface of the film. It should be noted that this cluster appears to be very unstable, since some molecules can leave it.

It should be noted that this large spherical cluster is not stable. Our further calculations showed that water molecules leave the ion cluster when the collision time reaches 3 picos seconds. During this period of time the cluster becomes more stable.

In the following calculations, we observed the ion cluster itself for 30 picoseconds without taking into account the influence of the target. Our calculations showed that no decay of the ion cluster was observed during this time, indicating its stability.

We have studied the process of sputtering of an Au (111) surface covered with four layers of ice during bombardment with Ar^+ ions with an initial energy equal to E_0= 700 eV at normal incidence.

First, let's consider the case (Fig. 2a) when the Na^+ ion is on the top layer of the film. The following figure (Fig. 2b) shows the state of the target at t = 0.2 picoseconds after the collision of the falling particle with the surface. By this time, the incident ion destroys the structure of the film and the upper surface layers of the substrate. It should be noted that the incident ion in this case penetrates through three atomic layers of the substrate and stops at the fourth layer, forming cascades of collisions, which affects the periodicity of the central part of the substrate.

From Figure 3c (t=1.4 picoseconds) it is clear that after the ion cluster is completely removed from the surface, surface sputtering of the remaining film molecules. It also shows small open areas free of water molecules. This is due to the transfer of energy from the substrate atoms to the film molecules. By this time, both the vibration region and the amplitude of the vibrational motions of the substrate atoms in this region noticeably decrease.

Further development of the process showed that the vibrations of the substrate atoms gradually decrease due to the transfer of energy to the film.
From Fig. 3d (t=3 picoseconds) it is clear that the ion cluster has completely moved away from the surface and is absolutely not associated with it. At this point in time, molecules that have received energy from the substrate atoms are sprayed over the entire surface of the film in the form of molecular clusters consisting of 2-6 molecules, as well as in the form of individual molecules [253].

We examined the mass spectra and kinetic energy distributions of sputtered particles from the Au (111) surface covered with four layers of ice, which contains the Na^+ ion when bombarded with Ar^+ ions at normal incidence. The initial energy values were chosen to be E_0=400 eV and E_0 = 700 eV.

Our calculations showed that at an initial energy E_0 = 400 eV, destruction of ice layers is observed, and the substrate in this case is not subject to severe destruction.

It should be noted that in this case the falling particles do not penetrate into the substrate. In addition, as was described in previous chapters, it is in this case that a large ion cluster is formed.

In Fig. 4 shows the mass spectrum of sputtered particles at the initial energy E_0 = 400 eV, when Na^+ ion located on the surface of ice film.

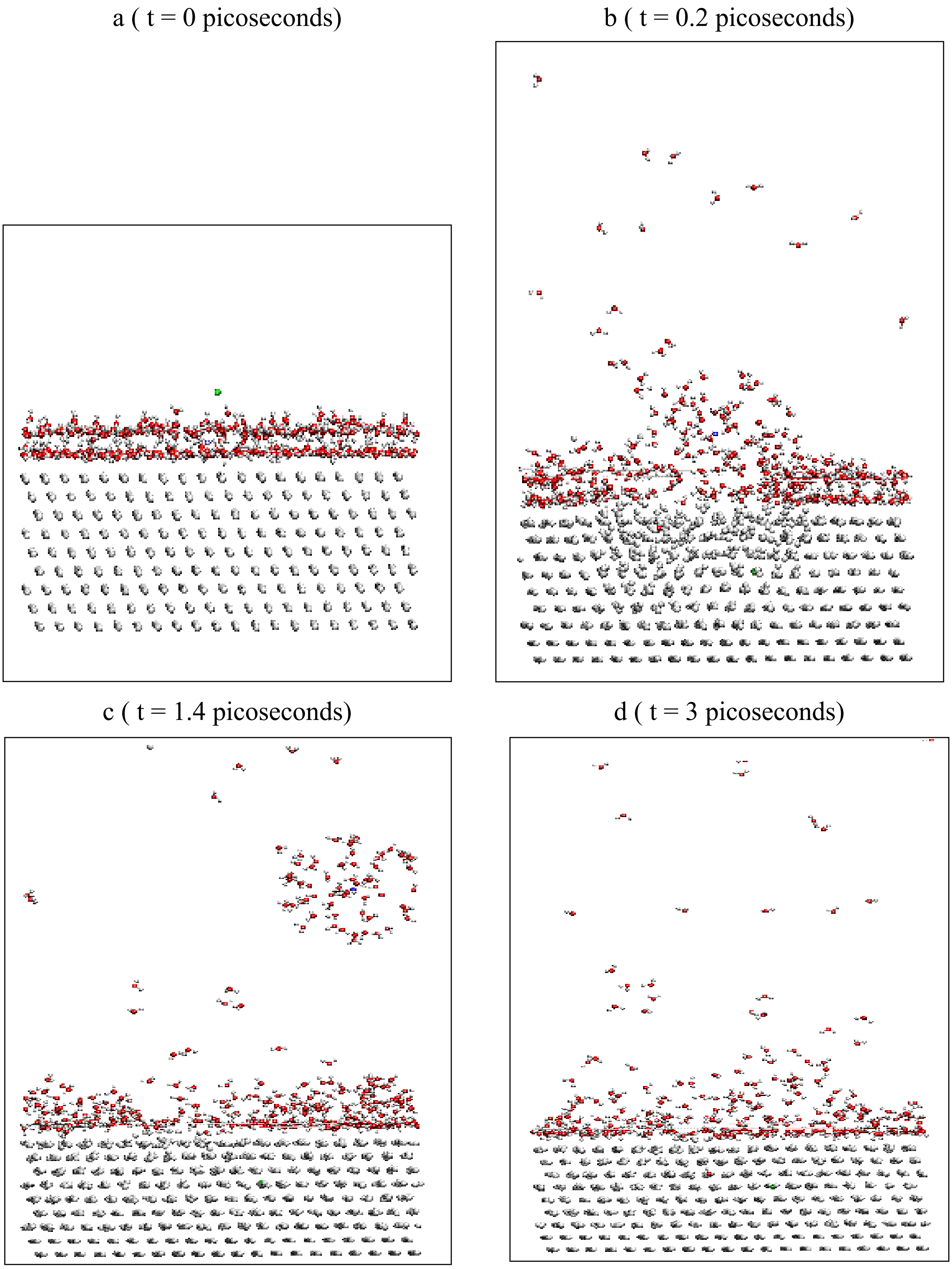

Figure 3. Snapshots of the sputtering process from 0 ps to 3.0 ps during bombardment with Ar^+ ions with E_0 = 700 eV of an ice film with an implanted Na^+ ion (ion at the top of the film), coated on the gold surface.

This mass spectrum can be divided into 3 groups. The first group is water molecules and molecular clusters containing from two to five water molecules. It should be noted that the highest peak belongs to water molecules, and the remaining peaks belong to molecular clusters located up to 100 a.m. m.e.

The second group of peaks located in the range of 200-240 amu belong to sputtered clusters that contain one gold atom and 1-2 water molecules.

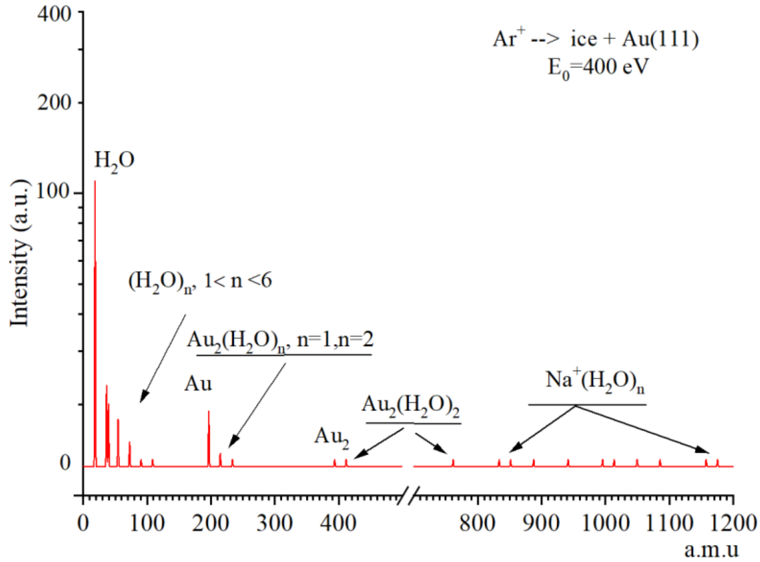

Figure 4. Mass spectrum of sputtered particles at normal incidence of Ar^+ ions on the Au (111) surface covered with four layers of ice film at $E_0 = 400$ eV

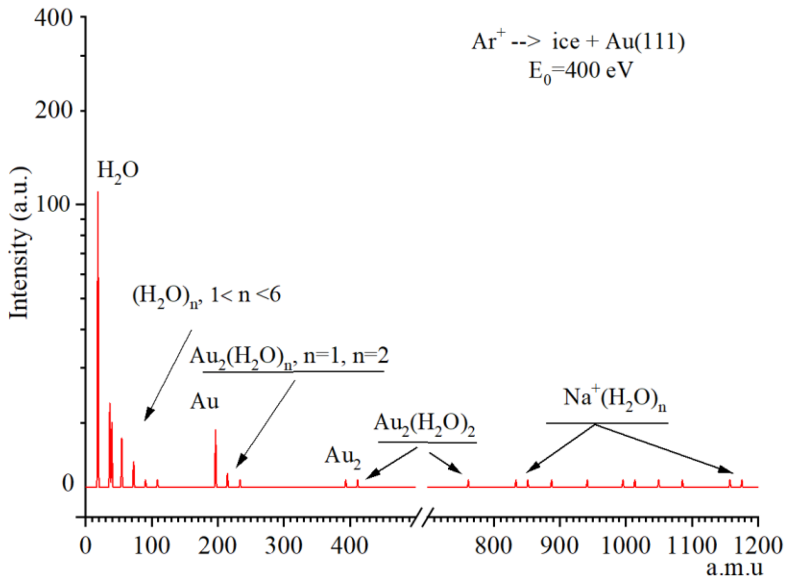

Figure 5. Mass spectrum of sputtered particles at normal incidence of Ar^+ ions on the Au (111) surface covered with four layers of ice ionic film at $E_0 = 700$ eV

The following peaks, which formed near 400 a.m.u. refer to sputtered clusters that contain one or two water molecules with two gold atoms.

Our calculations showed that gold atoms are sputtered when falling particles collide with an open part of the substrate. When a film is sputtered, these atoms capture one or two water molecules, since there is a certain interaction potential between them.

The third group of peaks belongs to the ion cluster, which has a range in the mass spectrum from 820 to 1176 amu. From the analysis of the trajectories of incidence particles it is clear that not all trajectories form a large ion cluster.

In Fig.5 presents the mass spectrum of sputtered particles is presented under the same conditions, but with an initial energy E_0 = 700 eV. In this case, 15 trajectories of incidence particles were considered. From this mass spectrum it is clear that there is a separation in the range of 408-920 amu. between the united first two groups and the third. The intensity of clusters consisting of 2-3 water molecules is greater compared to the previous case.
At 392 a.m.u. a peak was formed due to sputtering of clusters containing two gold atoms.

The third group, which was mentioned above, is formed due to large atomic masses, which indicates the formation of a large ionic cluster.

Conclusions

We have studied sputtering of an Au (111) surface covered with four layers of ice film at E_0=400 and 700 eV. It has been shown that the presence of an ion in the film leads to sputtering of an ion cluster consisting of the ion and large quantities of water molecules.

It is shown that due to the transfer of energy to the bombarding part, deformation is observed on the substrate. The resulting mass spectrum formed a peak of a large ion cluster. The results obtained are of great interest in understanding the mechanisms of sputtering films containing foreign ions and in the purification of water surfaces in technological processes.

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Galilean geometry with compressed coordinates.

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Abstract: The concept of distance is introduced on the plane, which is divided into two parts. Using the entered distance between points, the geometry of the plane is studied, called the geometry of the plane with compressed coordinates. The introduced distance has the properties of the Galilean distance and the geometry on this plane is the generalized geometry of the Galilean plane. By analogy, the curvature of a curve was calculated and the class of curves with constant curvature was determined.

Keywords: Galilean geometry; distance; parabolic angle; curvature; curve of constant curve.

Introduction.

The geometry of Galilean space belongs to the geometry of spaces with degenerate metrics. A. Artikbaev [1] and N.E. Pankina [2] were first engaged in solving specific problems of geometry “in general” in Galilean space.

After 2000, the geometry of Galilean space began to be widely studied. In this regard, it is worth noting the work of Professor M.E. Aydin of Firat University and his students [3], as well as Professor M. Dede of Arlin University (Turkey) and his students [4].

In the study of the geometry of non-Euclidean spaces we sometimes use the method of superimposed space, that is, the coordinate system of non-Euclidean space is considered as the coordinate system of Euclidean space. If the coordinate system of Galilean space is considered as a Euclidean system, then some of our results about isometry are a generalization of the notion of “isometry of surfaces by section” studied in the works of A. Sharipov [5,6].

Most recently there has been an ascending attention to bring in a geometric approach to a fractional derivative, mainly focusing on Differential Geometry, Fractal Geometry and Vector Calculus of which classic geometry are subclasses. See [7,8,9]. Despite diversity of fractional derivative operators used in these approaches, they mostly fail to perform Leibniz rule and various composition rules [10,11,12]. Nevertheless, these are indispensable tools to establish a theory in differential geometry of curves, which is our main interest in the present study.

The fractional order derivative has always been an interesting research topic in the theory of functional space for many years [13,14,15]. Various types of fractional derivatives were introduced, among which the following Riemann-Liouville and Caputo are the most widely used ones. We will study fractional derivatives using Galilean geometry with compression coordinates.

Definition of the Galilean plane.

Consider a two-dimensional affine plane with an affine coordinate system Oxy.

Let $A(x_1, y_1), B(x_2, y_2)$ points of this plane.

The distance between A and B is determined by the following formula:

$$AB = \begin{cases} |x_2 - x_1| & if \quad x_1 \neq x_2, \\ |y_2 - y_1| & if \quad x_1 = x_2. \end{cases} \tag{1}$$

This formula can define the distance between points because it satisfies the condition of symmetry, $AB = BA$.

If it can be proven that there exists a one-to-one mapping of the plane that preserves the conditions of isometry, then it can be asserted that there exists a geometry on the affine plane with the given distance [16,17].

Theorem 1 [16]. The affine transformation

$$\begin{cases} x' = x + a \\ y' = hx + y + b \end{cases} \tag{2}$$

is a motion of the affine plane that preserves the distance between points defined by formula (1).

According to Thurston's definition, there exists a geometry with the distance between points defined by formula (1) [18,19]. This geometry is called the Galilean geometry of the plane.

The motion of the Galilean plane defined by formula (2) is a special case of an affine transformation of the plane. The matrix of this transformation from the Heisenberg group $\begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}$- has determinants equal to one [20,21].

The basics of the geometry of the Galilean plane are presented in the book by Yaglom [22], where the planimetry of the Galilean plane is first systematically presented. The study of the geometry of the Galilean plane begins with the definition of the properties of the movements of this plane.

Transformation (2), which is the motion of the Galilean plane, consists of two parts

$$\begin{cases} x' = x + a, \\ y' = y + b, \end{cases} \quad \text{and} \quad \begin{cases} x' = x, \\ y' = hx + y. \end{cases}$$

The first part is a parallel translation, and the second is called the rotation of the Galilean plane.

To clarify the geometric meaning of the parameter h, the concept of an angle between lines is introduced.

By analogy with the Euclidean plane and the Minkowski plane, the Galilean plane can be defined using the scalar product.

Definition 1. An affine plane in which the scalar product of the vectors $A(x_1, y_1)$ and $B(x_2, y_2)$ is defined by the formula

$$\left(\vec{X}\cdot\vec{Y}\right)=\begin{cases} x_1x_2 & \\ y_1y_2 & if \quad x_1x_2=0\end{cases}$$

is called the Galilean plane.

We will prove that this definition is equivalent to Thurston's definition of the plane [23].

Let us calculate the distance between the points A and B, which are the ends of the vectors $\vec{X}$ and $\vec{Y}$, as the norm of the vector $\left(\vec{Y}-\vec{X}\right)$ connecting these points.

The vector $\left(\vec{Y}-\vec{X}\right)$- has coordinates (x_2-x_1, y_2-y_1),

$$AB=|Y-X|=|x_2-x_1|.$$

When $x_1=x_2$,

$$AB=|Y-X|=|y_2-y_1|.$$

The norm of the vector is obvious $|Y-X|$ - it coincides with the distance between points defined by formula (1).

The equivalence of the two definitions of the Galilean plane makes it possible to define a space with a degenerate distance between points as an affine space with a degenerate scalar product [24,25,26].

We use this possibility when defining the concept of semi-Euclidean space.

With naturals α, Newton's binomial formula takes the form

$$(a+b)^{\alpha}=\binom{0}{\alpha}a^{\alpha}+\binom{1}{\alpha}a^{\alpha-1}b+\binom{2}{\alpha}a^{\alpha-2}b^2+\ldots+\binom{n}{\alpha}a^{\alpha-n}b^n+\ldots \quad (3)$$

where do we have that

$$\binom{n}{\alpha}=\frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-(n-1))}{n!},\ \alpha\neq 0$$

the numbers are binomial coefficients [27].

In analytical geometry, a curve is usually understood as the geometric locus of the coordinate points of which satisfies an equation $F(x,y)=0$ in some coordinate system. The coordinate system can be affine. On the Galilean plane, it is better to consider the curves given by the equation $y=f(x)\quad x\in[a,b]$. In the theory of curves, second-order curves occupy a special place. In general, the study of second-order curves is based on an affine coordinate system. Moreover, the “curves” are determined by a bottom-type equation. When a second order equation with two variables can be an ellipse, hyperbola, parabola or special cases of these curves.

The study of the geometry of second-order curves was first encountered in the work of N. Makarova [2], where the definition of an ellipse and a hyperbola on the Galilean plane is given.

These definitions are based on the geometric characteristics of these curves and are completely different from the Euclidean definition.

We derived the equation of one representative of the second-order curve, the circle, which has a second-order equation and decomposes into two special lines. At the same time, we used the definition given on the Euclidean plane. Thus, another geometric locus of points was obtained. Therefore, the study of curves on the Galilean plane is of great interest.

Results and Discussion

Galilean geometry with compression coordinates.

Let $Oxy-$ the Cartesian coordinate system be on a plane.

The distance between points $A(x_1,y_1)$ and $B(x_2,y_2)$ is determined by the formula.

$$d^* = |AB|^* = \begin{cases} |x_2{}^\alpha - x_1{}^\alpha|, & x_1 \neq x_2 \\ |y_2 - y_1|, & x_1 = x_2 \end{cases} \quad 0<\alpha<1 \tag{4}$$

When $\alpha = 1$ this distance coincides with the distances of the Galilean plane G_2. Therefore, the geometry of the plane studied using distance (4) will be called Galilean geometry with compressed coordinates, and the distance determined by formula (4) will be called compressed distance. Geometric quantities determined using compressed distance are given with an asterisk.

Lemma 1. Compressed distances between points $A(x_1,y_1)$ and $B(x_2,y_2)$ no greater than the distances between these points on the Galilean plane.

According to the above definition, Lemma 1 can be confirmed.

We study the geometry of the plane using the introduced compressed distance (1). We define a circle as the geometric locus of points of the compressed plane equidistant from a given point called the center of the circle.

If the center of the circle is at a point (a,b) and the radius is equal to r, then the equation has the form $|x^\alpha - a^\alpha| = r$.

Obviously, the equation of the circle does not include the coordinate at the center of the circle. This shows the center of the circle is a straight line $x = a$ parallel to the axis Oy. Similar to the circle of the Galilean plane, in the compressed plane the same circle is parallel to the straight axis Oy.

We compare circles in the sense of Galilean distance with circles in the sense of compressed distance.

Lemma 2. A concentric circle with equal radii in terms of compressed distance is contained within the Galilean circle.

The proof follows from Lemma 1.

But when the center of the circle coincides with the axis Oy and the radius is equal to one, these circles coincide.

By analogy with the Galilean plane, we define the angle between straight lines as the length of the arc of a unit circle when the vertex of the angle is at the origin. Since in this case the circles coincide, the magnitude of the angle in the Galilean and compressed cases coincide.

If the sides of the angle lie on straight lines $y=k_1x$ and $y=k_2x$ then the angle h^α between them is calculated using formula (4) equal to

$$h^\alpha = |k_2 - k_1|$$

because the point of intersection of the sides of a triangle with the arc of a circle has coordinates $(1,k_1)$ and $(1,k_2)$

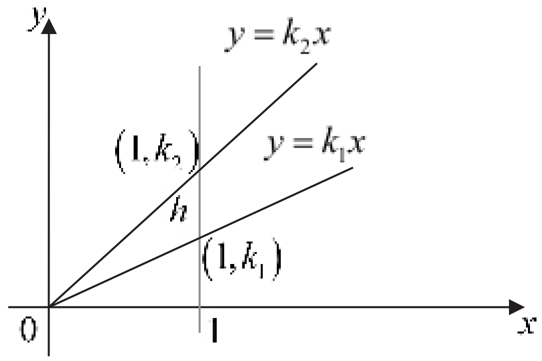

Fig.1. Compressed angle .

It is known [28] that the angle h is called a parabolic angle and takes a value in the interval $[0,\infty)$.

Consider a triangle with vertices $A(x_1,y_1), B(x_2,y_2)$ and $C(x_3,y_3)$ lying on different special planes, then if $x_1 < x_2 < x_3$.

It can be proven that $AB+BC=AC$ и $\angle A+\angle C=\angle B$ (fig.2).

The equality of the connected sides of the triangle is satisfied because these segments coincide and the magnitude of the angle of the triangle does not differ from the angle in the sense of the Galilean plane.

The elements of a triangle, median, bisector and altitude on a compressed plane do not differ in these concepts on the Galilean plane.

Let's determine the area of the triangle ABC:

It is known that the area of a triangle in the Galilean plane is found as follows

$$S=\frac{1}{2}abh,$$

where a and b are the sides of the triangle, and h is the angle between them. Let's create a triangular area in a compressed coordinate system.

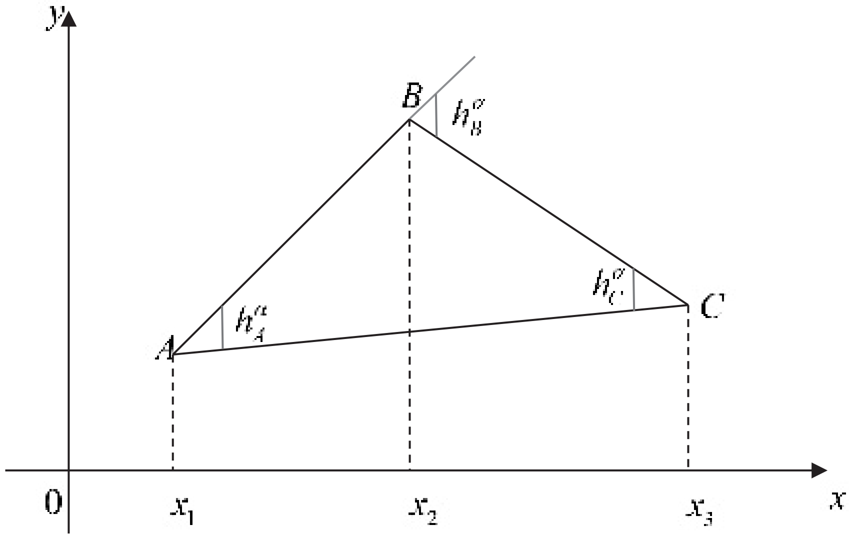

Fig.2. Triangle

Then, sides of the triangle

$$\begin{cases} a = x_2^\alpha - x_1^\alpha, \\ b = x_3^\alpha - x_1^\alpha. \end{cases}$$

We find the angle:

$$x^\alpha - x_1^\alpha = 1,$$

$$x = \sqrt[\alpha]{1 + x_1^\alpha},$$

$$\frac{x^\alpha - x_2^\alpha}{x_2^\alpha - x_1^\alpha} = \frac{y - y_2}{y_2 - y_1},$$

$$y = \frac{x^\alpha - x_2^\alpha}{x_2^\alpha - x_1^\alpha}\left(y_2 - y_1\right) + y_2.$$

From here

$$h = |y - y_1| = \left|\frac{x^\alpha - x_2^\alpha}{x_2^\alpha - x_1^\alpha}\left(y_2 - y_1\right) + y_2 - y_1\right| = |y_2 - y_1|\left|\frac{x^\alpha - x_2^\alpha}{x_2^\alpha - x_1^\alpha} + 1\right|,$$

we get

$$S = \frac{1}{2}\left|x_2^\alpha - x_1^\alpha\right|\left|x_3^\alpha - x_2^\alpha\right|\left|y_2 - y_1\right|\left|\frac{x^\alpha - x_2^\alpha}{x_2^\alpha - x_1^\alpha} + 1\right| =$$

$$= \frac{1}{2}\left|x_3^\alpha - x_2^\alpha\right|\left|y_2 - y_1\right|\left|1 + x_2^\alpha - x_1^\alpha\right|.$$

Theory of curves.

Let a curve γ with equations be given

$$y = f(x), \quad x \in [a,b] \tag{5}$$

Let us find the length of the arc of the curve as the limit of the length of the perimeter inscribed in the estuary curve. It is easy to prove that the length of the curve will always be equal to the length of the segment $[a,b]$ on the axis Ox, which is the projection of the ends of the curve on the axis Ox (fig.3).

$$e = \left| b^{\alpha} - a^{\alpha} \right|$$

Let's find the derivative of function (5) at a given point $x_0 \in [a,b]$.

We understand the derivative of a function in the sense of defining the derivative on the Euclidean plane according to the formula

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0)$$

But the increment of the argument Δx is determined by the formula distance between points

$$\Delta x_i = x_i^{\alpha} - x_{i-1}^{\alpha}$$

Function increment

$$\Delta y = f(x_i) - f(x_{i-1})$$

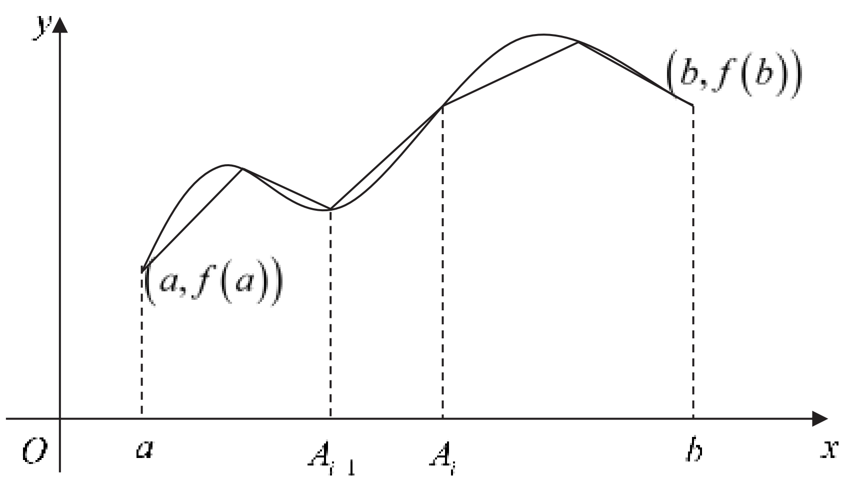

Fig.3.

Hence

$$\lim_{\Delta x\to\infty}\frac{f(x_i)-f(x_{i-1})}{x_i^{\alpha}-x_{i-1}^{\alpha}}=\begin{vmatrix}x_{i-1}=x\\ x_i=x+\Delta x\end{vmatrix}=\lim_{\Delta x\to\infty}\frac{f(x+\Delta x)-f(x)}{(x+\Delta x)^{\alpha}-x^{\alpha}}=$$

$$=\lim_{\Delta x\to\infty}\frac{f(x+\Delta x)-f(x)}{x^{\alpha}+\alpha x^{\alpha-1}\Delta x+\frac{\alpha(\alpha-1)}{2!}x^{\alpha-2}\Delta x^{2}+\ldots-x^{\alpha}}=$$

$$=\lim_{\Delta x\to\infty}\frac{1}{\alpha}\frac{f(x+\Delta x)-f(x)}{\Delta x}\frac{1}{x^{\alpha-1}+\Delta x\left(\frac{\alpha(\alpha-1)}{2!}+\ldots\right)}=$$

$$\frac{1}{\alpha x^{\alpha-1}}f'(x)=\frac{1}{\alpha}x^{1-\alpha}f'(x)$$

Since the derivative of a function depends on, α we call it α the -derivative of function (5) on the plane.

Now let's determine the curvature of the curve given by equation (5).

Theorem 2. The curvature of the curve (5) is calculated by the formula

$$k=\frac{1}{\alpha}x^{1-\alpha}f''(x)$$

Proof: The curvature of a curve is also determined as in the Euclidean plane [29]. Let us calculate the increment of the angle between the tangents at the points under consideration

$$M(x_0,f(x_0))\,\text{And}\,N(x_0+\Delta x,f(x_0+\Delta x))$$

$$\Delta\varphi=r(s_0+\Delta s)-r(s_0)$$

and the arc length increment

$$\Delta s=(s_0+\Delta s)^{\alpha}-s_0^{\alpha}.$$

By definition,

$$k=\lim_{\Delta s\to 0}\frac{r'(x_0+\Delta x)-r'(x_0)}{(s_0+\Delta s)^{\alpha}-s_0^{\alpha}}.$$

We finally get the curvature of the curve

$$k=\lim_{\Delta s\to 0}\frac{r'(x_0+\Delta x)-r'(x_0)}{(s_0+\Delta s)^{\alpha}-s_0^{\alpha}}=$$

$$\lim_{\Delta s\to 0}\left[\frac{1}{\alpha}\frac{r'(s_0+\Delta s)+r'(s_0)}{\Delta s}\frac{1}{s_o^{\alpha-1}+\Delta s\left(\binom{1}{\alpha}s_0^{\alpha-1}+\binom{2}{\alpha}s_0^{\alpha-2}\Delta s+\ldots\right)}\right]=$$

$$=\frac{1}{\alpha}\frac{\Delta h}{\Delta s}\lim_{\Delta s\to 0}\frac{1}{s_o^{\alpha-1}+\Delta s\left(\binom{1}{\alpha}s_0^{\alpha-1}+\binom{2}{\alpha}s_0^{\alpha-2}\Delta s+\ldots\right)}$$

$$=\frac{1}{\alpha}\frac{\Delta h}{\Delta s}\frac{1}{s_o^{\alpha-1}}=\frac{r''(s_0)}{\alpha s_0^{\alpha-1}}$$

$$k=\frac{1}{\alpha}x^{1-\alpha}f''(x)$$

We define a curve on the plane with constant curvature. This curve is the solution to the following differential equation

$$\frac{1}{\alpha}x^{1-\alpha}f''(x)=a$$

$$f''(x)=a\alpha x^{\alpha-1}$$

$$f'(x)=\alpha a\frac{x^{\alpha}}{\alpha}+b=ax^{\alpha}+b$$

$$f(x)=\frac{ax^{\alpha+1}}{\alpha+1}+bx+c.$$

By analogy with Galilean geometry, we call this curve α a -cycle- of the Galilean plane with compressed coordinates.

Conclusion

In this paper, the compressed coordinate plane was studied. The distance between points has the properties of the Galilean distance and represents the generalized geometry of the Galilean plane. Formulas are presented for calculating the area of a triangle and the curvature of a curve in this compressed coordinate system.

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Integral Formulas Related to the Cauchy-Fantappie Integral Formula

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Abstract

The abstract serves both as a general introduction to the topic and as a brief, non-technical summary of the main results and their implications. Authors are advised to check the author instructions for the journal they are submitting to for word limits and if structural elements like subheadings, citations, or equations are permitted.

Keywords: Cauchy-Fantappiè integral, Bochner-Hua Lo-ken integral, holomorphic function

MSC Classification: 32A26 , 32A40 , 32N15 , 32A25.

1 Introduction

In classical complex analysis, the significance of Cauchy's integral formula is well-known, which has the following remarkable properties: Firstly, it is true for any domain with a smooth or piecewise smooth boundary and does not depend on the type of domain (property of universality). Secondly, the kernel of this formula is holomorphic with respect to the outer variable.

In multidimensional complex analysis, there are numerous analogues of Cauchy's integral formula, but they do not simultaneously have these two properties. For example, Cauchy's multiple integral formulas for a polydisc, Leray's formula for ball, the Weil formula for polyhedra and the Bochner-Hua Lo-Ken integral formula for classical domains are with holomorphic kernels and are not universal, but the integral formula Martinelli-Bochner is a universal formula with a non-holomorphic kernel (see [1–4]). In [6] the properties of the functions from H^p class are given in the polydisk, it is given by descriptions of traces for several concrete functional classes on polyballs defined with the help of Bergman metric ball. These results are new even in polydisk. In [7] the eigenfunctions and eigenvalues of the Bochner-Martinelli operator in a half-space are investigated. The work [5] is devoted to the regularity of the Cauchy-Fantappiè integral on strictly convex domains and the monograph [4] is devoted to integral representations of holomorphic several complex variable functions, such as integral formulas of Bochner-Martinelli, Cauchy-Fantappiè, Koppelman and multidimensional logarithmic residue, etc., and their boundary properties. The applications under consideration are problems of analytic continuation of functions from the boundary of a bounded domain in $\mathbb{C}^n$. The Cauchy-Fantappiè integral formula, which contains all the most commonly used integral formulas, depends on an unknown function, associated with the domain, i.e. this formula has an unknown kernel. We present the Cauchy-Fantappiè integral formula (see [1]).

Theorem 1 *For any domain $D \subset \mathbb{C}^n$ with piecewise smooth boundary for and any function $f(z) \in \mathcal{A}(D)$*[1] *holds*

$$f(z) = \frac{(n-1)!}{(2\pi i)^n} \int\limits_{\partial D} f(\zeta) \frac{\delta(\lambda(\zeta)) \wedge d\zeta}{\langle \zeta - z, \lambda(\zeta) \rangle^n}, \tag{1}$$

where $\lambda(\zeta)$ is an arbitrary smooth vector function on ∂D such that

$$\langle \zeta - z, \lambda(\zeta) \rangle \neq 0 \quad \text{for all } z \in D \text{ and } \zeta \in \partial D,\ d\zeta = d\zeta_1 \wedge d\zeta_2 \wedge \ldots \wedge d\zeta_n,$$

$$\zeta - z = (\zeta_1 - z_1, \zeta_2 - z_2, \ldots, \zeta_n - z_n), \delta(w) = \sum_{\nu=1}^{n} (-1)^{\nu-1} w_\nu dw[\nu]\ ,$$

$$dw[\nu] = dw_1 \wedge dw_2 \wedge \ldots \wedge dw_{\nu-1} \wedge dw_{\nu+1} \wedge \ldots \wedge dw_n,\ \langle z, w \rangle = \sum_{\nu=1}^{n} z_\nu w_\nu.$$

2 Cauchy-Fantappiè integral formula in some domain

Despite the great generality of the Cauchy-Fantappiè formula, the question of finding integral representations with a holomorphic kernel for specific domains from $\mathbb{C}^n$ is not removed. In this article, the formula (1) integral formulas with holomorphic kernels are obtained for some domains from $\mathbb{C}^n$.

[1]The function f belongs to the function space $\mathcal{A}(D)$, if f is holomorphic in D and continuous on the closure of $\overline{D}$, i. e. $f(z) \in \mathcal{O}(D) \cap C(\overline{D})$.

) Consider the domain $D_1 \subset \mathbb{C}^n$ of the following type:

$$D_1 = \left\{ z \in \mathbb{C}^n : \sum_{\nu=1}^{n} |z_\nu|^2 < 1 \right\}.$$

For D_1 in formula (1) we choose the vector function $\lambda(\zeta)$ in the form

$$\lambda(\zeta) = \left(\frac{1}{\zeta_1}, \frac{1}{\zeta_2}, \ldots, \frac{1}{\zeta_n} \right), \ \zeta \in \partial D_1.$$

Then

$$\langle \zeta - z, \lambda(\zeta) \rangle = \left\langle (\zeta_1 - z_1, \zeta_2 - z_2, \ldots, \zeta_n - z_n), \left(\frac{1}{\zeta_1}, \ldots, \frac{1}{\zeta_n} \right) \right\rangle = = n - \sum_{\nu=1}^{n} \frac{1}{\zeta_\nu} z_\nu \neq 0$$

for $z \in D_1$ and $\zeta \in \partial D_1$.

Further,

$$\delta(\lambda(\zeta)) \wedge d\zeta = \sum_{\nu=1}^{n} (-1)^{\nu-1} |\zeta_\nu|^2 d\overline{\zeta}[\nu] \wedge d\zeta.$$

Now formula (1) takes the form

$$f(z) = \frac{(n-1)!}{(2\pi i)^n} \int_{\partial D_1} f(\zeta) \frac{\sum\limits_{\nu=1}^{n} (-1)^{\nu-1} |\zeta_\nu|^2 d\overline{\zeta}[\nu] \wedge d\zeta}{\left(n - \sum\limits_{\nu=1}^{n} \frac{1}{\zeta_\nu} z_\nu \right)^n}. \tag{2}$$

b) We consider the domain

$$D_2 = \{ z \in \mathbb{C}^n : |z_1 z_2|^2 + |z_2 z_3|^2 + \ldots + |z_n z_1|^2 < 1 \}.$$

In this case, choosing

$$\lambda(\zeta) = \left(\overline{\zeta}_1 |\zeta_2|^2, \ldots, \overline{\zeta}_{n-1} |\zeta_n|^2, \overline{\zeta}_n |\zeta_1|^2 \right), \ \zeta \in \partial D_2,$$

we have

$$\begin{gathered} \langle \zeta - z, \lambda(\zeta) \rangle = \\ = \left\langle (\zeta_1 - z_1, \zeta_2 - z_2, \ldots \zeta_n - z_n), \left(\overline{\zeta}_1 |\zeta_2|^2, \ldots, \overline{\zeta}_{n-1} |\zeta_n|^2, \overline{\zeta}_n |\zeta_1|^2 \right) \right\rangle = \\ = \left(|\zeta_1 \zeta_2|^2 + |\zeta_2 \zeta_3|^2 + \ldots + |\zeta_n \zeta_1|^2 \right) - \left(\overline{\zeta}_1 |\zeta_2|^2 z_1 + \overline{\zeta}_2 |\zeta_3|^2 z_2 + \ldots + \overline{\zeta}_n |\zeta_1|^2 z_n \right) = \\ = 1 - \sum_{\nu=1}^{n} \overline{\zeta}_\nu |\zeta_{\nu+1}|^2 z_\nu \neq 0, \end{gathered}$$

for $\zeta \in \partial D_2$ and $z \in D_2$ (where $\zeta_{n+1} = \zeta_1$).

Now, let's calculate

$$\delta(\lambda(\zeta)) \wedge d\zeta = \left(\sum_{\nu=1}^{n} (-1)^{\nu-1} \overline{\zeta}_\nu |\zeta_{\nu+1}|^2 \partial_{\overline{\zeta}_1} (\overline{\zeta}_1 |\zeta_2|^2) \wedge \ldots [\nu] \ldots \wedge \partial_{\overline{\zeta_n}} (\overline{\zeta}_n |\zeta_1|^2) \wedge d\zeta \right) =$$

$$= \prod_{k=1}^{n} |\zeta_k|^2 \sum_{\nu=1}^{n} (-1)^{\nu-1} \overline{\zeta}_\nu d\overline{\zeta}[\nu] \wedge d\zeta.$$

Therefore, formula (1) in this case has the form

$$f(z) = \frac{(n-1)!}{(2\pi i)^n} \int\limits_{\partial D_2} f(\zeta) \frac{\prod\limits_{k=1}^{n} |\zeta_k|^2 \sum\limits_{\nu=1}^{n} (-1)^{\nu-1} \overline{\zeta}_\nu d\overline{\zeta}[\nu] \wedge d\zeta}{\left(1 - \sum\limits_{\nu=1}^{n} \overline{\zeta}_\nu |\zeta_{\nu+1}|^2 z_\nu \right)^n}. \tag{3}$$

Note that for $n = 1$ formulas (2) and (3) coincide with the Cauchy's integral formula for the unit circle.

c) Let's consider a domain with a more difficult configuration:

$$D_3 = \{z \in \mathbb{C}^n : \alpha_1 |z_1 z_2|^{2\beta_1} + \alpha_2 |z_2 z_3|^{2\beta_2} + \ldots + \alpha_n |z_n z_1|^{2\beta_n} < 1\},$$

$\alpha_\nu > 0, \beta_\nu \geq 1, \nu = 1, \ldots, n.$

For this domain $\lambda(\zeta)$ can be taken as a vector - function of the following form:

$$\lambda(\zeta) = \left(\alpha_1 \zeta_1^{\beta_1 - 1} \overline{\zeta}_1^{\beta_1} |\zeta_2|^{2\beta_1}, \ldots, \alpha_\nu \zeta_\nu^{\beta_\nu - 1} \overline{\zeta}_\nu^{\beta_\nu} |\zeta_{\nu+1}|^{2\beta_\nu}, \ldots, \alpha_n \zeta_n^{\beta_n - 1} \overline{\zeta}_n^{\beta_n} |\zeta_1|^{2\beta_n} \right), \zeta \in \partial D_3.$$

In this case

$$\langle \zeta - z, \lambda(\zeta) \rangle = 1 - \sum_{\nu=1}^{n} \alpha_\nu \zeta_\nu^{\beta_\nu - 1} \overline{\zeta}_\nu^{\beta_\nu} |\zeta_{\nu+1}|^{2\beta_\nu} z_\nu \neq 0,$$

where $\zeta \in \partial D_3$ and $z \in D_3$ $(\zeta_{n+1} = \zeta_1)$.

The expression $\delta(\lambda(\zeta)) \wedge d\zeta$ has the form:

$$\delta(\lambda(\zeta)) \wedge d\zeta =$$

$$= \prod_{k=1}^{n} (\alpha_k \beta_k) |\zeta_1|^{2\beta_1} \cdots |\zeta_n|^{2\beta_n} \sum_{\nu=1}^{n} (-1)^{\nu-1} \frac{1}{\alpha_\nu \beta_\nu} |\zeta_\nu|^{2(\beta_\nu - 1)} \overline{\zeta}_\nu d\overline{\zeta}[\nu] \wedge d\zeta.$$

Therefore, formulas (1) gives the following integral representation:

$$f(z) = \frac{(n-1)!}{(2\pi i)^n} \int\limits_{\partial D_3} f(\zeta) \times$$

$$\times \frac{\prod\limits_{k=1}^{n} (\alpha_k \beta_k) |\zeta_1|^{2\beta_1} \cdots |\zeta_n|^{2\beta_n} \sum\limits_{\nu=1}^{n} (-1)^{\nu-1} \frac{1}{\alpha_\nu \beta_\nu} |\zeta_\nu|^{2(\beta_\nu - 1)} \overline{\zeta}_\nu d\overline{\zeta}[\nu] \wedge d\zeta}{\left(1 - \sum\limits_{\nu=1}^{n} \alpha_\nu \zeta_\nu^{\beta_\nu - 1} \overline{\zeta}_\nu^{\beta_\nu} |\zeta_{\nu+1}|^{2\beta_\nu} z_\nu\right)^n}. \quad (4)$$

It should be noted that for $\alpha_\nu = \beta_\nu = 1, \nu = 1, \ldots, n$, representation (3) follows from formula (4).

The Cauchy-Fantappiè representation has proven to be very useful and has many applications in multidimensional complex analysis. For example, receiving integral representations have holomorphic kernels by variable z, which makes it possible to uniformly approximate holomorphic functions in corresponding domains by polynomials.

3 Multiple integral Bochner-Hua Lo-ken formula as a generalized Cauchy-Fantappiè formula in matrix domains

The group of automorphisms can be used to find integral formulas for homogeneous domains. Domains with rich automorphism groups are often realize as matrix domains (see [12], [11]). These domains turned out to be useful in solving various problems in the theory of several complex variable functions.

Complex homogeneous bounded domains represent great interest from different points of view. This is explained by the fact that they are a relatively wide class of domains in $\mathbb{C}^n$, for which a number of meaningful, essentially multidimensional results have been obtained in [8, 19].

In 1935 E.Cartan (see [9]) initiated a systematic study of homogeneous domains and found all bounded homogeneous domains in the space $\mathbb{C}^2$ and $\mathbb{C}^3$. It is shown that in the space $\mathbb{C}^2$ any bounded homogeneous domain can be biholomorphically mapped into the ball

$$\mathbb{B}^2(1) = \left\{ z \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1 \right\},$$

or bicircle

$$\mathbb{U}^2 = \left\{ z \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1 \right\}.$$

There are some differences for the space $\mathbb{C}^3$, in this space any bounded homogeneous domain can be mapped biholomorphically into one of the following four domains:

1) the ball

$$\mathbb{B}^3(1) = \left\{ z \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 < 1 \right\};$$

2) the domain

$$\mathbb{G} = \mathbb{B}^2(1) \times \mathbb{U}^1 = \left\{ z \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 < 1, |z_3| < 1 \right\};$$

3) the polydisc

$$\mathbb{U}^3 = \left\{ z \in \mathbb{C}^3 : |z_1| < 1, |z_2| < 1, |z_3| < 1 \right\};$$

4) a bounded domain, which is obtained by a biholomorphic mapping from the domain

$$\tau^{+}(2)=\left\{z\in\mathbb{C}^3:(\operatorname{Im}z_3)^2>(\operatorname{Im}z_1)^2+(\operatorname{Im}z_2)^2,\operatorname{Im}z_3>0\right\}$$

to the future tube (see[10]).

In multidimensional complex analysis, E. Cartan [9] proposed a classification of all bounded symmetric domains. With respect to biholomorphic mappings, these bounded symmetric domains are divided into equivalence classes. Each such class can be specified by specifying one domain belonging to it. After this, it is obvious that it suffices to consider only irreducible classes, i.e., classes of domains that are inexpressible as products of bounded symmetric domains of lower dimensions. E. Cartan [9] established that there are six types of classes of irreducible bounded symmetric domains. Domains belonging to four of these types are called classical because their automorphism groups are classical semisimple Lie groups. Two of these types are special in the sense that each of them occurs in the space $\mathbb{C}^n$ of only one dimension n, respectively for $n=16$ and $n=27$.

Consider the classical domains (according to E. Cartan's classification) (see [8, 9]):

$$\Re_I(m,k)=\left\{Z\in\mathbb{C}[m\times k]:I^{(m)}-Z\bar{Z}'>0\right\},$$

$$\Re_{II}(m)=\left\{Z\in\mathbb{C}[m\times m]:I^{(m)}-Z\bar{Z}>0,\forall Z'=Z\right\},$$
$$\Re_{III}(m)=\left\{Z\in\mathbb{C}[m\times m]:I^{(m)}+Z\bar{Z}>0,\forall Z'=-Z\right\},$$
$$\Re_{IV}(n)=\left\{z\in\mathbb{C}^n:|\langle z,\bar{z}\rangle|^2-2|z|^2+1>0,|\langle z,\bar{z}\rangle|<1\right\},$$

where $I^{(m)}$ is the identity matrix of order m, $\bar{Z}'-$ is the complex conjugate matrix of the transposed matrix Z'. ($H>0$ for a Hermitian matrix H means, as usual, that H is positive definite). All these domains are homogeneous, symmetric, convex, complete, circular domains centered at O (O is zero matrix). All these domains are biholomorphically non-equivalent, so the complex analysis for them is constructed in different ways.

The E.Cartan domains $\Re_V$ and $\Re_{VI}$ in $\mathbb{C}^{16}$ and $\mathbb{C}^{27}$, respectively, are quite essential. The question of an efficient description of these two domains is still open.

In the theory of functions of a single complex variable, we often study the theory of functions in the unit circle $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$, since in the general case all symmetric domains are equivalent to the unit circle, therefore, the above four classes (symmetric classical domains) play an important role in multidimensional complex analysis. By explicitly writing out the transitive automorphism group of the four types of classical domains and matrix balls (see e.g. [13], [14]) associated with classical domains, one can find the Bergman and Cauchy-Szegő kernels for these domains by direct calculation. Then, using the properties of the Poisson kernel, we find a formula that restores the value of the holomorphic function in the domain itself from its values on some boundary sets of uniqueness. In [18] the volumes of a matrix ball of the third type and a generalized Lie ball are calculated. To find the kernels of integral formulas for these domains (Bergman, Cauchy-Szegő, Poisson kernels, etc.), the total volumes of these domains are needed, and these volumes are also used for the integral representation of holomorphic functions in these domains, in the mean value theorem, and in other important concepts (see for exam. [16], [17] [20, 21], [23]).

Now, let us consider the holomorphic continuation of the classical domain of the first type $\Re_I(m,k)$ and its skeleton $\mathbb{X}_I$. Consider the space $L^2(\mathbb{X}_I, d\mu)$, i.e., the space of square-integrable functions f, with respect to the normalized Lebesgue measure $d\mu$. It is the Haar measure on the skeleton $\mathbb{X}_I$, and hence is invariant under rotations. As is known, the Hardy class $H^2(\Re_I(m,k))$ consists of all functions f, that are holomorphic in the domain $\Re_I(m,k)$ for which

$$\|f\|_{H^2} = \sup_{0<r<1} \left(\int_{\mathbb{X}_I} |f(rZ)|^2 d\mu \right)^{\frac{1}{2}} < \infty.$$

Since $\Re_I(m,k)$ is a bounded complete circular domain, functions f of class $H^2(\Re_I(m,k))$ has the following properties (see [11], [15], [24]):

1^0. The slice functions $f_Z(\lambda) = f(\lambda Z)$ (in measure μ) belong to the space H^2 in the unit circle $\Delta = \left\{\lambda \in \mathbb{C}^1 : |\lambda| < 1\right\}$, for almost all $Z \in \mathbb{X}_I$;

2^0. The function f has radial boundary values

$$\lim_{r\to 1-0} f(rZ) = f^*(Z), Z \in \mathbb{X}_I,$$

and these boundary values f^* belong to the class $L^2(\mathbb{X}_I, d\mu)$;

3^0. The following formula is valid

$$\lim_{r\to 1-0} \int_{\mathbb{X}_I} |f(rZ)|\, d\mu = \int_{\mathbb{X}_I} |f^*(Z)|\, d\mu;$$

4^0. If slice functions $f_Z(\lambda)$ of some function holomorphic in $\Re_I(m,k)$ the function f belong to the Hardy class H^2 in the unit circle for almost all $Z \in \mathbb{X}_I$ and radial boundary values f^* lie down in $L^2(\mathbb{X}_I, d\mu)$, then $f \in H^2(\Re_I(m,k))$;

5^0. Any function $f \in H^2(\Re_I(m,k))$ can be represented by the Bochner-Hua Lo-ken formula as

$$f(Z) = \int_{\mathbb{X}_I} \det{}^{-k}\left(I^{(k)} - \langle Z, U\rangle\right) f(U) d\mu, \tag{5}$$

the function $f \in H^2(\Re_I(m,k))$ is restored to $\Re_I(m,k)$ by their radial boundary values f^* .

6^0. If the set $V \subset \mathbb{X}_I$ has positive measure ($\mu(V) > 0$), then V is a set of uniqueness for the Hardy class $H^2(\Re_I(m,k))$;

7^0. The Hardy class $H^2(\Re_I(m,k))$ is invariant under automorphisms of the ball $\Re_I(m,k)$.

4 Conclusion remarks

The Bochner-Hua Lok-en integral formula (5) given above is a general case of the Cauchy-Fantappiè formula considered in the previous section. If $m = 1$, then $\mathbb{B}^k(1) = \{z \in \mathbb{C}^k : |z| < 1\}$ – represents the Cauchy-Fantappiè formula for the unit ball, that is, the kernel of formula (5) appears as

$$\det{}^k(I - \langle Z, U\rangle) = \{1 - \langle z, u\rangle\}^k.$$

Hence the required integral formula $\mathbb{B}^k(1)$ becomes the Cauchy-Fantappiè formula for the ball:

$$f(Z) = \int\limits_{\mathbb{X}_I} \frac{f(U)}{\det^n(I - \langle Z, U\rangle)} d\mu(U) = \int\limits_{\mathbb{S}^k(1)} \frac{f(u)}{\{1 - \langle z, u\rangle\}^n} d\mu(u) \ ,$$

where $z = (z_1, z_2, ..., z_k) \in \mathbb{B}^k(1)$ and $w = (w_1, w_2, ..., w_k) \in \mathbb{S}^k(1)$.

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ESTIMATION OF UNKNOWN PARAMETERS OF GAMMA AND WEIBULL DISTRIBUTIONS IN INCOMPLETE MODELS OF STATISTICS

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Abstract

In this paper, we present an analysis of the estimation results for the parameters of the Gamma and Weibull distributions under Type I right-censoring using numerical optimization techniques. The Maximum Likelihood Estimation (MLE) approach, combined with both the Quasi-Newton and Nelder-Mead algorithms, is employed to estimate unknown parameters, with MATLAB used for numerical simulations. In the case of the Gamma distribution, we focus on the estimation of shape (α) and scale (β) parameters through simulations with sample sizes ranging from 100 to 2000. The estimation results reveal that the Quasi-Newton method achieves higher estimation accuracy in uncensored states, while the Nelder-Mead method shows better results under fixed censoring. For the Weibull distribution, random samples with sizes of 500 and 1000 are generated to estimate the shape (k) and scale (λ) parameters. The Quasi-Newton algorithm outperforms Nelder-Mead under uncensored conditions, whereas the latter shows superior performance in censored scenarios. Overall, the results underscore the importance of selecting an appropriate estimation method based on the censoring level and sample size. The Quasi-Newton algorithm tends to provide more accurate parameter estimates for uncensored data, while the Nelder-Mead approach is more suitable for censored data. Future research could expand on these findings by exploring additional numerical techniques and enhancing the integration of statistical modeling methods.

Keywords: *Gamma distribution, Weibull distribution; Quasi-Newton method; Nelder-Mead method; Maximum Likelihood Estimation; random censoring.*

1. Introduction

The estimation of parameters for Gamma and Weibull distributions in the presence of censored observations represents one of the key tasks in mathematical statistics. These distributions are widely used for modeling the time until a specific event occurs, reliability analysis, and risk assessment. However, the presence of censored data complicates the parameter estimation process, requiring the application of modern numerical methods to ensure high accuracy and robustness of the estimates. The maximum likelihood estimation (MLE) method is traditionally used for parameter estimation in such distributions.

Cohen [1] was the first to apply MLE to the two-parameter Weibull distribution, demonstrating its effectiveness for analyzing data with progressive censoring. Nevertheless, solving likelihood equations analytically becomes infeasible under complex censoring schemes, necessitating the use of numerical optimization methods. Balakrishnan and Kateri [2] proposed numerical approaches for estimating the parameters of the Weibull distribution based on complete and censored data. Their studies showed that numerical methods provide more accurate parameter estimates compared to analytical approaches when handling large datasets.

A comparison of optimization methods, such as Quasi-Newton and Nelder-Mead, indicates that their efficiency depends on the level of censoring and the data structure. Lagarias et al. [3] noted that the Nelder-Mead method exhibits instability in high-dimensional problems but remains popular due to its simplicity of implementation. Fletcher [4] demonstrated that Quasi-Newton methods have faster convergence rates and higher accuracy, especially when processing data with low levels of censoring.

Praveen Agarwal and others [8-11] investigated various practical problems related to differential equations and statistical modeling. The methods presented in their works mathematically substantiate the existence of solutions to differential equations, which is crucial for modeling Gamma and Weibull distributions. These studies contribute to a better understanding of the application of numerical optimization methods under censored data conditions.

The Weibull distribution is traditionally used for modeling time-to-failure in engineering and reliability analysis [5]. The Gamma distribution is applied in medicine and insurance for analyzing time series with censoring [6]. Kazemi and Azizpour [7] explored hybrid censoring schemes for the Weibull distribution, demonstrating that Bayesian methods outperform MLE under high levels of censoring. In the context of random censoring models, issues of statistical efficiency have been studied in [12-13]. Additionally, in the author's work [14], the problem of parameter estimation for the Gamma distribution using numerical methods is addressed.

In the present study, we conduct a comparative analysis of the Quasi-Newton and Nelder-Mead algorithms for estimating the parameters of Gamma and Weibull distributions under right censoring. The use of numerical methods allows us to evaluate the performance of these algorithms across various censoring models. The results obtained indicate that the Quasi-Newton algorithm is more effective for complete observations, whereas the Nelder-Mead method demonstrates high accuracy for incomplete observations under a fixed level of censoring.

2. Likelihood construction and MLE for censored data

We observe $X=(X_1,...,X_n)$ which are independent and identically distributed (iid) and have a continuous distribution with the probability density function (pdf) $f(x)$ and the cumulative distribution function (CDF) $F(x)$.[5] Data from experiments involving random censoring can be conveniently represented by pairs $\left(z_i,\delta_i\right)$ with $z_i=\min(x_i,T_i)$:

$$\delta_i=\begin{cases}0, & if\ x_i>T_i\ censored\\ 1, & if\ x_i\le T_i\ uncensored\end{cases}\qquad \text{for } i=1,...,n,$$

where δ_i is a censoring indicator variable and T_i is a censoring time of test unit *i*. Denote the vector of unknown parameters by $\theta=(\theta_1,...,\theta_p)$. We have a sample size *n*: $C^{(n)}=\left\{\left(z_i,\delta_i\right),1\le i\le n\right\}$ and we have

$$L(X,\theta)=\prod_{i=1}^{n}[f(X_i;\theta)]^{\delta_i}[1-F(X_i;\theta)]^{1-\delta_i}. \tag{1}$$

The likelihood function (1) effectively combines uncensored and censored observations, in that if an individual is not censored, the probability of the event is $f(X_i)$, and if the individual is censored at X_i, the probability of the event is $S(X_i)=1-F(X_i)$,[7,13,14] the survivorship function evaluated at X_i. Taking the natural log of L, the objective is to maximize the expression.

$$\ln(L(Z^n;\theta))=\sum_{i=1}^{n}\delta_i\ln f(X_i;\theta)+\sum_{i=1}^{n}(1-\delta_i)\ln(1-F(X_i;\theta)) \tag{2}$$

We can substitute variables like as follow $T_i=x_i$, with unknown parameter θ usually survival function $S(x)$ is defined by CDF $F(x)$:

$$S(x) = P(X > x) = 1 - F(x),$$

$$\ln[L(X,\theta)] = \sum_{i=1}^{n} \delta_i \ln f(x_i,\theta) + \sum_{i=1}^{n} (1-\delta_i) \ln(1 - F(x_i,\theta))$$

where,

$$\delta_i = \begin{cases} 1, \ if\ X_i \le x_i\ uncensored \\ 0, \ if\ X_i > x_i\ censored \end{cases}$$

To simplify computations, the log-likelihood function $l(X,\theta)$ is often used, as it converts products into sums.

$$l(X,\theta) = \ln[L(X,\theta)]$$

MLE finds parameter values that maximize the likelihood function. This is achieved by taking the derivative of the likelihood function with respect to the parameters, setting it to zero, and finding the parameters:

$$\frac{\partial l(X,\theta)}{\partial \theta} = 0$$

3. Analytical approach with MLE method for Gamma and Weibull distributions

The Gamma distribution can be parameterized with shape α and rate β is denoted $X \sim \Gamma(\alpha,\beta) = Gamma(\alpha,\beta)$.

The corresponding probability density function (pdf) in the shape-rate parameterization for distributions Gamma and Weibull respectively are:

$$f(x;\alpha,\beta) = \frac{x^{\alpha-1} e^{-\beta x} \beta^{\alpha}}{\Gamma(\alpha)} \quad \text{for } x > 0,\ \alpha > 0,\ \beta > 0,$$

where $\Gamma(\alpha)$ is the gamma function; for all positive integers, $\Gamma(\alpha) = (\alpha - 1)!$. The cumulative distribution function (cdf) is the regularized gamma function:

$$F(x;\alpha,\beta) = \int_0^x f(t;\alpha,\beta)dt = \frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}$$

where $\gamma(\alpha,\beta x)$ is the lower incomplete gamma function.

$$f(x;\lambda,k) = \begin{cases} \dfrac{k}{\lambda}\left(\dfrac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}, \ x \ge 0 \\ 0\ ,\ x < 0 \end{cases}$$

where $k>0$ is the shape parameter and $\lambda>0$ is the scale parameter of the Weibull distribution. The cdf for the Weibull distribution is:

$$F(x;k,\lambda)=\int_0^x f(t,\lambda,k)dt=\int_0^x \frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1} e^{-(t/\lambda)^k}dt=1-e^{-(x/\lambda)^k}, x\geq 0$$

In the context of right-censored data, selecting an appropriate probability distribution is crucial for accurate parameter estimation. Common choices include the Gamma and Exponential distributions, both widely applied in survival analysis and reliability engineering. [10,16]

Let's consider the Gamma distribution and the case when $\alpha=1,\ \beta=\frac{1}{\lambda}$ for simplicity [1]:

$$F(x;1,\frac{1}{\lambda})=\int_0^x \frac{1}{\lambda}e^{-t/\lambda}dt=\frac{1}{\lambda}\int_0^x e^{-t/\lambda}dt=\frac{1}{\lambda}\cdot(-\lambda)\cdot e^{-t/\lambda}\big|_0^x=1-e^{-x/\lambda}.$$

For right-censored data, the likelihood function is derived by considering both the probability of observing the uncensored data and the probability of right-censored data. Let δ_i be the indicator variable for censorship $\delta_i=I(x_i<T_i)$. The likelihood function $L(X,\alpha,\beta)$ for a set of observations x_i and censorship indicators δ is given by:

$$L(X,\alpha,\beta)=\prod_{i=1}^{n}\left[f(x_i;\alpha,\beta)^{\delta_i}(1-F(x_i;\alpha,\beta))^{1-\delta_i}\right].$$

Now, we can take the natural logarithm of the likelihood function:

$$l(X,\lambda)=\sum_{i=1}^{n}\left[\delta_i\ln f(x_i;\lambda)+(1-\delta_i)\ln(1-F(x_i;\lambda))\right]=$$
$$=\sum_{i=1}^{n}\left(\delta_i\ln\left[\frac{1}{\lambda}e^{-x_i/\lambda}\right]+(1-\delta_i)\ln\left[1-(1-e^{-x_i/\lambda})\right]\right)=$$
$$=\sum_{i=1}^{n}\left(-\delta_i\cdot\ln\lambda-\delta_i\cdot\frac{x_i}{\lambda}-(1-\delta_i)\cdot\frac{x_i}{\lambda}\right).$$

To maximizing the Log-Likelihood function we should calculate the partial derivatives of the log-likelihood with respect to the parameters $\alpha=1$ and $\beta=\frac{1}{\lambda}$:

$$\frac{\partial l(X,\lambda)}{\partial \lambda}=0;$$

$$\frac{\partial l(X,\lambda)}{\partial \lambda}=\sum_{i=1}^{n}\left(-\frac{\delta_i}{\lambda}+\delta_i\cdot\frac{x_i}{\lambda^2}+(1-\delta_i)\cdot\frac{x_i}{\lambda^2}\right)=0;$$

$$\hat{\lambda}_{MLE}=\frac{\sum_{i=1}^{n}x_i}{\sum_{i=1}^{n}\delta_i};$$

When $\alpha=2,\beta=\beta$ we can find out cdf for Gamma function

$$F(x;2,\beta)=\int_0^x\frac{te^{-\beta t}\beta^2}{\Gamma(2)}dt=\left[\Gamma(\alpha)=(\alpha-1)!,\ \Gamma(2)=1\right]=\int_0^x t\,e^{-\beta t}\beta^2dt=$$

$$=\begin{bmatrix} we\ take\ substitute\ by\ parts\ formula \\ \int udv=uv-\int vdu \\ u=t,\ dv=e^{-\beta t}dt \\ du=dt,\ v=-\dfrac{1}{\beta}e^{-\beta t} \end{bmatrix}=\beta^2t\left(-\frac{1}{\beta}e^{-\beta t}\right)\Big|_0^x+\beta^2\int_0^x\frac{1}{\beta}e^{-\beta t}dt=$$

$$=-\beta xe^{-\beta x}+\beta\left(-\frac{1}{\beta}e^{-\beta t}\right)\Big|_0^x=1-e^{-\beta x}(\beta x+1).$$

Now, we can find the natural logarithm of the likelihood function and estimate with respect to β on MLE method:[1]

$$l(\beta\,|\,X,\delta)=\sum_{i=1}^{n}\left[\delta_i\ln f(x_i;\beta)+(1-\delta_i)\ln(1-F(x_i;\beta))\right]=$$

$$=\sum_{i=1}^{n}\left(\delta_i\ln\left[x\,e^{-x\beta}\beta^2\right]+(1-\delta_i)\ln\left[1-(1-e^{-\beta x}(\beta x+1))\right]\right)$$

$$=\sum_{i=1}^{n}\left(\delta_i\ln x_i-\delta_i\beta x_i+2\delta_i\ln\beta-\beta x_i+\delta_i\beta x_i+\ln(\beta x_i+1)-\delta_i\ln(\beta x_i+1)\right)$$

$$=\sum_{i=1}^{n}\left(\delta_i\ln x_i+2\delta_i\ln\beta-\beta x_i+\ln(\beta x_i+1)-\delta_i\ln(\beta x_i+1)\right).$$

To maximizing the Log-Likelihood function we should calculate the partial derivatives of the log-likelihood with respect to the parameters:

$$\frac{\partial l(\beta\,|\,X,\delta)}{\partial\beta}=0$$

$$\frac{\partial l(\beta \mid X,\delta)}{\partial \beta}=\sum_{i=1}^{n}\left(\frac{2\delta_i}{\beta}-x_i+\frac{x_i}{\beta x_i+1}-\frac{\delta_i x_i}{\beta x_i+1}\right)=0$$

$$\sum_{i=1}^{n}\left(\frac{2\delta_i}{\beta}-x_i+\frac{x_i}{\beta x_i+1}-\frac{\delta_i x_i}{\beta x_i+1}\right)=0\,.$$

After some algebra we define $\hat{\beta}_{MLE}$:

$$\hat{\beta}_{MLE}=\frac{n\left[\sum_{i=1}^{n}x_i-2\sum_{i=1}^{n}\delta_i\right]}{\left[2\sum_{i=1}^{n}\delta_i\sum_{i=1}^{n}x_i-\left(\sum_{i=1}^{n}x_i\right)^2+\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\delta_i x_i\right]}.$$

We use the estimated value $\hat{\beta}_{MLE}$ to find the $\hat{\alpha}_{MLE}$ and go back to find cdf of Gamma distribution when $\alpha=\alpha$ $,\beta=\hat{\beta}_{MLE}$

$$F(x;\alpha,\hat{\beta}_{MLE})=\int_{0}^{x}\frac{t^{\alpha-1}e^{-\hat{\beta}_{MLE}\cdot t}\hat{\beta}_{MLE}^{\alpha}}{\Gamma(\alpha)}dt,\ \left[\Gamma(\alpha)=(\alpha-1)!\right];$$

Due to complexity of the $F(x;\alpha,\hat{\beta}_{MLE})$ integral function for any values of the gamma distribution parameters α and β, it is common to employ numerical techniques or specialized software tools (including mathematical software or programming libraries like Python, MATLAB, or R) to perform the integration process and derive the cdf, rather than relying on analytical solutions.

We can also apply this process to the Weibull distribution to estimate its unknown parameters, λ and k, using the MLE method in an analytical way. Let's find the cdf $F(x;\lambda,k)$ for the given pdf of the Weibull distribution:

$$f(x;\lambda,k)=\begin{cases}\dfrac{k}{\lambda}\left(\dfrac{x}{\lambda}\right)^{k-1}e^{-(x/\lambda)^k}, & x\geq 0,\\ 0\ , & x<0,\end{cases}$$

the cdf is defined as:

$$F(x;k,\lambda)=\int_{0}^{x}f(t,\lambda,k)dt,\ \ x\geq 0$$

Substituting the expression for $f(x;\lambda,k)$:

$$F(x;k,\lambda)=\int_0^x f(t,\lambda,k)dt=\int_0^x \frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1} e^{-(t/\lambda)^k}dt.$$

To solve this integral analytically we can use $u=\left(\frac{t}{k}\right)^k$ substitution, so we have:

$$t=\lambda u^{1/k},\ dt=\frac{\lambda}{k}u^{(1-k)/k}du$$

We can update the limits: when $t=0$, $u=0$; when $t=x$, $u=\left(\frac{x}{\lambda}\right)^k$.

After some algebra the integral becomes as follow:

$$F(x,\lambda,k)=\int_0^{\left(\frac{x}{\lambda}\right)^k} e^{-u}du=\left(-e^{-u}\right)\Big|_0^{\left(\frac{x}{\lambda}\right)^k}=1-e^{-\left(\frac{x}{\lambda}\right)^k}.$$

Thus, the cdf of the given pdf is: $F(x,\lambda,k)=1-e^{-(x/\lambda)^k}$, $x\geq 0$.

Here, log-likelihood function is:

$$l(X,\lambda,k)=\sum_{i=1}^n\left[\delta_i\ln f(x_i;\lambda,k)+(1-\delta_i)\ln(1-F(x_i;\lambda,k))\right]=$$

$$=\sum_{i=1}^n\left[\delta_i\left[\ln\left(\frac{k}{\lambda}\right)+(k-1)\ln\left(\frac{x_i}{\lambda}\right)-\left(\frac{x_i}{\lambda}\right)^k\right]+(1-\delta_i)\ln(1-(1-e^{-(x_i/\lambda)^k}))\right]$$

$$=\sum_{i=1}^n\left[\delta_i\left(\ln\left(\frac{k}{\lambda}\right)+(k-1)\ln\left(\frac{x_i}{\lambda}\right)-\left(\frac{x_i}{\lambda}\right)^k\right)+(1-\delta_i)\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right].$$

To maximizing the log-likelihood function we should calculate the partial derivatives of the log-likelihood with respect to the parameters λ and k:

Partial derivative with respect to λ:

$$\frac{\partial l}{\partial\lambda}=\frac{k}{\lambda}\sum_{i=1}^n\left(-\delta_i+\lambda^{-k}x_i^k\right).$$

Partial derivative with respect to k:

$$\frac{\partial l}{\partial k}=\sum_{i=1}^n\left[\lambda^{-k}x_i^k\left(\ln(\lambda)-\ln(x_i)\right)+\delta_i\left(\ln(x_i)-\ln(\lambda)+\frac{1}{k}\right)\right].$$

To find the MLEs for λ and k, we should solve the system of equations:

$$\frac{\partial l}{\partial \lambda} = 0 \text{ and } \frac{\partial l}{\partial k} = 0.$$

$$\begin{cases} \dfrac{k}{\lambda} \displaystyle\sum_{i=1}^{n} \left(-\delta_i + \lambda^{-k} x_i^k \right) = 0 \\ \displaystyle\sum_{i=1}^{n} \left[\lambda^{-k} x_i^k \left(\ln(\lambda) - \ln(x_i) \right) + \delta_i \left(\ln(x_i) - \ln(\lambda) + \frac{1}{k} \right) \right] = 0 \end{cases}$$

The symbolic solution to the system of equations does not yield explicit results due to the complexity of the equations. This is expected for nonlinear likelihood functions like this one, as they often require numerical optimization rather than analytical solutions.

Thus, to estimate the unknown parameters of both Gamma and Weibull distributions, we can use numerical optimization methods such as the Quasi-Newton and Nelder-Mead simplex algorithm in MATLAB. We then compare their estimated results below in tables and graphs relatively below.

Table 1

Estimated of unknown parameters of Gamma distribution with two rules in MLE

n	*Censoring level, Constant* T_i $T_i \in [0,10]$	*p%*	*Gamma Distribution*	
			Initial Guess α =2; β =1	
			MLE by Quasi-Newton rule	*MLE by Nelder-Mead rule*
100	*10*	*0*	*α =1.928; β=1.106*	*α =1.906; β=1.107*
	4	*10*	*α =1.848; β=1.218*	*α =2.121; β=1.210*
	2.5	*30*	*α =1.359; β=2.107*	*α =1.380; β=2.056*
	2	*50*	*α =1.125; β=3.185*	*α =1.129; β=2.984*
500	*10*	*0*	*α =2.082; β=0.971*	*α =2.095; β=1.110*
	4	*10*	*α =1.896; β=1.108*	*α =2.102; β=0.969*
	2.5	*30*	*α =1.384; β=1.999*	*α =1.395; β=1.910*
	2	*50*	*α =1.113; β=3.003*	*α =1.101; β=2.659*
1000	*10*	*0*	*α =2.016; β=0.991*	*α =1.98; β=0.987*
	4	*10*	*α =1.895; β=1.303*	*α =2.201; β=1.205*
	2.5	*30*	*α =1.332; β=2.187*	*α =1.391; β=1.992*
	2	*50*	*α =1.101; β=2.945*	*α =1.109; β=2.730*
1500	*10*	*0*	*α =2.013; β=0.995*	*α =1.985; β=1.012*
	4	*10*	*α =1.899; β=1.319*	*α =1.908; β=1.328*
	2.5	*30*	*α =1.383; β=2.002*	*α =1.398; β=1.976*
	2	*50*	*α =1.112; β=3.00*	*α =1.154; β=2.943*
2000	*10*	*0*	*α =1.992; β=1.016*	*α =2.002; β=1.0085*
	4	*10*	*α =1.831; β=1.115*	*α =1.847; β=1.189*
	2.5	*30*	*α =1.381; β=1.969*	*α =1.317; β=1.529*
	2	*50*	*α =1.126; β=3.103*	*α =1.113; β=2.936*

** Note: The sample data, n, is simulated randomly. Therefore, if the code is executed, it may display different results compared to those presented in Table 1.*

Similar results were obtained for the Weibull distribution. Here we present only the results of estimating unknown parameters.

Table 2

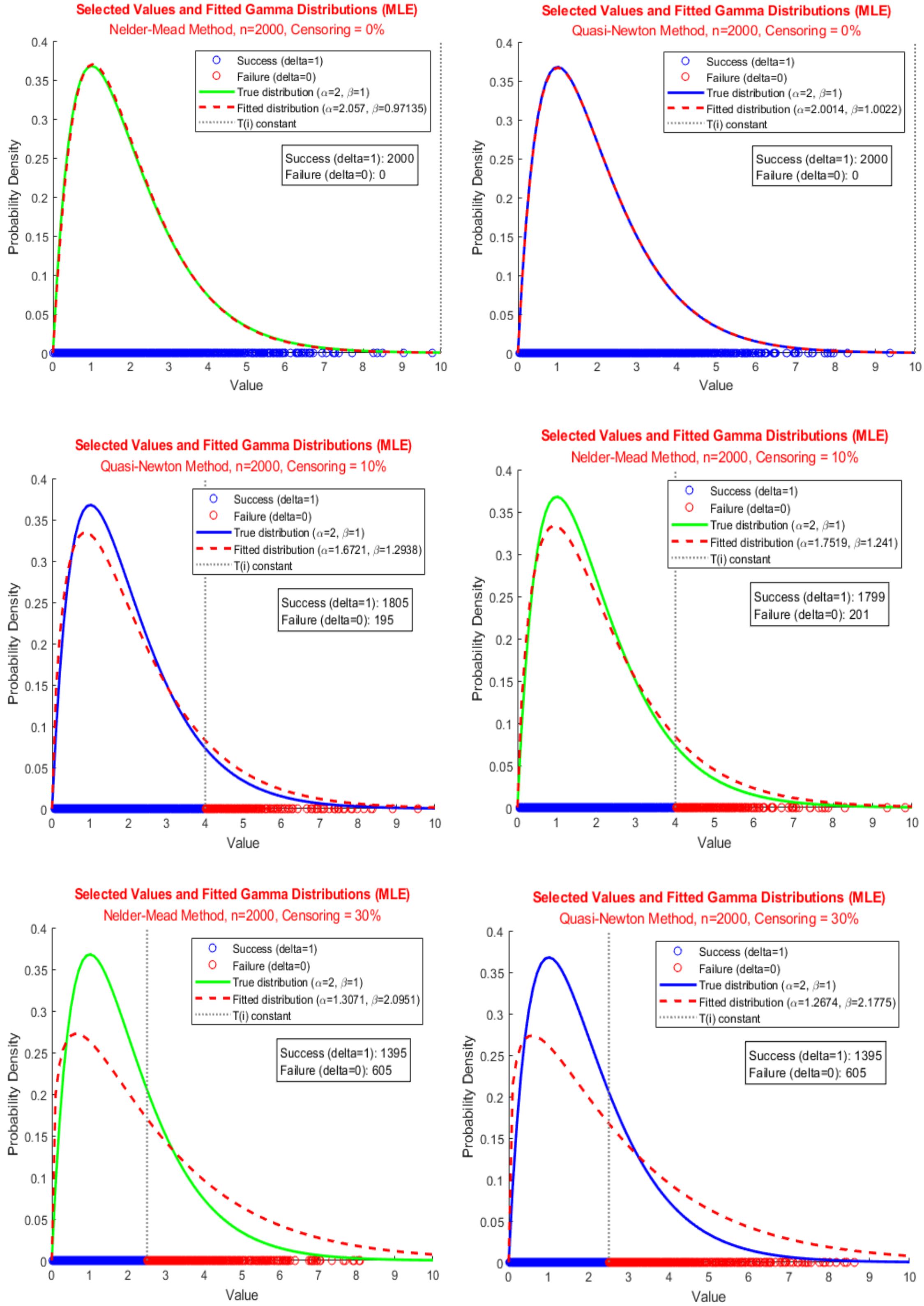
Estimated of unknown parameters of Weibull distribution with two rules in MLE

Interval [0;10]			
T	p%	Weibull Distribution	
		Initial Guess k =2; λ=3	
		$T_i \in [0;10]$ fixed constant	
		MLE by Quasi-Newton rule	MLE by Nelder-Mead rule
***n*=500**			
10	0%	k =1.9815; λ =3.0215;	k =1.9118; λ =3.0412;
2.7	10%	k =2.0913; λ =2.705;	k =2.0742; λ =2.809;
2.2	30%	k =2.323; λ =2.2342;	k =2.1821; λ =2.401;
***n*=1000**			
10	0%	k =1.9898; λ =3.0358;	k =1.9764; λ =3.0723;
2.7	10%	k =2.0816; λ =2.6103;	k =2.0646; λ =2.6207;
2.2	30%	k =2.2259; λ =2.2236;	k =2.1957; λ =2.3091;

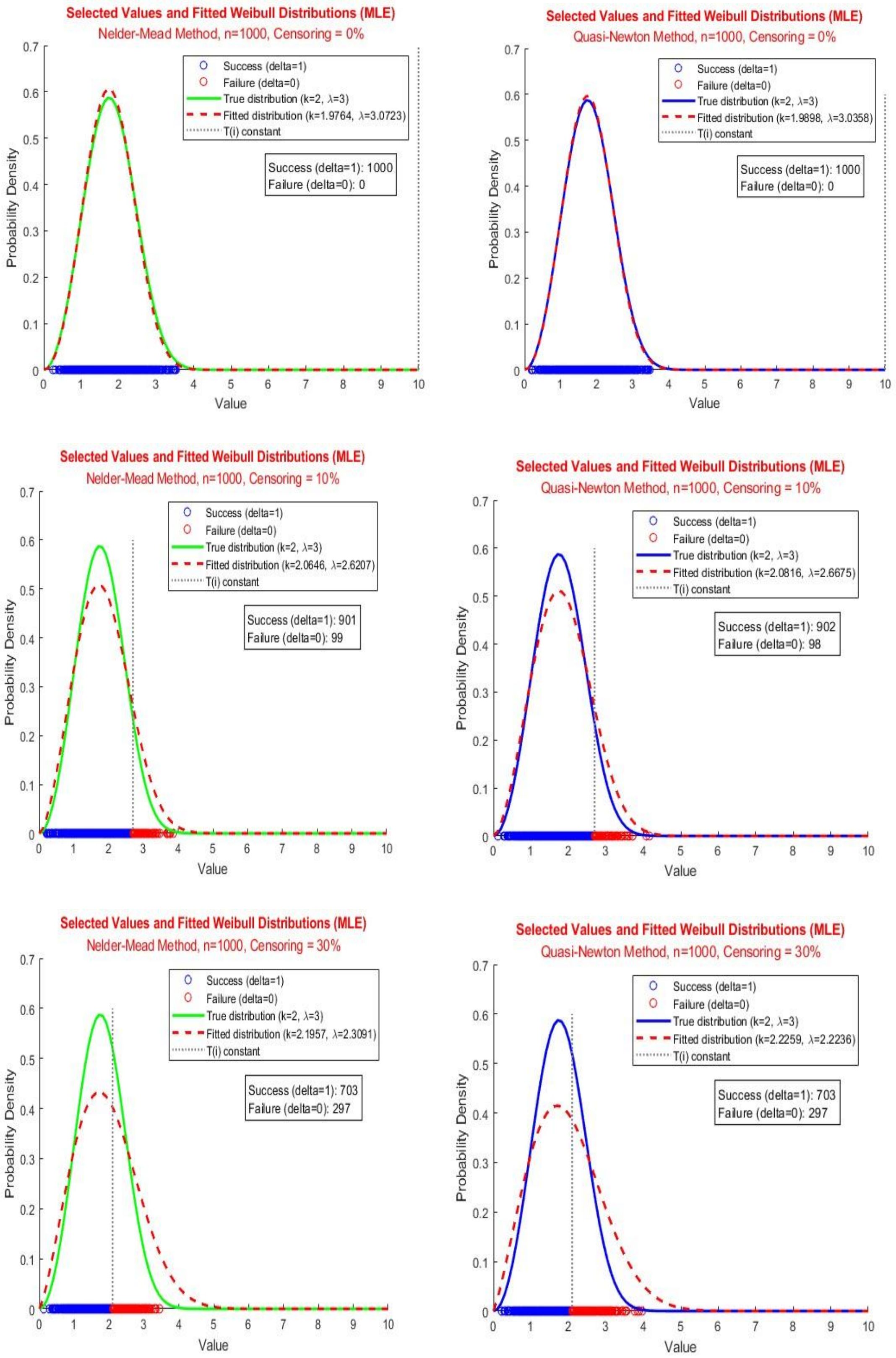
** Note: The sample data, n, is simulated randomly. Therefore, if the code is executed, it may display different results compared to those presented in Table 2.*

To illustrate the results for both distributions we obtained in Table 1 and 2 respectively, we have included some graphics below relatively.

Random selected values and fitted Gamma distribution using MLE



Random selected values and fitted Weibull distribution using MLE



4. Conclusion

Based on simulation experiments and the presented tables for estimating the parameters of Gamma and Weibull distributions, the following conclusions can be drawn:

1. The Quasi-Newton method often demonstrates higher accuracy and faster convergence for low levels of censoring or complete observations. This method effectively utilizes gradient information, enabling a rapid approach to the optimal solution.
2. The Nelder–Mead method is simpler to implement as it does not require the computation of derivatives of the likelihood function. However, it can be less stable and less accurate in high-dimensional tasks or when dealing with complex likelihood functions.

The results presented in the tables show that the Nelder–Mead method provides better parameter estimates under a fixed (high) level of censoring. This is due to the fact that under strong censoring, the shape of the likelihood function becomes more complex, and gradient-independent methods may better handle such problems.

Thus, the choice of a specific numerical method depends on the problem structure, sample size, and level of censoring. The Quasi-Newton method is preferable under relatively simple initial conditions and low levels of censoring, offering high accuracy and fast convergence. The Nelder–Mead method demonstrates better adaptability for partially observed (censored) data and larger sample sizes, although it may require more iterations and face convergence challenges in high-dimensional tasks.

The practical significance of the results lies in their applicability to a wide range of applied problems, including equipment reliability, failure analysis, biostatistics, and economic time series. Efficient parameter estimation methods under censored data conditions enable more accurate modeling and forecasting of system and process behaviors across various fields of science and technology.

The importance of this study lies in the necessity of precise and robust parameter estimation for Gamma and Weibull distributions in the presence of censored data. The comparative analysis of the Quasi-Newton and Nelder-Mead methods allows for selecting the most effective approaches to solving this task, ensuring high accuracy and reliability of the estimates.

Future research prospects include the consideration of other estimation methods, such as the EM algorithm and Bayesian approaches, in various random censoring models. Comparative analysis of these methods will help determine their efficiency and applicability under different conditions, as well as develop more universal and robust approaches to parameter estimation for distributions with censored data.

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Vulnerability Assessment and Phishing Analysis

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Abstract - The paper considers aspects of the use of certain types of vulnerability assessment and phishing analysis. This paper delves into the essential concepts surrounding vulnerability assessment and suggests effective prevention techniques to fortify digital data. By conducting comprehensive vulnerability assessments, organizations can gain valuable insights into potential entry points for malicious actors and preemptively address them before exploitation occurs. We discuss various vulnerability scanning tools, methodologies, and frameworks employed to assess the susceptibility of digital infrastructures. Additionally, this paper highlights the importance of maintaining continuous monitoring and response capabilities as an integral part of vulnerability prevention. In the dynamic landscape of cybersecurity, threats evolve rapidly, necessitating real-time monitoring to detect and respond to potential breaches promptly. Furthermore, the paper emphasizes the need for a holistic approach to vulnerability management, considering both technical and human aspects. Addressing the human element, including employee education and awareness training, proves critical in preventing social engineering attacks and fostering a security-aware culture within organizations.

Keywords: Attacker, cyber frauds, vulnerability, malware, phishing.

I. INTRODUCTION

In 1820, the first cybercrime was noted. So far, bearing in mind that electronic machines have come a long way, this is certainly not so scary. In an increasingly interconnected world where data flows like water and technology serves as the backbone of our daily lives, the security of our digital landscapes has never been more critical. With the rapid expansion of cyberspace and the relentless evolution of cyber threats, understanding and mitigating vulnerabilities in our digital infrastructure has become a paramount concern.

The internet, an intricate web of interconnected systems and devices, has woven itself into the very fabric of modern life. It has revolutionized communication, commerce, and countless aspects of daily existence. The Internet, which connects billions of people around the world, is the main pillar of the modern information society. In 2023, Northern Europe took first place among the world's regions in terms of the percentage of the population using the Internet. In Norway, Saudi Arabia, and the United Arab Emirates, 99 percent of the population currently uses the Internet. Internet users in China, India and the United States are ahead of others. Also, other developed and developing countries are recording very high growth in this regard. However, this digital marvel is far from invincible; it is, in fact, riddled with chinks in its virtual armor. These chinks, often referred to as vulnerabilities, have emerged as one of the most critical and defining aspects of today's internet landscape.

Technology, the driving force behind the rapid evolution of our modern world, stands as a double-edged sword, possessing the remarkable ability to serve both virtuous and sinister ends [1]. Attacker will use it for bad purpose. Devices available in IT are also no exceptions; like other tool, they are used as either prey of crime or means for committing a crime. In the contemporary era of the Internet and interconnected computer systems, criminal activities can easily transcend international boundaries, often cloaked in a deceptive shroud of anonymity. Unwittingly, we frequently divulge vast troves of personal information. The question that arises is whether we can be confident that this wealth of data will never fall into the wrong hands or be exploited for nefarious purposes.

One significant avenue through which cybercrime proliferates is by capitalizing on vulnerable or flawed software systems. In the intricate web of the digital world, attackers often pinpoint and manipulate weaknesses or imperfections within software applications, thereby gaining unauthorized access, compromising data, and unleashing potential devastation. These vulnerabilities can arise from various sources, including coding errors, outdated software, or even undisclosed "zero-day" vulnerabilities that hackers exploit before developers have a chance to patch them [2]. Cybercriminals frequently employ a wide array of techniques, such as malware injections, system breaches, or data breaches, to take advantage of these software vulnerabilities. “Fig. 1” gives us information about when Attackers target vulnerabilities.

II. MATERIALS AND METHODS

There are diversified types of cybercrime recorded across the globe, and some of the noteworthy examples are credit card fraud, email fraud, deception fraud, fiscal fraud, crypto-virus attacks, cyber spying, identity theft, user interface redressing, and malware [3]. Let's explore how these criminal activities are carried out. This is shown in “Fig. 2”.

A. Phishing

Phishing involves fraudulent attempts to obtain sensitive information such as usernames, passwords, credit card details, and personal information by impersonating a trustworthy entity or organization.

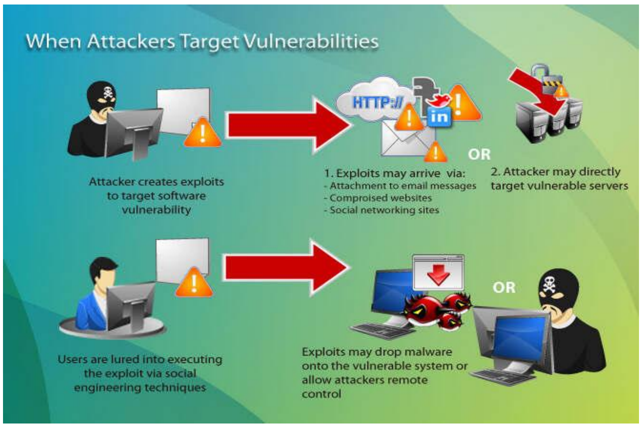

Fig. 1. When Attackers Target Vulnerabilities.

Cybercriminals typically use various tactics, such as email, fake websites, or messages on social media, to trick individuals into believing they are interacting with a legitimate source [4-7]. These messages often contain urgent or enticing language to encourage recipients to click on malicious links, download malicious attachments, or disclose their confidential information. In “Fig. 3.” shows the definition of phishing attack.

Fig. 2. Cyber frauds.

B. Misusing identity

Identity theft, the malicious act of misusing another individual's personal information for fraudulent purposes, has emerged as a significant threat in the digital age. Identity theft is a concept that can be categorized into two primary methods: Application fraud and account takeover. In the context of identity theft, attackers assume the identity of others and misuse it. They achieve this by exploiting applications, particularly those that request permission to access information from social networking sites. This implies that identity theft involves fraudulent activities where individuals or entities manipulate personal information, often obtained through these applications, for unauthorized and malicious purposes.

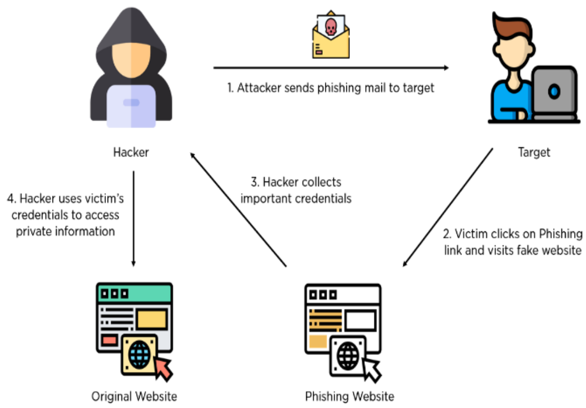

Fig. 3. Definition of phishing attack.

C. Keylogger

An electronic device or compact software application designed to track and record every keypress made by a user on a particular computer's keyboard. We investigate the different actors who may deploy keyloggers, including cybercriminals seeking financial gain, state-sponsored hackers engaged in cyber espionage, and disgruntled insiders seeking to leak sensitive information. Understanding these motivations is crucial for developing effective countermeasures and preventive strategies [8].

III. RESULTS AND DISCUSSION

Remember that cybercriminals are constantly evolving their techniques, so maintaining a cautious and informed mindset is crucial to avoiding phishing attacks [9].

Here are some key steps to help you avoid falling victim to a phishing attack:

- Be Skeptical: Always approach unsolicited emails, messages, or requests for information with caution. Cybercriminals often impersonate trusted organizations or individuals to gain your trust.
- Think Before You Click: Do not click on links or download attachments in emails or messages unless you are absolutely certain of their legitimacy. Glide your cursor over hyperlinks to get a sneak peek of the URL prior to selecting.
- Verify the Sender: Check the sender's email address or phone number carefully. Be cautious if it seems unusual, misspelled, or doesn't match the organization it claims to be from.
- Check the URL: Examine the website's URL before entering any personal or financial information. Ensure it starts with "https://" for a secure connection and that the domain name matches the legitimate organization's domain.
- Beware of Urgent or Threatening Language: Phishing emails often create a sense of urgency or fear to manipulate you into taking quick action. Be cautious of emails claiming your account will be locked or deleted unless you act immediately.
- Double-Check Requests for Personal Information: Legitimate organizations usually won't ask you to provide sensitive information like passwords, Social Security numbers, or credit card details via email or text message.
- Avoid Pop-Up Windows: Be cautious of pop-up windows that ask for personal information or login credentials. These can be used to steal your data.
- Verify with the Source: If you receive an email or message requesting sensitive information or actions (e.g., transferring money), verify the request through a trusted channel. Contact the organization or individual directly using contact information you find independently (not from the email).
- Expand your understanding: Keep up-to-date on the most current phishing tactics and trends. Knowledge is your best defense.
- Use Email Filters: Enable spam filters in your email client to automatically detect and quarantine suspicious emails.
- Report Phishing: If you receive a phishing email, report it to your email provider and the appropriate authorities. Such action can aid in safeguarding others from becoming targets.
- Employee Training: If you're part of an organization, ensure that employees receive training on recognizing and reporting phishing attempts. Many successful attacks originate from phishing emails targeting employees.

A. How to Spot a Counterfeit Phishing Website [10]?

- Confirm the authority of the webpage's web address.
- Check the Padlock symbol.
- Establish the authenticity of the website by verifying its digital certificate.
- Perform a double-click action on the padlock icon located at either the top right or bottom corner of your browser window.

Examples of phishing websites:

- www.gmai1.com
- www.icici6ank.com
- www.bank0findia.com
- www.yah00.com

B. Identity thieves frequently seek the following types of data [11]:

- Passwords
- Details pertaining to bank accounts

- Credit card digits
- Data housed on a user's computer or mobile devices, such as contacts, videos, images, confidential documents, and more

Preventing identity misuse. Understanding the theoretical underpinnings of identity misuse is essential for developing effective preventive measures, fostering responsible digital citizenship, and promoting a safer online environment. As technology continues to advance, continuous research and interdisciplinary collaboration are critical to stay ahead of the evolving landscape of identity misuse.

Preventing identity misuse, the following must be observed:

- Safeguard Personal Information
- Protect your passwords
- Be credit/debit card smart
- Destroy/Shred receipts not required
- Review your records regularly
- Be cautious with Phishing Links
- Monitor your online accounts
- Be careful with public Wi-Fi

C. Common methods used by keyloggers. How to identify keyloggers

A keylogger possesses the capability to utilize virtually any means of communication for transmitting the data it has captured back to the malicious actor [12].

Here are some common methods employed by keyloggers for this purpose:

- FTP Upload: Keyloggers may opt to send data via FTP, a file transfer protocol.
- Email: Some keyloggers discreetly forward the gathered information through email.
- IRC (Internet Relay Chat): Communication can be facilitated through IRC channels.
- HTTP POST: Keyloggers can use HTTP POST requests to send data surreptitiously.
- Connect-back: In certain instances, the attacker initiates a connection to a service running on your compromised device.
- P2P Network: Keyloggers might leverage Peer-to-Peer (P2P) networks like Gnutella or BitTorrent for data transfer.
- Custom Protocols: Custom communication protocols operating over TCP or UDP can be employed to transmit data directly to the attacker.

It's crucial to understand that these services can be configured to operate on non-standard ports. This is done deliberately to evade detection, meaning an IRC server might run on port 50321 instead of the usual 6667, or an FTP server could operate on port 80 rather than the standard 21. This adaptability enables keyloggers to remain hidden and pose a more significant cybersecurity threat.

How to identify keyloggers:

- Employ up-to-date anti-spyware software.
- Execute a scan with your antivirus program, as it may potentially detect keyloggers on your system.
- Prevent yourself from keyloggers, Use Virtual key Board.

IV. CONCLUSIONS

Based on the information provided, here is a general conclusion:

Cybercrime is a widespread issue with various forms, including phishing, identity theft, and keyloggers, posing significant threats in the digital age. Phishing involves deceptive tactics to acquire sensitive information, while identity theft misuses personal data for fraudulent purposes.

Keyloggers, on the other hand, silently record keystrokes, potentially enabling various malicious activities.

To protect yourself from these threats:

Phishing Awareness: Be skeptical of unsolicited messages and requests, think before clicking on links or downloading attachments, verify sender information, and avoid falling for urgent or threatening language.

Counterfeit Phishing Websites: Confirm the legitimacy of website addresses, check for the padlock symbol, verify digital certificates, and double-check the authenticity of websites.

Identity Theft Prevention: Safeguard personal information, protect passwords, monitor accounts, and exercise caution with public Wi-Fi.

Keylogger Detection: Utilize up-to-date anti-spyware and antivirus software to scan for keyloggers. Consider using a virtual keyboard for added security.

Additionally, staying informed about evolving cyber threats, phishing tactics, and trends is crucial in maintaining your digital safety [13]. Employing spam filters, reporting phishing attempts, and providing employee training in organizations are proactive measures to combat cyber threats effectively.

Overall, a cautious and informed approach, combined with cybersecurity best practices, is essential to protect yourself and your data from the ever-evolving landscape of cybercrime.

Precautions for using a public computer safely:

- Avoid storing your login details.
- Never leave the computer unattended when sensitive information is displayed.
- Turn off any password-saving features.
- Clear your digital footprint.
- Refrain from inputting banking information on a public computer.
- Request a computer equipped with the latest antivirus software from the Cyber Cafe Owner.

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Traveling wave solutions for the loaded Korteweg-de Vries equation with variable coefficients using the generalized (G'/G)-expansion method

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Abstract

In this article, a generalized (G'/G) -The expansion method is used to find the exact traveling wave solutions of the loaded Korteweg-de Vries equation with variable coefficients. As a result, hyperbolic, trigonometric, and rational function solutions with parameters are obtained. When these parameters are taking special values, the solitary wave solutions are derived from the hyperbolic function. It is shown that the proposed method is a direct and effective method, which can be applied to many other nonlinear evolution equations in mathematical physics.

Keywords: Nonlinear evolution equations; generalized (G'/G)-expansion method; hyperbolic solution; trigonometric solution; rational solution; loaded Korteweg-de Vries equation.

INTRODUCTION

The seeking exact solutions of nonlinear evolution equations (NLEEs) is important significance in mathematical physics and becomes one of the most exciting and extremely active areas of research investigation. In the past several decades, many effective methods for obtaining exact solutions of NLEEs have been presented, such as the inverse scattering method [1]. In [2] the Burger's equation with variable coefficients is studied.

Alternatively, the (G'/G) - The expansion method [3-6] is also an effective method in finding traveling wave solutions of nonlinear evolution equations. In the work[7], the application of the (G'/G)-expansion method for the Burgers, Burgers–Huxley, and modified Burgers–KdV equations were investigated. (G'/G)-expansion method was used for the integration of the loaded Korteweg-de Vries (KdV) equation, for the loaded modified Korteweg-de Vries (mKdV) ,for the loaded nonlinear Degasperis-Procesi equation, for the loaded Burger's and for the loaded shallow water wave equation in [8-12]. In the work [13] travelling wave solutions for the loaded Burgers equation with variable coefficients using the generalized (G'/G)- expansion method is studied.

In this paper, the solution of the loaded Korteweg-de Vries equation with variable coefficients is studied by usage of (G'/G)-expansion method.

Let us consider the following loaded Korteweg-de Vries equation with variable coefficients

$$u_t + g(t)uu_x + f(t)u_{xxx} + \gamma(t)u(0,t)u_x = 0 \tag{1}$$

where $g(t)$ and $f(t)$ are differentiable functions of t, $x \in R$, $t \ge 0$, $\gamma(t)$ - is the given real continuous function.

2. Description of the generalized $\left(G'/G\right)$ - expansion method.

For a given nonlinear evolution equation with independent variables $X=\left(x,y,z,t\right)$ and dependent variable u, we consider the partial differential equation (PDE)

$$F\left(u,u_x,u_y,u_t,u_z,u_{xt},u_{yt},u_{zt},u_{xx},u_{yy},u_{tt},u_{zz},...\right)=0 \ . \tag{2}$$

The solution of (2) can be expressed by a polynomial in as $\left(G'/G\right)$ follows:

$$u\left(X\right)=\alpha_0\left(X\right)+\sum_{i=1}^{m}\alpha_i\left(X\right)\left(\frac{G'}{G}\right)^i \ , \ \alpha_m\left(X\right)\neq 0 \ , \tag{3}$$

where $G=G(\xi)$ satisfies the following second-order ordinary differential equation

$$G''+\lambda G'+\mu G=0, \tag{4}$$

while $\xi=\xi\left(X\right)$ and $\alpha_i\left(X\right)$ are all functions of X to be determined later. To determine $u\left(X\right)$ explicitly, we take the following footsteps:

Step1. Determine the integer m by balancing the highest-order nonlinear term(s) and the highest-order partial derivatives of $u\left(X\right)$ in (3).

Step2. Substitute (4) along with (1) into (3) and collect all terms in the same order of $\left(G'/G\right)$ together. The left-hand side of (3) is converted into a polynomial in $\left(G'/G\right)$. Then set each coefficient of this polynomial to zero to derive a set of over-determined differential equations for $\alpha_0\left(X\right)$, $\alpha_i\left(X\right)$, and ξ .

Step3. Solve the system of over-determined differential equations obtained in Step 2 for $\alpha_i\left(X\right)$ and ξ by using the software Mathematica.

Step4. Use the results obtained in the above steps to derive a series of fundamental solutions of (3) depending on $\left(G'/G\right)$, since the solutions of (1) have been well known for us, then we can obtain the exact solutions of (3).

3. Applications

In this section, we determine the exact traveling wave solutions of the nonlinear loaded Korteweg-de Vries equation with variable coefficients which are attracted much attention.

By balancing u_{xxx} with uu_x in (2), we get $m=2$. In order to search for explicit solutions, we suppose that (2) has the following formal solution:

$$u\left(x,t\right)=\alpha_2\left(t\right)\left(\frac{G'}{G}\right)^2+\alpha_1\left(t\right)\left(\frac{G'}{G}\right)+\alpha_0\left(t\right), \ \alpha_2\left(t\right)\neq 0 \ , \tag{5}$$

where $G=G\left(\xi\right)$ satisfies (1), $\xi=p\left(t\right)x+q\left(t\right)$ while $p\left(t\right)$ and $q\left(t\right)$ are functions to be determined.

From (1) and (5) we have:

$$u_t=-2\alpha_2\left(t\right)\left[x\frac{dp\left(t\right)}{dt}+\frac{dq\left(t\right)}{dt}\right]\left(\frac{G'}{G}\right)^3+\left\{\frac{d\alpha_2\left(t\right)}{dt}-2\lambda a_2\left(t\right)\left[x\frac{dp\left(t\right)}{dt}+\frac{dq\left(t\right)}{dt}\right]\right.$$

$$-\alpha_1(t)\left[x\frac{dp(t)}{dt}+\frac{dq(t)}{dt}\right]\Bigg\}\left(\frac{G'}{G}\right)^2+\left\{\frac{d\alpha_1(t)}{dt}-2\mu\alpha_2(t)\left[x\frac{dp(t)}{dt}+\frac{dq(t)}{dt}\right]-\right.$$

$$-\lambda\alpha_1(t)\left[x\frac{dp(t)}{dt}+\frac{dq(t)}{dt}\right]\Bigg\}\left(\frac{G'}{G}\right)+\frac{d\alpha_0(t)}{dt}-\mu\alpha_1(t)\left[x\frac{dp(t)}{dt}+\frac{dq(t)}{dt}\right] \tag{6}$$

$$u_x=-2\alpha_2(t)p(t)\left(\frac{G'}{G}\right)^3+\left[-2\lambda a_2(t)p(t)-\alpha_1(t)p(t)\right]\left(\frac{G'}{G}\right)^2+$$

$$+\left[-2\mu\alpha_2(t)p(t)-\lambda\alpha_1(t)p(t)\right]\left(\frac{G'}{G}\right)-\mu\alpha_1(t)p(t) \tag{7}$$

$$uu_x=-2\alpha_2^2(t)p(t)\left(\frac{G'}{G}\right)^5+\left[-2\lambda\alpha_2{}^2(t)p(t)-3\alpha_1(t)\alpha_2(t)p(t)\right]\left(\frac{G'}{G}\right)^4+$$

$$\left\{-2\mu\alpha_2{}^2(t)p(t)-3\lambda\alpha_1(t)\alpha_2(t)p(t)-\alpha_1{}^2(t)p(t)-\right.$$

$$\left.-2\alpha_0(t)\alpha_2(t)p(t)\right\}\left(\frac{G'}{G}\right)^3+\left\{-3\mu\alpha_1(t)\alpha_2(t)p(t)-\lambda\alpha_1{}^2(t)p(t)-\right.$$

$$\left.-2\lambda\alpha_0(t)\alpha_2(t)p(t)-\alpha_0(t)\alpha_1(t)p(t)\right\}\left(\frac{G'}{G}\right)^2+\left\{-\mu\alpha_1{}^2(t)p(t)-\right.$$

$$\left.-2\mu\alpha_0(t)\alpha_2(t)p(t)-\lambda\alpha_0(t)\alpha_1(t)p(t)\right\}\left(\frac{G'}{G}\right)-\mu\alpha_0(t)\alpha_1(t)p(t) \tag{8}$$

$$u_{xxx}=-24\alpha_2(t)p^3(t)\left(\frac{G'}{G}\right)^5+\left[-54\lambda\alpha_2(t)p^3(t)-6\alpha_1(t)p^3(t)\right]\left(\frac{G'}{G}\right)^4+$$

$$+\left[-40\mu\alpha_2(t)p^3(t)-38\lambda^2\alpha_2(t)p^3(t)-12\lambda^2\alpha_1(t)p^3(t)\right]\left(\frac{G'}{G}\right)^3+$$

$$+\left[-52\mu\lambda\alpha_2(t)p^3(t)-8\mu\alpha_1(t)p^3(t)-8\lambda^3\alpha_2(t)p^3(t)-7\lambda^2\alpha_1(t)p^3(t)\right]\left(\frac{G'}{G}\right)^2$$

$$+\left[-16\mu^2\alpha_2(t)p^3(t)-14\mu\lambda^2\alpha_2(t)p^3(t)-8\mu\lambda\alpha_1(t)p^3(t)-\lambda^3\alpha_1(t)p^3(t)\right]\left(\frac{G'}{G}\right)$$

$$-6\mu^2\lambda\alpha_2(t)p^3(t)-2\mu^2\alpha_1(t)p^3(t)-\mu\lambda^2\alpha_1(t)p^3(t). \tag{9}$$

Substituting (6) – (9) into (1) and collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together, the left-hand side of (1) is converted into a polynomial in $x^j\left(\frac{G'}{G}\right)^i$, $(j=0,1,i=0,1,2,3,4,5)$. Setting each coefficient of this polynomial to zero, we get the following set of over-determined differential equations:

$$x^0\left(\frac{G'}{G}\right)^5 :-24f(t)\alpha_2(t)p^3(t)-2g(t)\alpha_2^2(t)p(t)=0\,,$$

$$x^0\left(\frac{G'}{G}\right)^4 : -54\lambda f(t)\alpha_2(t)p^3(t)-6f(t)\alpha_1(t)p^3(t)-$$

$$-2\lambda g(t)\alpha_2{}^2(t)p(t)-3g(t)\alpha_1(t)\alpha_2(t)p(t)=0,$$

$$x^0\left(\frac{G'}{G}\right)^3 :-2\alpha_2(t)\frac{dq(t)}{dt}-40\mu f(t)\alpha_2(t)p^3(t)-38\lambda^2 f(t)\alpha_2(t)p^3(t)-$$

$$-12\lambda f(t)\alpha_1(t)p^3(t)-2\mu g(t)\alpha_2{}^2(t)p(t)-3\lambda g(t)\alpha_1(t)\alpha_2(t)p(t)-$$

$$-g(t)\alpha_1^2(t)p(t)-2g(t)\alpha_0(t)\alpha_2(t)p(t)-2\gamma(t)u(0,t)\alpha_2(t)p(t)=0,$$

$$x^0\left(\frac{G'}{G}\right)^2 : \frac{d\alpha_2(t)}{dt}-2\lambda\alpha_2(t)\frac{dq(t)}{dt}-\alpha_1(t)\frac{dq(t)}{dt}-52\mu\lambda f(t)\alpha_2(t)p^3(t)-$$

$$-8\mu f(t)\alpha_1(t)p^3(t)-8\lambda^3 f(t)\alpha_2(t)p^3(t)-7\lambda^2 f(t)\alpha_1(t)p^3(t)-$$

$$-3\mu g(t)\alpha_1(t)\alpha_2(t)p(t)-\lambda g(t)\alpha_1^2(t)p(t)-2\lambda g(t)\alpha_0(t)\alpha_2(t)p(t)-$$

$$-g(t)\alpha_0(t)\alpha_1(t)p(t)-\left[2\lambda\alpha_2(t)p(t)+\alpha_1(t)p(t)\right]\gamma(t)u(0,t)=0,$$

$$x^0\left(\frac{G'}{G}\right)^1 : \frac{d\alpha_1(t)}{dt}-2\mu\alpha_2(t)\frac{dq(t)}{dt}-\lambda\alpha_1(t)\frac{dq(t)}{dt}-$$

$$-16\mu^2 f(t)\alpha_2(t)p^3(t)-14\lambda^2\mu f(t)\alpha_2(t)p^3(t)-8\mu\lambda f(t)\alpha_1(t)p^3(t)-$$

$$-\lambda^3 f(t)\alpha_1(t)p^3(t)-\mu g(t)\alpha_1^2(t)p(t)-2\mu g(t)\alpha_0(t)\alpha_2(t)p(t)-$$

$$-\lambda g(t)\alpha_0(t)\alpha_1(t)p(t)-\left[2\mu\alpha_2(t)p(t)+\lambda\alpha_1(t)p(t)\right]\gamma(t)u(0,t)=0,$$

$$x^0\left(\frac{G'}{G}\right)^0 : \frac{d\alpha_0(t)}{dt}-\mu\alpha_1(t)\frac{dq(t)}{dt}-6\mu^2\lambda f(t)\alpha_2(t)-$$

$$-6\mu^2\lambda f(t)\alpha_2(t)p^3(t)-2\mu^2 f(t)\alpha_1(t)p^3(t)-\mu\lambda^2 f(t)\alpha_1(t)p^3(t)-$$

$$-\mu g(t)\alpha_0(t)\alpha_1(t)p(t)-\mu\alpha_1(t)p(t)\gamma(t)u(0,t)=0,$$

$$x\left(\frac{G'}{G}\right)^3 :-2\alpha_2(t)\frac{dp(t)}{dt}=0,$$

$$x\left(\frac{G'}{G}\right)^2 :-2\lambda\alpha_2(t)\frac{dp(t)}{dt}-\alpha_1(t)\frac{dp(t)}{dt}=0,$$

$$x\left(\frac{G'}{G}\right)^1 :-2\mu\alpha_2(t)\frac{dp(t)}{dt}-\lambda\alpha_1(t)\frac{dp(t)}{dt}=0,$$

$$x\left(\frac{G'}{G}\right)^0 :-\mu\alpha_1(t)\frac{dp(t)}{dt}=0\,. \tag{10}$$

Solving the system (10), we have

$$p(t)=P,\alpha_1(t)=R,\alpha_0(t)=Q,\alpha_2(t)=\frac{R}{\lambda},g(t)=\frac{-12\lambda P^2 f(t)}{R},$$

$$q(t)=-P^3\left(\lambda^2+8\mu-\frac{12\lambda Q}{R}\right)\int_0^t f(\tau)d\tau-P\int_0^t \gamma(\tau)u(0,\tau)d\tau\ , \quad (11)$$

where P, Q and R are arbitrary constants.

Substituting (11) into (5), we have

$$u(x,t)=\frac{R}{\lambda}\left(\frac{G'}{G}\right)^2+R\left(\frac{G'}{G}\right)+Q, \quad (12)$$

where

$$\xi=Px-P^2\left(\lambda^2+8\mu-\frac{12\lambda Q}{R}\right)\int_0^t f(\tau)d(\tau)d\tau-P\int_0^t \gamma(\tau)u(0,\tau)d\tau.$$

From the general solution of (1) we can find the ratio (G'/G). Consequently, we have the following three types of exact solutions (1):

Case 1. When $\lambda^2-4\mu>0$, we obtain the hyperbolic function solution in the form

$$u(x,t)=\frac{R}{\lambda}\left[-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\cdot\frac{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi}{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi}\right]^2+$$

$$+R\left[-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\cdot\frac{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi}{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\right)\xi}\right]+Q \quad (13)$$

Case 2. When $\lambda^2-4\mu<0$, we obtain the trigonometric function solution in the form

$$u(x,t)=\frac{R}{\lambda}\left[-\frac{\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\cdot\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi+C_2\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi}\right]^2+$$

$$+R\left[-\frac{\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\cdot\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi+C_2\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\right)\xi}\right]+Q \quad (14)$$

Case 3. When $\lambda^2 - 4\mu = 0$, we obtain the rational solution in the form

$$u(x,t) = \frac{R}{\lambda}\left[\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right]^2 + R\left[\frac{C_2}{C_1 + C_2\xi} - \frac{\lambda}{2}\right] + Q \tag{15}$$

Finally, we note that, if $\mu = 0, \lambda > 0, C_2 = 0$, and $C_1 \neq 0$ then we deduce from (13) that

$$u(x,t) = \frac{R\lambda}{4}\operatorname{csch}^2\left(\frac{\lambda\xi}{2}\right) + Q \tag{16}$$

while if $\mu = 0, \lambda > 0, C_2 \neq 0$ and $C_2^2 > C_1^2$ we get

$$u(x,t) = \frac{-R\lambda}{4}\operatorname{sech}^2\left(\xi_0 + \frac{\xi\lambda}{2}\right) + Q \tag{17}$$

where

$$\xi_0 = \tanh^{-1}\left(\frac{C_1}{C_2}\right).$$

Note that (13) represents the solitary wave solutions of the nonlinear loaded Korteweg-de Vries equation with variable coefficients (1).

Conclusion

In this article, the generalized (G'/G) -expansion method is used to obtain more general exact solutions for NLEEs. By using the proposed method we have successfully obtained exact solutions with parameters of the nonlinear loaded Korteweg-de Vries equation with variable coefficients. When these parameters are taking special values, the solitary wave solutions are derived from the hyperbolic solutions.

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