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CONSTRUCTIVE MATHEMATICAL ANALYSIS

3RD INTERNATIONAL CONFERENCE: CONSTRUCTIVE MATHEMATICAL ANALYSIS

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PROCEEDINGS BOOK

EDITORS

PROF. DR. TUNCER ACAR &
PROF. DR. IOAN RAŞA

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Message from Editors

Dear Participants,

It is our pleasure to chair the 3rd International Conference on Constructive Mathematical Analysis, an activity of the journal Constructive Mathematical Analysis. The first two events in this series were held as workshops; following requests from international researchers working in constructive mathematical analysis, the series was expanded into a full conference. Our 2025 meeting was supported by the Scientific Research Projects Coordinatorship of Selçuk University, the Scientific and Technological Research Council of Türkiye, the Republic of Turkey Ministry of Youth and Sports, and BEYSU.

The purpose of the conference is to bring together established experts and early-career analysts from around the world who work in various areas of mathematics and its applications, so they may present their research, exchange ideas, discuss challenging problems, foster future collaborations, and interact with one another. The main objective is to promote and encourage academic exchange on recent research across all fields of analysis and function theory. The program featured keynote lectures and contributed short talks that presented new results and highlighted future challenges.

We would like to thank the invited speakers—**Prof. Francesco Altomare, Prof. Erdal Karapınar, Prof. Harun Karslı, Prof. Ioan Raşa, Prof. Gianluca Vinti, Prof. Xiaoming Wang,** and **Prof. Ferenc Weisz**—for their valuable contributions to the conference.

This booklet contains the titles and proceedings of selected papers at the *The Third International Conference: Constructive Mathematical Analysis* and is available on the conference website.

Thanks.

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Contents

Neimark-Sacker bifurcation and control of chaos in prey-predator system subject to Harvesting effect on prey

Özlem Ak Gümüş

Abstract. This article aims to examine the dynamics of the positive coexistence fixed point of the discrete-time model given in [3] by taking the conversion efficiency of prey into predator as the bifurcation point. We show that the system is exposed to Neimark-Sacker bifurcation. We also control the Neimark-Sacker bifurcation with a feedback control strategy. Additionally, computer simulations are presented to verify the theoretical results obtained with the stability, bifurcation and chaos control strategy.

2020 Mathematics Subject Classifications: 39A30, 37G35, 39A33, 91B76, 92B05, 92D25

Keywords: Predator-Prey System, Neimark-Sacker Bifurcation, Stability, Bifurcation Theory, Harvesting Effect, Chaos Management Strategies

1. Introduction

Numerous mathematical models have been developed to study populations mathematically. These models have become the main tool for analyzing interactions between different populations, primarily predator-prey models which are created by two pioneers, Lotka-Volterra [29, 40]. To reach more realistic conclusions, other factors affecting the population need to be known. There is broad interest to the use of bioeconomic models in order to gain insight about the optimal management of renewable resources [8, 9]. For instance, exploitation of biological resources is widely practiced in fisheries management. Therefore, it is critical to investigate predator-prey models with harvest effects by incorporating population harvest into dynamic systems. Harvesting has an important function in population control [10, 27]. Recently, there has been a great deal of research presenting the effects of harvest in predator-prey models [7, 10, 15, 22, 27, 30, 31, 37, 42, 43, 44].

Population behavior can be analyzed using models based on difference or differential equations. [35, 41]. The literature contains a sufficient number of continuous-time predator-prey models that describe the intricate relationship between species [6, 33, 35]. However, in ecology, populations evolve in distinct time steps since many species' generations do not overlap. Difference equations can be appropriate for describing such population dynamics. Using discrete-time models of predator-prey interaction is another option for understanding the behavior of these populations. Analyses on discretization of continuous predator-prey models are examined by researchers [14, 16, 32, 33, 35]. The rich dynamic behaviors exhibited by such models have led to the discrete version becoming an active area of research [5, 34]. At the same time, it is of biological importance that discrete-time models produce unpredictable dynamical behavior. For this reason, bifurcation theory is widely used in mathematical studies of dynamical systems to predict population behavior [17, 26]. Numerous studies have examined bifurcation behavior in discrete-time models of population dynamics (e.g. see [1, 2, 3, 4, 6, 11, 12, 15, 18, 19, 20, 21, 23, 27, 45] and references therein). However, there is still a need for more refined analyses that explore how harvesting efforts influence the emergence of complex dynamics such as bifurcations and chaos, particularly when control mechanisms are incorporated into the system. This is an effective technique to emphasize the potential that the principles governing natural systems are very simple and discoverable. Also, in order to avoid this complex behavior, some chaos control strategies are presented [11, 12, 13, 25, 36, 38, 39, 45]. In this respect, these strategies delay or completely prevent chaotic behavior.

In [6], it is considered Lotka-Volterra model without harvesting effect. The modified Lotka-Volterra model exposed to the harvesting effect in this prey population is presented as follows:

$$\dot{x} = ax(1-x) - cxy - hx \quad (1)$$
$$\dot{y} = bxy - ey.$$

where x_t and y_t represent the population sizes of prey and predator, respectively. The term cx corresponds to the predator's functional response, indicating the number of prey consumed per unit area and time by a single predator. The parameter b denotes the efficiency with which consumed prey are converted into predator biomass, and bxy characterizes the predator's numerical response. The parameter e accounts for predator mortality, while h denotes harvesting effect. Moreover, all parameters a, b, c, e, h as well as the initial values x_0, y_0 are assumed to be positive real numbers.

In this study, we reconsider the model introduced in [3], which results from applying the forward Euler discretization to the continuous system described in (1).

$$x_{n+1} = x_n + \delta[ax_n(1-x_n) - cx_ny_n - hx_n] \quad (2)$$
$$y_{n+1} = y_n + \delta[bx_ny_n - ey_n]$$

where δ is a step size. The authors in [3] conduct a detailed investigation into the fixed point of the model (2), their stability criteria, the set of multiple bifurcations, and also codimension-1 and codimension-2 bifurcations depending on the step size δ. Our aim in this study is to examine in detail the existence of Neimark-Sacker bifurcation (codim-1) of the system depending on the efficiency of the conversion of prey into predators b, and to control the chaos behavior of the model. Our analysis based on the bifurcation parameter b provides a new perspective by relating biological efficiency to the transformation of prey into predator. This approach not only reflects ecological interpretations, but also provides a deeper understanding of how changes in predation efficiency affect the stability of the system and the emergence of Neimark-Sacker bifurcations. Therefore, this study complements and extends previous results by revealing a distinct bifurcation structure under biologically relevant conditions. Moreover, by incorporating numerical simulations supported by interactive mathematical tools [24] and Mathematica program, we not only confirm the analytical results but also visualize the transition from order to chaos as the bifurcation parameter varies. This integration of theoretical and computational techniques strengthens the biological applicability of the model and enables the identification of critical thresholds for stability loss. In particular, we show how appropriate state feedback control can effectively suppress chaotic dynamics and stabilize the system near a positive coexistence equilibrium. These findings highlight the model's potential in managing predator-prey interactions under varying ecological pressures.

It is clear that the points satisfying the following equations are fixed points:

$$\overline{x} = \overline{x} + \delta[a\overline{x}(1-\overline{x}) - c\overline{x}\,\overline{y} - h\overline{x}] \quad (3)$$
$$\overline{y} = \overline{y} + \delta[b\overline{x}\,\overline{y} - e\overline{y}].$$

The Jacobian matrix of a system (2) evaluated at any fixed point $(\overline{x}, \overline{y})$ is given by

$$J_{(\overline{x},\overline{y})} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad (4)$$

and the characteristic polynomial of matrix $J_{(\overline{x},\overline{y})}$ can be given with

$$F(\lambda) = \lambda^2 - trJ_{(\overline{x},\overline{y})}\lambda + \det J_{(\overline{x},\overline{y})}. \quad (5)$$

Assume that λ_1 and λ_2 be two roots of $F(\lambda) = 0$. If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then the fixed point $(\overline{x}, \overline{y})$ is locally asymtotically stable (sink), and attracts nearby trajectories. If $|\lambda_1| > 1$ and $|\lambda_2| > 1$, $(\overline{x}, \overline{y})$ is always unstable (source), and repels nearby trajectories. And, the fixed point $(\overline{x}, \overline{y})$ exhibits saddle point behavior, with stable and unstable manifolds if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or vice versa). If either $|\lambda_1| = 1$ or $|\lambda_2| = 1$, the fixed point $(\overline{x}, \overline{y})$ is non-hyperbolic point, and linear stability analysis is insufficient to determine its behavior. (To have knowledge about the dynamics of a positive coexistence fixed point depending on the characteristic equation of a given matrix, see Lemma 4 in [19] and Lemma 2.2 in [28])

The existence of fixed points of the system (2) has been investigated in [3]: The system (3) has a trivial fixed point $E_0 = (0, 0)$ and an exclusion fixed point $E_1 = (\frac{a-h}{a}, 0)$ when $a > h$.

If $ab > ae + bh$, then the system (2) has only one positive coexistence fixed point $E_2 = (\frac{e}{b}, \frac{ab-ae-bh)}{bc})$. Also, the local dynamics of the fixed points E_0 and E_1 are given by obtaining the Jacobian matrix evaluating E_0 and E_1.

To analyze the dynamics associated with the unique positive coexistence fixed point E_2, we can write the Jacobian matrix at the fixed point E_2 as

$$J_{E_2} = \begin{pmatrix} 1 - \frac{ae\delta}{b} & -\frac{ce\delta}{b} \\ \frac{[a(b-e)-bh]\delta}{c} & 1 \end{pmatrix} \tag{6}$$

and its characteristic polynomial is

$$F(\lambda) = \lambda^2 + [-2 + \frac{ae\delta}{b}]\lambda + [\frac{-ae\delta(1+e\delta) + b(1 + ae\delta^2 - eh\delta^2)}{b}].$$

This study is composed of three parts. In Section II, we present our main results: First, we analyze the local asymptotic stability of the positive coexistence fixed point ot the system (2) according to the parameter b in $\mathbb{R}_+^2$. Then, we show that a Neimark-Sacker bifurcation occurs in the system (2). Then, we apply a chaos control method to stabilize the dynamics of the system (2). We present supporting numerical simulations to verify the theoretical findings. Finally, the paper concludes with a summary of the main results and conclusions.

2. Main results

In this section, we evaluate the local dynamics of the unique positive coexistence fixed point E_2 of system (2) located in the closed first quadrant $\mathbb{R}_+^2$: The stability conditions of the positive coexistence fixed point of system (2), the existence and direction of the Neimark-Sacker bifurcation are analyzed. If system (2) satisfies the conditions of eigenvalue placement, transversality and non-resonance, a Neimark-Sacker bifurcation emerges at a certain bifurcation point, as discussed in [17, 26, 38, 41]. A feedback control strategy is used to manage the Neimark-Sacker bifurcation. Furthermore, numerical simulations are provided to verify the theoretical findings. Our focus is on the parameter b, which represents the efficiency of prey conversion into predators.

Positivity Conditions for System (2). In order to preserve the biological feasibility of system (2), it is essential that the prey and predator populations x_n and y_n remain strictly positive for all $n \in \mathbb{N}$. We now derive sufficient conditions to ensure the positivity of the solution sequences $\{x_n\}$ and $\{y_n\}$.

Sufficient Conditions for Positivity:

(P1) **Initial positivity:**

$$x_0 > 0, \quad y_0 > 0.$$

(P2) **Influence of the conversion rate b:** To ensure predator positivity in the y-update, the conversion efficiency must satisfy

$$b > \frac{e}{x_0}.$$

This guarantees that predator reproduction exceeds natural mortality at initial time.

(P3) **Lower bound on initial prey population:**

$$x_0 > \frac{e}{b} - \frac{1}{\delta b}.$$

(P4) **Bounded effect of predator and mortality on prey:**

$$1 + \delta\,(a(1-x_0) - cy_0 - h) > 0.$$

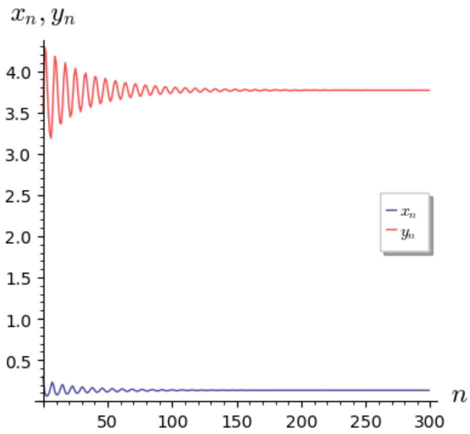
When conditions (P1)–(P4) are satisfied, the recursive sequences $\{x_n\}$ and $\{y_n\}$ remain in the positive orthant $\mathbb{R}_+^2$, ensuring that the population levels stay biologically meaningful throughout the iteration process. System (2) models a biologically motivated prey-predator interaction under discrete-time dynamics. The positivity of the solution sequences $\{x_n\}$ and $\{y_n\}$ is crucial to ensure the ecological relevance of the model. The sufficient conditions (P1)–(P4) collectively provide a practical guideline for selecting parameter values and initial conditions to guarantee that the model remains well-defined in the positive orthant. In particular, the condition (P2) highlights the importance of the conversion efficiency parameter b in sustaining predator populations, while (P4) ensures that the prey population does not collapse due to over-predation or excessive mortality. These insights contribute to the reliable implementation of the model in simulations and further analytical studies.

2.1. **Stability of the Coexisting Positive Fixed Point.** By applying Lemma 2.2 in [28], the topological nature of the positive coexistence fixed point E_2 can be characterized as follows:

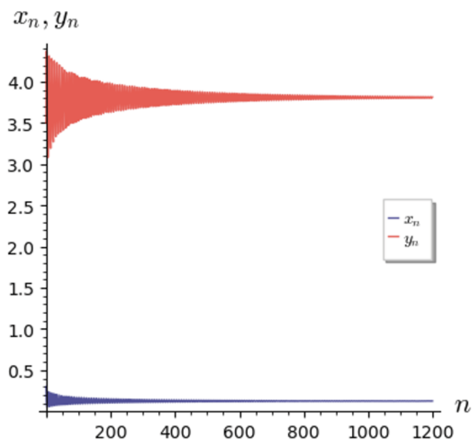
Lemma 2.1. *Assume that $ab > ae + bh$. For the positive coexistence fixed point E_2, the following situations hold:*

- *(i) E_2 is a sink if : $\frac{2ae\delta+ae^2\delta^2}{4+ae\delta^2-eh\delta^2} < b < \frac{a(1+e\delta)}{(a-h)\delta}$,*
- *(ii) E_2 is a source if: $b > \max\left\{ \frac{2ae\delta+ae^2\delta^2}{4+ae\delta^2-eh\delta^2}, \frac{a(1+e\delta)}{(a-h)\delta} \right\}$,*
- *(iii) E_2 is a saddle if: $b < \frac{2ae\delta+ae^2\delta^2}{4+ae\delta^2-eh\delta^2}$,*
- *(iv) E_2 is non-hyperbolic if: $b = \frac{a+ae\delta}{(a-h)\delta}$.*

Example 2.2. *With respect to the selected parameter values $a = 5.5$, $b = 1.5$, $e = 0.2$, $c = 1$, $h = 1$, $\delta = 0.9$ and the initial conditions $(x_0, y_0) = (0.3, 3.5)$, E_2 exhibits locally asymptotically stable. Moreover, at $b = 1.60247$, the fixed point E_2 of the model (2) becomes non-hyperbolic. The parameter values which are $e = 0.2$, $c = 1$, $h = 1$, $\delta = 0.9$ have been taken from* [19]*. To validate the theoretical findings, the trajectories corresponding to model (2) are illustrated in Figure 1 .*



(a) Sink behavior for $b = 1.50$



(b) Sink behavior for $b = 1.59$

FIGURE 1. Trajectories of the predator-prey system (2) with the parameter values $a = 5.5$, $e = 0.2$, $c = 1$, $h = 1$, $\delta = 0.9$ and initial condition $(x_0, y_0) = (0.3, 3.5)$ for two different b values.

2.2. **Neimark-Sacker Bifurcation.** To investigate the bifurcation behavior, we address the impact of the parameter b as the bifurcation parameter and define the parameter space for bifurcation occurrence:

$$(7) \qquad NSB_{E_2} = \left\{ \begin{array}{c} a, b, c, h, \ e, \delta \ \in \ \mathbb{R}^+ : b > \frac{1}{2}\left(\frac{ae}{a-h} + \frac{a}{a-h}\sqrt{e(a-h+e)}\right) \\ \text{and } b = b_1 = \frac{a+ae\delta}{(a-h)\delta} \end{array} \right\}.$$

When $b = b_1$, the Jacobian matrix J_{E_2} has a pair of complex conjugate eigenvalues with modulus equal to one. These eigenvalues derived from equation (5) are given by:

$$(8) \qquad \lambda, \overline{\lambda} = 1 - \frac{ae\delta}{2b} \mp i\delta \frac{\sqrt{e[4ab^2 - ae(a+4b) - 4b^2h]}}{2b}\Big|_{b=b_1}$$

such that

$$|\lambda| = \left|\overline{\lambda}\right| = 1.$$

For values of b in the Neimark–Sacker bifurcation set NSB_{E_2}, the following transversality condition holds:

$$\frac{\partial\,|\lambda_i(b)|}{\partial b}\,|_{b=b_1}\neq 0\ ,\ i=1,2. \tag{9}$$

Additionally, if

$$trJ_{E_2}\,|_{b=b_1}\neq 0,-1, \tag{10}$$

then the condition is satisfied

$$\lambda^k(b_1)\neq 1\ ,\ k=1,2,3,4. \tag{11}$$

Here, the trace of the Jacobian matrix J_{E_2} evaluated at $b=b_1$ is given by

$$trJ_{E_2}=\frac{2+e\delta(2-a\delta+h\delta)}{1+e\delta}.$$

Moreover, the eigenvalues of J_{E_2} at $b=b_1$ are calculated as

$$\lambda,\overline{\lambda}=1-\frac{a\delta^2e+he\delta^2}{2+2e\delta}\mp\frac{i\delta\sqrt{e(a-h)[4-e\delta(-4+a\delta-h\delta)]}}{2+2e\delta}.$$

Let $q,p\in\mathbb{C}^2$ denote eigenvectors corresponding to the eigenvalues λ and $\overline{\lambda}$ of the matrix $J(NSB_{E2})$ and its transpose, respectively. These eigenvectors are given by

$$q\sim\left(\frac{-1+H-i\sqrt{-H^2+2H-4KT-1}}{2T},1\right)^T \tag{12}$$

and

$$p\sim\left(\frac{-1+H+i\sqrt{-H^2+2H-4KT-1}}{2K},1\right) \tag{13}$$

where $H=\frac{1+e\delta(1-a\delta+h\delta)}{1+e\delta}$, $K=-\frac{ce(a-h)\delta^2}{a+ae\delta}$ and $T=\frac{a}{c}$.

By using the scalar product in $\mathbb{C}^2$, defined as $<p,q>=\overline{p_1}q_1+\overline{p_2}q_2$, we normalize p according to q, obtaining:

$$p\sim\left(\begin{array}{c}-\frac{Ti}{\sqrt{(-1+H)^2-4KT}}+\frac{-1+H-i\sqrt{(-1+H)^2-4KT}}{2K}\\ \frac{1}{1+\frac{(-1+H+i\sqrt{-H^2+2H-4KT-1})^2}{4KT}}\end{array}\right) \tag{14}$$

where $<p,q>=1$.

To translate the fixed point E_2 of system (2) to the origin $(0,0)$, we define the coordinate transformation

$$u_n=x_n-\frac{e}{b},\ v_n=y_n-\frac{ab-ae-bh}{bc}. \tag{15}$$

With this change of variables, the system becomes the following discrete map:

$$\begin{pmatrix}u\\ v\end{pmatrix}\to J_{E_2}\begin{pmatrix}u\\ v\end{pmatrix}+\begin{pmatrix}F_1(u,v)\\ F_2(u,v)\end{pmatrix}, \tag{16}$$

where the nonlinear components are given by

$$F_1(u,v)=-c\delta uv-a\delta u^2+O(\|U\|^3)$$

$$F_2(u,v)=b\delta uv+O(\|U\|^3).$$

such that $U_n=(u_n,v_n)^T$. Additionally, the system (2) can be reformulated in the following form

$$\begin{pmatrix}u_{n+1}\\ v_{n+1}\end{pmatrix}\to J_{E_2}\,(U_n)+\frac{1}{2}B(u_n,u_n)+\frac{1}{6}C(u_n,u_n,u_n)+O(\|U_n\|^4), \tag{17}$$

where $u;v;w\in\mathbb{R}^2$, and B and C are multilinear vector functions defined as

$$B(u,v)=\begin{pmatrix}B_1(u,v)\\ B_2(u,v)\end{pmatrix}$$

and

$$C(u,v) = \begin{pmatrix} C_1(u,v,w) \\ C_2(u,v,w) \end{pmatrix}.$$

The components of these vector functions are explicitly given by

$$B_1(u,v) = \sum_{j,k=1}^{2} \frac{\partial^2 F_1}{\partial \xi_j \partial \xi_k} |_{\xi=0}\, u_j v_k = -2a\delta u_1 v_1 - c\delta(u_2 v_1 + u_1 v_2),$$

$$B_2(u,v) = \sum_{j,k=1}^{2} \frac{\partial^2 F_2}{\partial \xi_j \partial \xi_k} |_{\xi=0}\, u_j v_k = b\delta(u_2 v_1 + u_1 v_2),$$

$$C_1(u,v,w) = \sum_{j,k=1}^{2} \frac{\partial^3 F_1}{\partial \xi_j \partial \xi_k \xi_l} |_{\xi=0}\, u_j v_k w_l = 0,$$

$$C_2(u,v,w) = \sum_{j,k=1}^{2} \frac{\partial^3 F_2}{\partial \xi_j \partial \xi_k \xi_l} |_{\xi=0}\, u_j v_k w_l = 0.$$

For any vector $U \in \mathbb{R}^2$, it can be uniquely expressed in the form

(18) $$U = zq + \overline{zq}$$

where $z \in \mathbb{C}$ is defined by $z =< p, U >$, and $\overline{z}$ is its complex conjugate. For sufficiently small $|b|$, the system (2) can be reduced to a normal form as,

(19) $$z \to \lambda(b)z + g(z, \overline{z}, b),$$

where $\lambda(b) = (1 + \omega(b))e^{i \arctan(b)}$, satisfying $\omega(b_1) = 0$. Here, $g(z, \overline{z}, b)$ is a smooth complex-valued function, and its Taylor expansion in terms of z and $\overline{z}$ is given by:

(20) $$g(z, \overline{z}, b) = \sum_{k+l\geq 2} \frac{1}{k!l!} g_{kl}(b) z^k \overline{z}^l,$$

where the coefficients are computed as

$$\begin{aligned} g_{20}(b_1) &=< p, B(q,q) > \\ g_{11}(b_1) &=< p, B(q,\overline{q}) > \\ g_{02}(b_1) &=< p, B(\overline{q},\overline{q}) > \\ g_{21}(b_1) &=< p, C(q,q,\overline{q}) > . \end{aligned}$$

The coefficient $\varphi(b_1)$, which dictates the direction in which the invariant closed curve emerges in the system (16), is given by the following expression:

(21)
$$\varphi(b_1) = Re(\frac{e^{-iar\tan(b_1)}}{2} g_{21}) - Re(\frac{(1-2e^{iar\tan(b_1)})e^{-2iar\tan(b_1)}}{2(1-e^{iar\tan(b_1)})} g_{20} g_{11}) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2$$

where $e^{iar\tan(b_1)} = \lambda(b_1)$. (For more information on argument and theorem, see [26, 17, 38, 41])

As a result, we can establish the following theorem regarding the Neimark–Sacker bifurcation:

Theorem 2.3. *Assume that condition (10) holds and that $\varphi(b_1) \neq 0$. If the bifurcation parameter b varies in a sufficiently small neighborhood of NSB_{E2}, then the system (2) undergoes a Neimark–Sacker bifurcation at the unique fixed point E_2. Moreover, if $\varphi(b_1) < 0$ ($\varphi(b_1) > 0$), then a unique attracting (repelling) invariant closed curve bifurcates from E_2.*

2.3. **Chaos control.** In recent years, the stabilization of chaotic dynamics in ecological models has gained increasing attention, particularly in systems that exhibit bifurcations such as the Neimark–Sacker type. The unpredictable nature of such systems can lead to sudden and persistent oscillations in population levels, which may not be biologically sustainable or desirable in resource management contexts. To mitigate this, various chaos control strategies have been developed in the literature (see [13, 25, 36, 39, 45]). Among these, the state feedback approach stands out due to its simplicity and adaptability, allowing real-time corrections based on deviations from the desired equilibrium. In the context of the predator-prey model studied here, we apply a linear state feedback control strategy aimed at stabilizing the system near the coexistence equilibrium that becomes unstable after a Neimark–Sacker bifurcation. By properly tuning the control gains, the chaotic attractor can be suppressed, restoring the system to regular and predictable behavior. This section presents both the analytical formulation of the controlled system and the associated stability region in the feedback gain parameter space, followed by simulations that demonstrate the success of this control mechanism. We will try to regulate chaotic behavior using a chaos control strategy based on state feedback control [13, 25]. To stabilize the system (2) which undergoes a Neimark–Sacker bifurcation at the fixed point $(\overline{x}, \overline{y})$, we consider the controlled system:

$$\begin{aligned} x_{n+1} &= x_n + \delta[ax_n(1 - x_n) - cx_ny_n - hx_n] + K \\ y_{n+1} &= y_n + \delta[(bx_ny_n - ey_n] \end{aligned} \tag{22}$$

where the control input is defined by $K = -k_1(x_n - \overline{x}) - k_2(y_n - \overline{y})$, with k_1 and k_2 representing the feedback gains. The Jacobian matrix $J(\overline{x}, \overline{y})$ of the controlled system (22) is given by:

$$J(\overline{x}, \overline{y}) = \begin{bmatrix} 1 - \frac{ae\delta}{b} - k_1 & -\frac{ce\delta}{b} - k_2 \\ \frac{[a(b-e)-bh]\delta}{c} & 1 \end{bmatrix}.$$

The characeristic equation of the Jacobian matrix $J(\overline{x}, \overline{y})$ is expressed as

$$\mu^2 + [-2 + \frac{ae\delta}{b} + k_1]\mu + \frac{c[b + be(a-h)\delta^2 - ae\delta(1 + e\delta) - bk_1] + b\delta k_2[a(b-e) - bh]}{bc} = 0. \tag{23}$$

Let μ_1 and μ_2 denote the roots of the characteristic equation (23). Then, the sum and product of the roots are given by

$$\mu_1 + \mu_2 = 2 - \frac{ae\delta}{b} - k_1 \tag{24}$$

and

$$\mu_1\mu_2 = \frac{c[b + be(a-h)\delta^2 - ae\delta(1 + e\delta) - bk_1] + b\delta k_2[a(b-e) - bh)]}{bc}. \tag{25}$$

To identify the marginal stability boundaries, we consider three cases based on specific values of the eigenvalues. First, assume $\mu_1\mu_2 = 1$. Then from Eq. (25), we derive the first stability boundary:

$$l_1 := \frac{e\delta(ab\delta - a - ae\delta - bh\delta)}{b} - k_1 + \frac{(ab - ae - bh)\delta k_2}{c} = 0. \tag{26}$$

Now, let $\mu_1 = 1$. Substituting into Eqs. (24) and (25), we obtain the second boundary:

$$l_2 := \frac{e\delta^2(ab - ae - bh)}{b} + \frac{(ab - ae - bh)\delta k_2}{c} = 0 \tag{27}$$

Finally, let us take $\mu_1 = -1$. Using the same equations, we find the third boundary

$$l_3 := \frac{-ae\delta(2 + e\delta) + b(4 + e(a-h)\delta^2)}{b} - 2k_1 + \frac{(ab - ae - bh)\delta k_2}{c} = 0 \tag{28}$$

The triangular region enclosed by the lines l_1, l_2 and l_3 in the k_1k_2-plane represents the set of feedback gain values for which the modulus of the eigenvalues remains less than 1, thereby ensuring local asymptotic stability of the controlled system.

2.4. **Numerical simulations.** To validate the theoretical findings on the stability and bifurcation behavior of the predator-prey system, a series of numerical simulations are performed. In Example 2.4, by varying the bifurcation parameter b, the system exhibits a Neimark–Sacker bifurcation at $b = 1.60247$ as confirmed by both the bifurcation diagram and the associated phase portraits. From Figure 2, it is observed that the fixed point $(\overline{x}, \overline{y})$ is stable for $b < 1.60247$, but loses stability at $b = 1.60247$. For $b > 1.60247$, a closed invariant curve forms around the fixed point, indicating the emergence of unstable behavior due to Neimark–Sacker bifurcation.The phase portaits of bifurcation diagrams in Figure 2 for different values of b are shown in Figure 3. These phase portraits clearly illustrate how a smooth invariant curve bifurcates from the previously stable fixed point and grows in radius. At $b = 1.60247$, the positive coexistence fixed point becomes unstable, and an attracting closed invariant curve surrounding the unstable fixed point appears, confirming the Neimark–Sacker bifurcation. These visualizations illustrate how the initially stable fixed point loses stability and gives rise to a closed invariant curve as b increases, indicating the onset of quasi-periodic behavior. The progressive growth of this invariant curve for increasing b values highlights the transition from regular dynamics to complex oscillatory behavior. In Example 2.5, a state feedback control strategy is applied to stabilize the chaotic dynamics induced by the bifurcation. A triangular region in the (k_1, k_2)-parameter space is identified, within which the controlled system maintains local asymptotic stability. This region represents the set of values for the control parameters k_1 and k_2 for which the controlled system (30) has eigenvalues with modulus less than one, ensuring local asymptotic stability of the equilibrium point. The triangular stability region bounded by lines l_1, l_2 and l_3 is depicted in Figure 4.(a). The stability region, determined through characteristic equations and marginal stability lines, is depicted graphically and confirms the effectiveness of the feedback control approach. Figures 4.(b) and 5 illustrate complementary aspects of the dynamical behavior of the controlled predator–prey system. Figure 4.(b) presents the time series of the prey and predator populations under state feedback control for specific parameter values. The trajectories clearly converge to the positive equilibrium point, confirming the stabilizing effect of the control mechanism. In contrast, Figure 5 shows the variation of the maximum Lyapunov exponent as the bifurcation parameter $b \in [1.3, 1.9]$ varies. This diagram indicates a transition from stable behavior (negative exponent) to chaotic dynamics (positive exponent), highlighting the onset of complex population oscillations beyond a critical bifurcation threshold. Together, these figures demonstrate both the potential for chaos in the uncontrolled system and the ability of feedback control to suppress such undesirable dynamics. These simulations not only corroborate the analytical results but also demonstrate the practical implementation of chaos control methods in ecological modeling.

Example 2.4. *Let consider the following system with parameter values* $e = 0.2$, $c = 1$, $h = 1, \delta = 0.9$ *taken from* [19],

$$
\begin{aligned}
x_{n+1} &= x_n + 4.95x_n(1 - x_n) - 0.9x_ny_n - 0.9x_n \\
y_{n+1} &= y_n + 1.44x_ny_n - 0.18y_n.
\end{aligned} \tag{29}
$$

where $a = 5.5$ *and* $b \cong 1.6$. *The positive coexistence fixed point is* $(\overline{x}, \overline{y}) = (0.125, 3.8125)$, *and the Jacobian matrix evaluated at this point is:*

$$J_{(\overline{x},\overline{y})} = \begin{pmatrix} 0.38125 & -0.1125 \\ 5.49 & 1 \end{pmatrix}.$$

The eigenvalues of the Jacobian matrix are:

$$\lambda_{1,2} = 0.690625 \mp 0.722435i$$

such that $|\lambda_{1,2}| \cong 1$. *The corresponding complex eigenvectors* $q, p \in \mathbb{C}^2$ *are*

$$q \sim (-0.1315910470 - 0.056352459i, i)^T$$

and

$$p \sim (6.421643092 - 2.75i, -i)^T.$$

The normalized adjoint vector p^* *satisfying* $< p^*, q >= 1$ *is:*

$$p^* \sim (-2.62199249 + 2.75i, 0.214119654 + 0.50i)^T.$$

By applying the coordinate transformation

$$u_n = x_n - 0.125, \; v_n = y_n - 3.8125$$

the system (29) becomes:

$$
\begin{aligned}
u_{n+1} &= 0.38125u_n - 0.1125v_n - 0.9u_nv_n - 4.95u_n^2 \\
y_{n+1} &= 5.49u_n + v_n + 1.44u_nv_n.
\end{aligned}
$$

The coefficients in the normal form expansion (from (20)) are calculated as:

$$
\begin{aligned}
g_{20}(b_1) &= 0.725879 + 0.265549i \\
g_{11}(b_1) &= 0.231208 + 0.360091i \\
g_{02}(b_1) &= 0.609655 + 0.899999i \\
g_{21}(b_1) &= 0.
\end{aligned}
$$

From (21), we compute $\varphi(b_{NS}) = -0.487478 < 0$. *Thus, a supercritical Neimark–Sacker bifurcation occurs at* $b_{NS} = 1.60247$. *This result confirms that the system undergoes a qualitative change at the critical threshold, the fixed point loses its stability and a stable closed orbit appears, indicating the onset of sustained oscillatory behavior.*

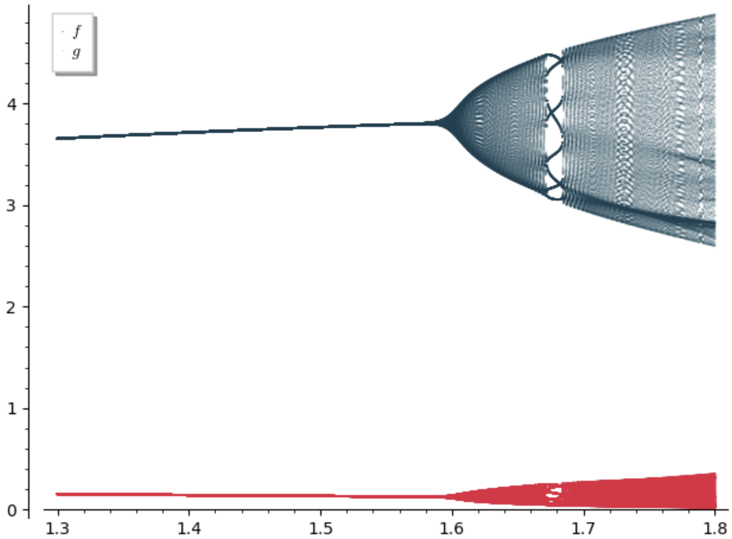
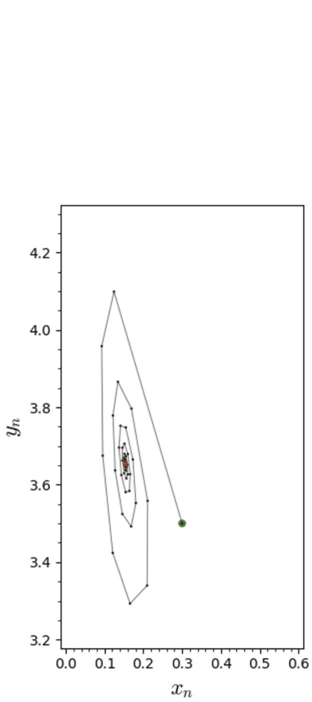
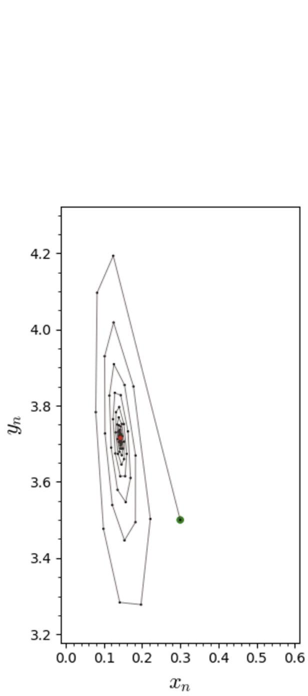


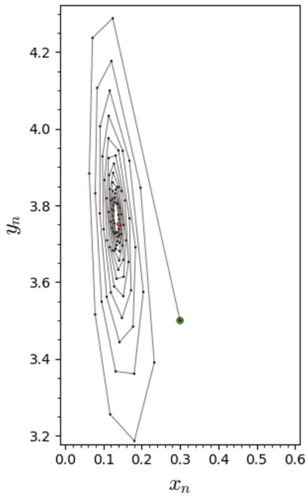
FIGURE 2. Bifurcations diagram of the system (2) for $b \in (1.3, 1.8)$ with the parameter values $a = 5.5$, $e = 0.2$, $c = 1$, $h = 1$, $\delta = 0.9$ and the initial conditions $(x_0, y_0) = (0.3, 3.5)$.



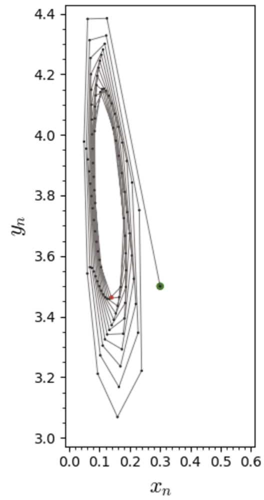
(a) $b = 1.3$



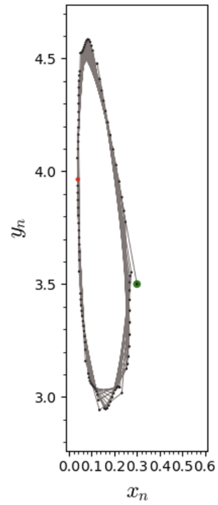
(b) $b = 1.4$



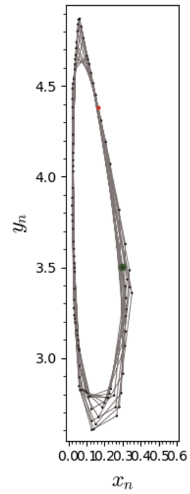
(c) $b = 1.5$



(d) $b = 1.60247$



(e) $b = 1.7$



(f) $b = 1.8$

FIGURE 3. The phase portraits of system (2) for various values of b, demonstrating the evolution of the invariant curve as b increases. Parameters: $a = 5.5$, $e = 0.2$, $c = 1$, $h = 1$, $\delta = 0.9$, and the initial condition $(x_0, y_0) = (0.3, 3.5)$.

Example 2.5. *In this example, a state feedback control strategy is applied to stabilize the system* (2)*. The parameter values are taken as* $(e, a, c, h, \delta, b) = (0.2, 5.5, 1, 1, 0.9, 1.9)$ *with the initial condition* $(x_0, y_0) = (0.3, 3.5)$*. The corresponding controlled system is given by:*

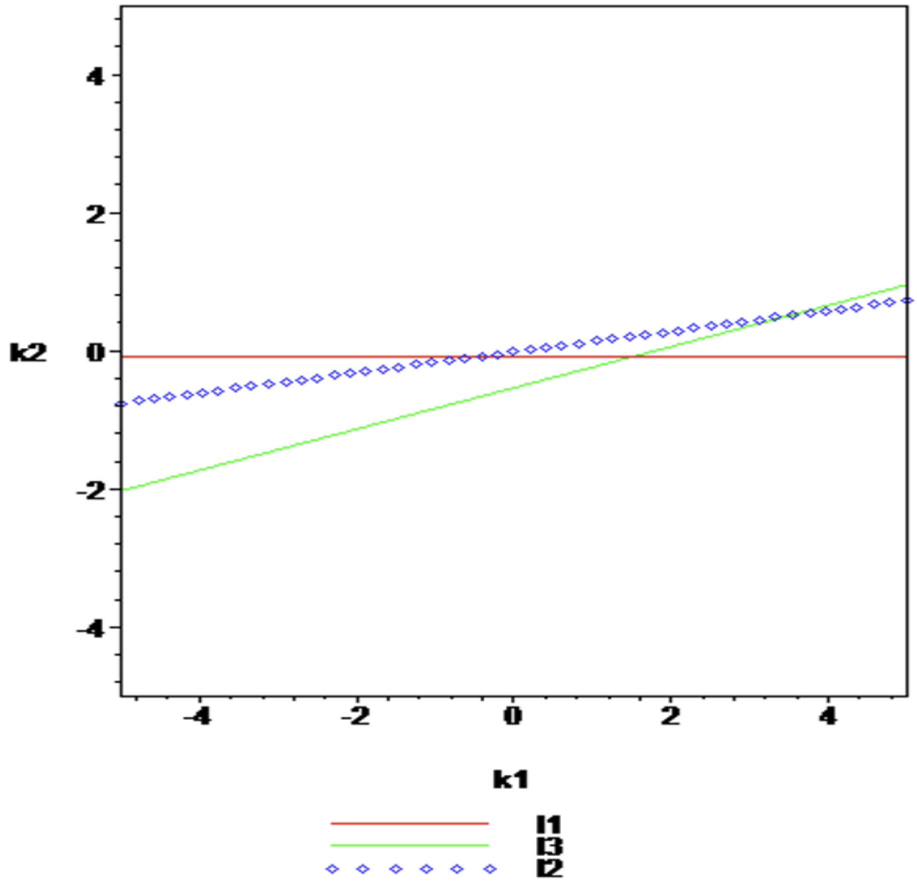
$$
\begin{aligned}
x_{n+1} &= x_n + 0.9(5.5x_n(1-x_n) - x_n y_n - x_n) - k_1(x_n - 0.105263) \\
&\quad - k_2(y_n - 3.92105) \\
y_{n+1} &= y_n + 0.9[(1.9x_n y_n - 0.2y_n].
\end{aligned} \tag{30}
$$

By analyzing the marginal stability conditions given in equations(26)-(28), we obtain the corresponding marginal lines in the k_1k_2*-plane:*

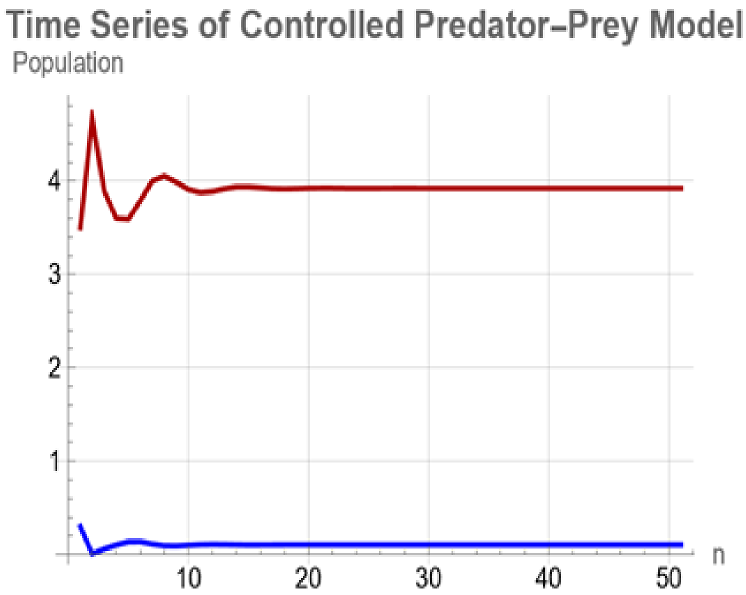
$$
\begin{aligned}
l_1 &= 0.114158 - k_1 + 6.705k_2 = 0 \\
l_2 &= 0.635211 + 6.705k_2 = 0
\end{aligned}
$$

and

$$
l_3 = 3.59311 - 2k_1 - 6.705k_2 = 0.
$$



(a)



(b)

FIGURE 4. (a) Stability region of the controlled system (30) in the k_1k_2 plane. (b) Time series of the prey and predator populations for the controlled system with parameters $a = 5.5$, $b = 1.9$, $c = 1$, $h = 1$, $e = 0.2$, $\delta = 0.9$, and control coefficients $k_1 = 0.61$, $k_2 = 0.005$. The system stabilizes at the positive equilibrium point $(x^*, y^*) = (0.105263,\ 3.92105)$, illustrating the effectiveness of the state feedback control.

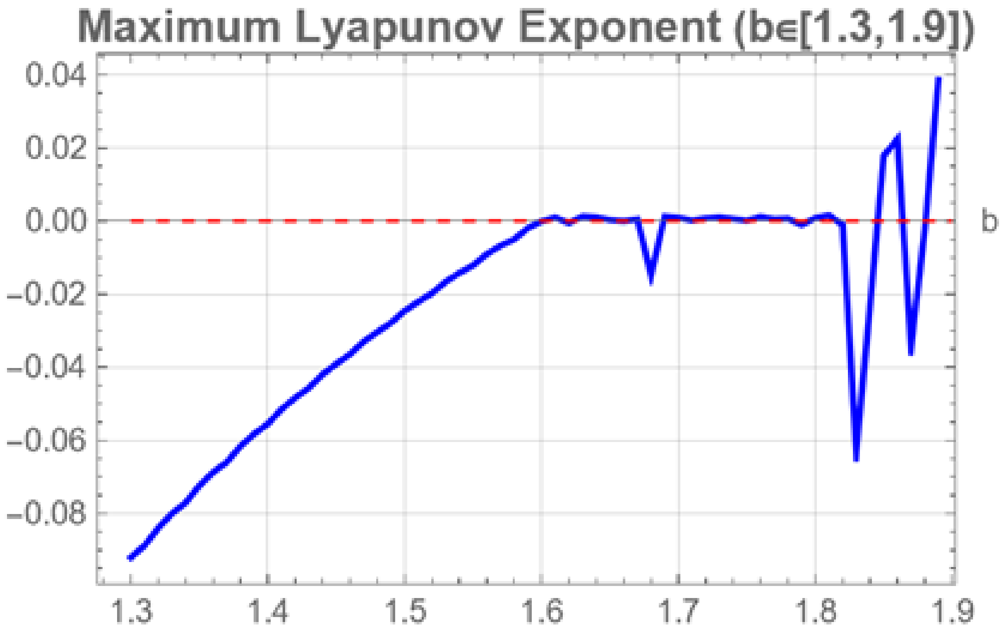


FIGURE 5. Maximum Lyapunov exponent of system (2) with respect to the bifurcation parameter $b \in [1.3, 1.9]$, illustrating the transition to chaos. The parameter values are $a = 2.5$, $e = 0.2$, $c = 1$, $h = 1$, and $\delta = 0.9$.

3. Conclusion

In this study, a more in-depth analysis of the effects of a different bifurcation parameter on the predator-prey model discussed in [3] was conducted; and a more comprehensive understanding of the dynamic behavior of the model was developed. Focusing specifically on the parameter "conversion efficiency of prey into predator", the stability of the positive coexistence equilibrium point, the Neimark-Sacker bifurcation, and chaos control were investigated. This advanced analysis, which was performed while preserving the same model structure, allowed the effects of parameter-dependent changes on the system dynamics to be identified in an isolated manner. Thus, the complex behavior patterns and chaotic transitions of the model were determined more clearly depending on the parameter b; and important inferences were obtained in terms of control mechanisms. It was also shown that the chaotic behavior of the system could be stabilized using the state feedback control method.

The system (2) exhibits a Neimark-Sacker bifurcation at the positive coexistence fixed point E_2. The asymptotic stability conditions of the fixed point $E_2 = (\frac{e}{b}, \frac{ab-ae-bh)}{bc})$, $ab > ae + bh$ are investigated by using the linearization method. Moreover, we demonstrate that system (2) undergoes a Neimark–Sacker bifurcation when the bifurcation parameter $b_1 = \frac{a+ae\delta}{(a-h)\delta}$ using bifurcation theory. It is evident that a Neimark–Sacker bifurcation emerges when the system parameters vary in the neighborhood $b = b_1$, $b > \frac{1}{2}\left(\frac{ae}{a-h} + \frac{a}{a-h}\sqrt{e(a-h+e)}\right)$. The dynamic properties of system (2) with the parameter $b = 1.60247$ (also, according to different values of b), $e = 0.2$, $a = 5.5, c = 1$, $h = 1$, $\delta = 0.9$ and the initial conditions $(x_0, y_0) = (0.3, 3.5)$ are represented by some Figures. On the other hand, parameter values are taken from the article [19] in order to ensure consistency with biological reality. By choosing the value of b as bifurcation parameter, the bifurcation state is observed. Chaos is controlled via a state feedback method for the parameter $e = 0.2$, $a = 5.5$, $b = 1.9$, $c = 1$, $h = 1$, $\delta = 0.9$.

Using the parameter values given in [19], the effect of harvest effort on the prey population instead of the predator population is displayed in Figure 6. For the parameter values $a = 2.5$, $e = 0.2$, $c = 1$, $h = 1$, $\delta = 0.9$, the bifurcation point of system (2) is $b = 2$. In the model studied in [19], we see that the bifurcation point is $b = 2.3111$ for the same values.We therefore conclude that harvesting effort in the prey population causes the system to reach the bifurcation point at a lower value of the bifurcation parameter b.

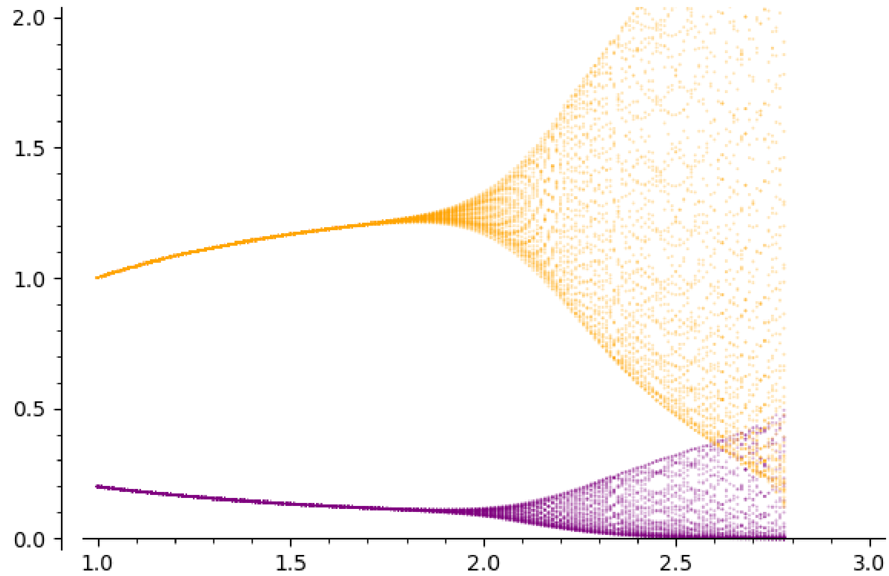


FIGURE 6. Bifurcation diagrams of the system (2) over the interval $b \in (1, 3)$ with parameter values $a = 2.5$, $e = 0.2$, $c = 1$, $h = 1$, $\delta = 0.9$, and the initial conditions $(x_0, y_0) = (0.3, 0.4)$.

The parameter h in the model represents the harvesting pressure on the prey species (e.g., fishing activity). If the harvesting rate is too high, the prey population declines, leading to a shortage of food for the predator (such as tuna or sharks). This can destabilize the ecosystem and potentially result in the collapse of fish stocks. Specifically, the control parameters k_1 and k_2 represent external interventions (e.g., controlled harvesting, or releasing natural enemies). The model can be used to predict when and how much intervention is needed to bring the populations to a sustainable level. In contrast to the results presented in [3], where bifurcation dynamics were investigated in terms of the step parameter δ, this study reveals qualitatively different dynamic behaviors by treating parameter b as the bifurcation parameter, thus providing new insights into the role of predation efficiency in system stability and control.

To assess the sensitivity of the system to parameter variations, we present in Table 1 both the parameter values used in the original study and newly adopted values from the literature. The corresponding time series of prey and predator populations based on the new parameter set is depicted in Figure 7. The intrinsic growth rate of prey ($a = 0.48$) is taken from Cendrero, O., Cort, J.L., and de Cardenas, E. (1981). *Revisión de algunos datos sobre la biología de la anchoa, Engraulis encrasicolus (L.) del Mar Cantábrico*. Bol. Inst. Esp. Oceanogr., **6**, 117–124. The other parameter values ($b = 0.5$, $c = 1$, $h = 0.2$, $e = 0.2$) are adopted from Xiao, Y., and Jennings, S. (2005). *Trade-offs between carrying capacity and productivity: Implications for managing marine fish. SIAM Journal on Applied Mathematics*, **65**(6), 2107–2126.

Figure 1.(a) shows the time series behavior of the discrete predator-prey system under the original parameter values used in the study. In this case, both prey and predator populations converge relatively quickly to a stable equilibrium point and exhibit moderately damped oscillations. The system shows a rapid approach to stability, with the fluctuations decaying over a short time horizon. In contrast, Figure 7 presents the system dynamics based on new parameters obtained from the literature. Under this configuration, the system exhibits a slower convergence to equilibrium, but stability persists for a very long time. The critical bifurcation value calculated for the new parameter set is approximately $b = 17.49$, which is far beyond biologically plausible limits. This indicates that the system remains in a sustainably stable regime.This comparison highlights how parameters can significantly affect the temporal evolution and stability properties of ecological systems.

Parameter	Description	Value in Study	New Value (Reference)
a	Intrinsic growth rate of prey	5.5	0.48 [Cendrero et al., 1981]
b	Conversion efficiency (prey → predator)	1.9	0.5 [Xiao & Jennings, 2005]
c	Predation rate	1	1 [Xiao & Jennings, 2005]
h	Harvest rate of prey	1	0.2 [Xiao & Jennings, 2005]
e	Mortality rate of predator	0.2	0.2 [Xiao & Jennings, 2005]
δ	Step size	0.9	0.1

Table 1. Comparison of parameter values used in the study and newly adopted values (Comparison of parameter values used in the study and newly adopted values (growth rate from Cendrero et al., 1981; other values from Xiao and Jennings, 2005)

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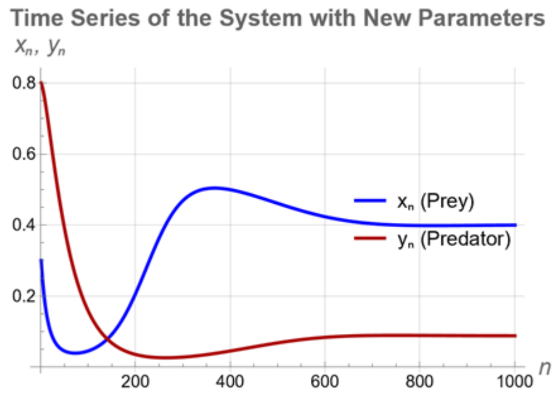

Figure 7. Time series of the prey (x_n) and predator (y_n) populations based on the newly adopted parameter set: $a = 0.48$, $b = 0.5$, $c = 1$, $h = 0.2$, $e = 0.2$, and $\delta = 0.1$, with initial conditions $(x_0, y_0) = (0.3, 0.8)$. The prey and predator populations converge to a biologically feasible steady state after a series of oscillations.

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Application of wavelet transformation in analyzing railway track profile irregularities

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Abstract. Wavelet transformation provides a time-frequency analysis of non-stationary signals. It enables the detection of local events occurring within short time intervals. This capability is particularly critical for the analysis of railway track profile geometry [1], which often exhibits rapid and abrupt vertical irregularities over specific distances due to various mechanical factors. These factors include local voids in the supporting soil, rapid variations in track layer thickness, ballast wear and contamination, the existence of rail-ends and turnouts, and the discontinuous support provided by sleepers. Inspection vehicles equipped with special measurement sensors can convert these geometric changes into a continuous signal.

Through multi-level wavelet decomposition, the dominant characteristics of the track profile signal can be extracted, along with detailed sub-signals of different frequencies and wavelengths [2]. These sub-signals represent the contribution of each underlying cause to the observed rail profile irregularities [3]. This study presents a case study where wavelet decomposition is applied to a track profile data collected from UK. The track profile is isolated into sub-parts, and possible causes of the overall deformation is investigated independently. The findings can contribute to developing effective maintenance strategies that enhance safety and extend the service life of railway tracks.

2020 Mathematics Subject Classifications: 42C40, 94A12

Keywords: Railway track analysis, Wavelet decomposition, Rail profile irregularities

1. Introduction

Wavelet transformation, frequently used in signal and image processing, enables complex data composed of different components to be expressed using discrete mathematical functions. Due to its ability to precisely reveal the time and frequency characteristics of the oscillations that form a signal, wavelets have a wide range of applications across various disciplines. Compression, filtering, and noise reduction can be performed on audio, image, and vibration data. Furthermore, the frequency bands corresponding to the individual components of a signal can be identified. Wavelet transformation is used for acoustic signal analysis, extraction of track deformations, and the detection of track damage in railway tracks.

Track measurement vehicles passing over a railway track record acceleration values resulting from irregularities along the track. These data are passed through a high-pass filter to remove the earth component and then double-integrated to obtain the track deformation values. Deformation values depending on location or time constitute a continuous signal. This signal is a combination of various track irregularities with different frequencies. These irregularities may arise from factors such as variations in track layer thickness along a specific distance, localised soil settlements, sleeper cracks, worn ballast particles, frost conditions, rail wear, insulated rail joints, or turnout areas. As a train passes over these irregularities, it transfers a dynamic force onto the track that exceeds its static force. The dynamic impact force is correlated with the horizontal track section (L) where the irregularity occurs and the irregularity's amplitude (h). The Extended Bezgin Equations were developed to estimate dynamic impact forces based on the application of the Bezgin Method -originally introduced in 2017- [4] to track profile irregularities. The equation results show up to 98 % agreement with the finite element model, particularly in the comparisons conducted at the rail ends and the turnouts [5, 6]. In this study, Wavelet Transformation is applied to raw geometry data collected from a railway track in the UK to extract deformations in different frequency bands. Then, the irregularity data obtained from the approximation signal, which reflects the geometric characteristics of the signal, will be used as input parameters for the Extended

Bezgin Equations, and the peak dynamic impact forces on the track will be estimated. This study establishes a methodological basis for analyzing the mechanical effects of distinct track irregularities.

2. Wavelet Transformation

The Wavelet Transform expresses the original main signal as the sum of sub-signals with different frequencies and wavelengths. The signal is first decomposed at a certain level to obtain the wavelet coefficients. Then, the signal is reconstructed to obtain approximation and detail signals. This process is accomplished by scaling and translation operations on a mother wavelet that can best represent the original signal. The scaled and shifted version of the mother wavelet is as shown in Equation 1.

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{t-b}{a}\right) \tag{1}$$

In this formulation, t represents the independent variable (such as time or position), $\psi(t)$ denotes the mother wavelet function, $a \in \mathbb{R}^+$ is the scale parameter, and $b \in \mathbb{R}^+$ is the translation parameter. The Continuous Wavelet Transform (CWT) of a signal $f(t)$ is expressed as:

$$W_f(a,b) = \int_{-\infty}^{\infty} f(t)\,\frac{1}{\sqrt{|a|}}\,\overline{\psi\left(\frac{t-b}{a}\right)}\,dt \tag{2}$$

The Continuous Wavelet Transform can often be unnecessary in practical applications in terms of functionality. Unlike the Continuous Wavelet Transform, the Discrete Wavelet Transform develops solutions using a limited number of scales. The selection of scales as powers of 2 can significantly increase computational efficiency. Nevertheless, the Discrete Wavelet Transform is continuous in terms of the translation of the wavelet. During the computation, the selected mother wavelet is shifted sequentially along the original signal, starting from its initial point, comparing the wavelet with the entire signal in this manner. Subsequently, the mother wavelet is scaled (resized), and the same process is repeated for each new scale until no further scales remain. This process yields approximation and detail signals.

3. Multi-Level Transformation

In Wavelet Transform, a decomposition level is selected based on a specific evaluation criterion or wave characteristic. As a result of decomposition at multiple levels, the original signal is broken down into numerous smaller, lower-resolution fragments. A Wavelet Decomposition Tree is constructed by representing these fragments and their subsequent sub-fragments in a hierarchical diagram, as shown in Figure 1.

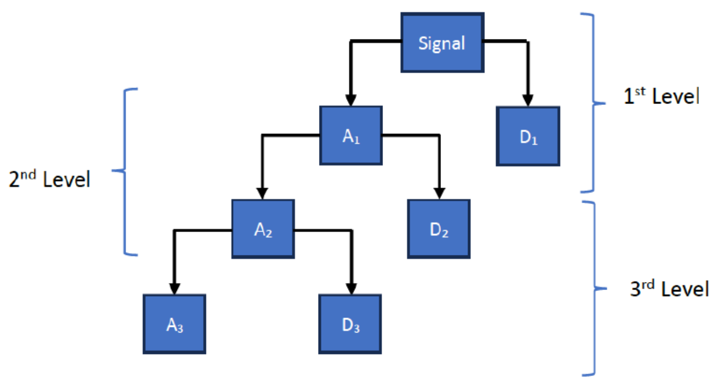

Figure 1. Sub-signals obtained from the 3rd-level wavelet transform [3].

When "n" decompositions are performed in multi-level decomposition, a final approximation signal and "n" detail signals are obtained. The relationship between the approximation signal (A), detail signals (D), and the original signal obtained as a result of multi-level decomposition is as shown in Equation 3.

$$S = A_n + \sum_{j=1}^{n} D_j \tag{3}$$

By performing multi-level decomposition on a railway track profile composed of multiple components, irregularities at different frequency bands can be separately identified. The geometric properties of the yielded signals can be used as input parameters in the Extended Bezgin Equations, along with relevant track and train parameters [7]. This enables the estimation of dynamic impact forces associated with each distinct irregularity.

4. Extended Bezgin Equations

Track stiffness as well as rail profile can vary along the railway track. Figure 2 shows the passage of a train wheel over a track irregularity with a wavelength of λ . The irregularity consists of a descending and ascending profile. The transition length for the descending and ascending profile is "L" and the vertical variation in the profile is denoted by "h." If there is a variation in track stiffness, the track initially deforms by an amount "a" under the static wheel load. By the end of the stiffness transition zone, the static deformation reaches "b." However, due to the dynamic forces generated during passage through this irregularity, it deforms by a value of "c" higher than "b."

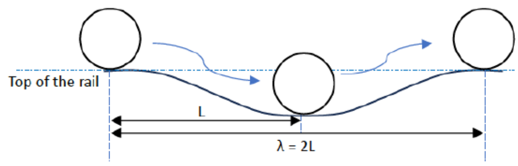

Figure 2. A track irregularity with a wavelength of λ [3].

The dynamic impact force factor is represented by $K'_{B1,d}$ for descending track profile case, while it is denoted by $K'_{B2,a}$ for the ascending track profile case. In addition, there is a secondary impact force, K_J, that occurs when the wheel momentarily loses contact after passing through the irregularity and then suddenly falls back onto the track.

The dynamic impact force factors obtained using the Bezgin Method are presented in Equations 4-6. The system stiffness is the combination of track stiffness and the stiffness of the rolling stock elements. Accordingly, a' and b' represent the static system deformations in regions with different system stiffness. The train speed is expressed as v, the system damping as s, the gravitational acceleration as g, the vertical profile variation as h, and the horizontal length at which the irregularity occurs as L. The impact reduction factor (f), which determines the level of impact that will be transferred to the track, is presented in Equation 7. This concept relates to the proportion of the wheels' theoretical free-fall time (t_{fall}) from a height to the passage time (t_{pass}) required to cover a distance L.

(4) $$K'_{B,d} = 1 + \sqrt{\frac{2h}{a'}(1 - f - s)}$$

(5) $$K'_{B,a} = 2\sqrt{\frac{h}{2a'}(1 - f - s) + 1} - 1$$

(6) $$K_j = \frac{2h_a}{g}\left(\frac{v}{L}\right)^2$$

(7) $$f = 1 - \frac{t_{\text{fall}}}{t_{\text{pass}}} = 1 - \frac{v}{L}\sqrt{\frac{2h}{g}}$$

The maximum dynamic impact force during the approach to a civil structure (ascending profile) is estimated using the terms $K'_{B2,a} + K_j$. While wheel departures from the structure (descending profile), the peak dynamic force factor is $K'_{B1,d}$.

5. Estimation of Dynamic Impact Force from Railway Track Data

Application of wavelet transformation on irregular track profile data yields an approximation signal that highlights the main characteristics of the original signal. Figure 3 presents the comparison of the profile data obtained from a railway track located in the East Midlands, UK, and the approximation signal yielded by the 5th-order wavelet transformation. The wavelengths and amplitudes of the track irregularities are visible in the approximation signal. This allows the application of the Bezgin Method using the "h" and "L" values, which are the vertical and horizontal variations between each peak and valley in the signal. The dynamic impact force estimations for the irregularities derived from approximation signal are shown in Figure 4. In the calculations, the track stiffness was considered to be 60 kN/mm, and stiffness values of the wheel and bogie suspension springs were taken as 5 kN/mm and 2.5 kN/mm. The static wheel force is 90 kN and the train speed is considered as 160.5 km/h.

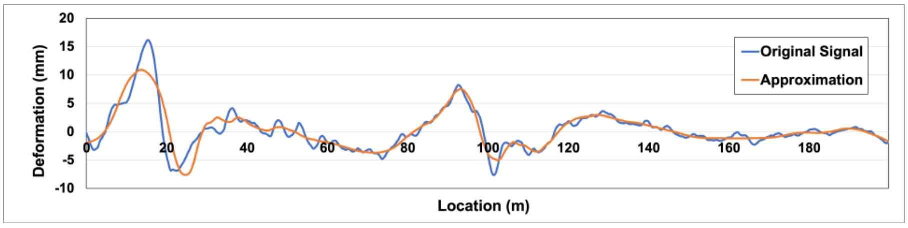

Figure 3. Track profile deformations from the East Midlands study area and the approximation signal obtained from the 5th-level wavelet transformation

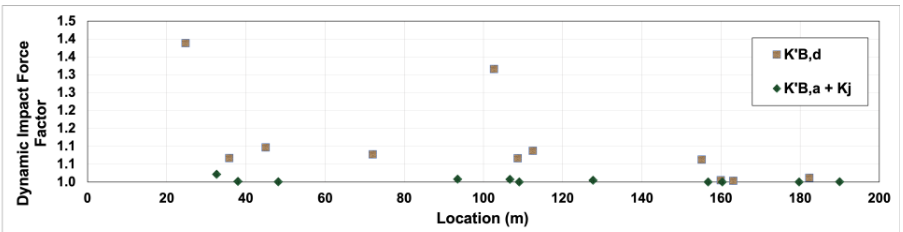

Figure 4. Estimated dynamic impact force factors derived from the approximation signal

The analysis conducted for the selected track section reveals that the dynamic impact force factors resulting from ascending and descending profile irregularities vary between 1.0 and 1.39. This indicates that the train has the potential to transfer a force 39% greater than its static force to the track. The irregular track data shows significant fluctuations around 20th m and 100th m. The maximum dynamic forces on the track occurred around these regions. The mechanical effects of track irregularities with shorter wavelengths can be revealed by using the geometric parameters extracted from detail signals as input in the Extended Bezgin

Equations. A combined evaluation of visual observations and the sub-signals obtained from the wavelet transform sheds light on the potential risks posed by irregularities at different frequencies.

6. Conclusion

The findings of this study highlight the value of Wavelet Transformation in understanding the mechanical behavior of railway tracks. Wavelet Transformation offers a simplified approach for applications where various geometric and mechanical irregularities with different wavelengths coexist, such as railway tracks. By separating the track profile signal into different frequency bands, it becomes possible to interpret the effects of irregularities in each frequency band separately.

Integrating this analytical approach with the Bezgin Method, the study provides a systematic framework to identify critical sections of the track subject to elevated mechanical stress. The methodology was demonstrated through a case study involving real-world data collected from a railway section in the UK, confirming the practical utility of the proposed approach. In fact, in areas with high profile variations, an additional dynamic force of up to 39% was observed. Furthermore, no significant dynamic force was observed in stable areas of the track. Long-wavelength irregularities can be analyzed with the approximity signal, while relatively smaller wavelength irregularities can be interpreted with the detail signals. By determining a wavelet transform level considering the wavelengths of the irregularities observed on the track, the dynamic impact forces caused by these irregularities can be estimated individually. This approach provides insights for effective maintenance planning and prioritizing track interventions. Therefore, it contributes to enhancing the safety, efficiency, and service life of the railway tracks.

Acknowledgment

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Fixed point results for generalized cyclic α-$f\pi\varpi$-contractions

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ABSTRACT. In this paper, we introduce a new class of such contractions and demonstrate new fixed point theorems that generalize and unify numerous fixed point results that currently exist. The $\alpha - f\pi\varpi-$contraction that we have introduced represents a new hybrid condition that incorporates control functions, admissibility, and cyclic structure in b-metric-like space, effectively permitting the generalization of existing fixed point principles.

2020 MATHEMATICS SUBJECT CLASSIFICATIONS: 47H10, 54H25.

KEYWORDS: b-metric-like space, Fixed point, Cyclic contractions.

1. INTRODUCTION

Fixed point theory is one of the most important areas of mathematical analysis and has a wide range of applications in nonlinear analysis, optimization and differential equations among many others. Many generalizations of the Banach contraction principle have been proposed in the past to cover more general mathematical objects. It is natural to investigate contractions in b-metric and b-metric-like spaces, which represents a generalization of the classical metric space by removing the triangle property (see [4]-[8]).

In [7], the concept of cyclic $\alpha - f\pi\varpi-$ contractions was introduced, and several fixed point theorems were established. Motivated by these developments, we extend this framework by introducing the notion of cyclic $\alpha - f\pi\varpi-$ contractive pairs of mappings. Our work further advances the theory by formulating these results within the setting of b-metric-like spaces, which generalize classical metric structures, we present two new fixed point theorems, one for $\alpha - f\pi\varpi-$ contractive pairs and another for cyclic $\alpha - f\pi\varpi-$ contractive pairs. These theorems are supported by new contractive conditions, and they apply to a broader class of mappings than those previously studied.

1.1. **Preliminaries.** We start with a series of definitions and notations that will serve for the rest of the analysis.

Definition 1.1. [1] *A b-metric-like on a nonempty set Λ is a function $d : \Lambda \times \Lambda \to [0, +\infty)$ such that for a value $1 \leq s$ and for all $x, y, z \in \Lambda$ the following three conditions hold true*
(D_1) *if $d(x, y) = 0$ then $x = y$;*
(D$_2$) $d(x, y) = d(y, x)$;
(D$_3$) $d(x, z) \leq s(d(x, y) + d(y, z))$.
The pair (Λ, d) is then called a b-metric-like space.

Example 1.2. [1] *Let $\Lambda = \{0, 1, 2\}$, and let*

$$d(x,y) = \begin{cases} 2, & x = y = 0 \\ \frac{1}{2}, & \textit{otherwise.} \end{cases}$$

Then (Λ, d) is a b-metric-like space with the constant $s = 2$.

A sequence $\{x_n\}$ in the b-metric-like space (Λ, d) converges to a point $x \in \Lambda$ if and only if $d(x, x) = \lim_{n\to+\infty} d(x, x_n)$.
A sequence $\{x_n\}$ in the b-metric-like space (Λ, d) is called a Cauchy sequence if there exists $\lim_{n,m\to+\infty} d(x_m, x_n)$ (and it is finite)(0-cauchy if the limit is also null).
A b-metric-like space is called d-complete if every Cauchy sequence $\{x_n\}$ in Λ converges to a point $x \in \Lambda$ such that $\lim_{n\to+\infty} d(x, x_n) = d(x, x) = \lim_{n,m\to+\infty} d(x_m, x_n)$. A b-metric-like space is called 0-d-complete if every 0-Cauchy sequence $\{x_n\}$ in Λ converges to a point $x \in \Lambda$ such that $d(x, x) = 0$ and $\lim_{n\to+\infty} d(x, x_n) = \lim_{n,m\to+\infty} d(x_m, x_n) = 0$.

Definition 1.3. [3, 6] *A C-class function is a continuous function $F:[0,+\infty)\times[0,+\infty)\to\mathbb{R}$ such that for any $x,y\in[0,+\infty)$, the following conditions hold:*
(C1) $F(x,y)\le x$.
(C2) $F(x,y)=x$ implies that either $x=0$ or $y=0$.

Definition 1.4. [6] *Let Υ be a self-mapping on a nonempty set Λ and $\alpha:\Lambda\times\Lambda\to[0,+\infty)$ be a function. Υ is called an α-admissible mapping if $\alpha(\Upsilon\iota,\Upsilon\kappa)\ge1$, for all $\iota,\kappa\in\Lambda$ with $\alpha(\iota,\kappa)\ge1$.*

Let $\Pi=\{\pi:[0,+\infty)\to[0,+\infty):\pi$ is increasing and continuous$\}$ and
$\coprod=\{\varpi:[0,+\infty)\to[0,+\infty):\varpi$ is increasing and lower semi-continuous$\}$.

Definition 1.5. [7] *A triple (π,ϖ,f) where $\pi\in\Pi,\varpi\in\amalg$ and f a C-class function is said to be monotone if*

$$\iota\le\kappa \text{ implies } f\left(\pi(\iota),\varpi(\iota)\right)\le f\left(\pi(\kappa),\varpi(\kappa)\right)$$

for any $\iota,\kappa\in[0,+\infty)$.

Example 1.6. [7] *Let $f(u,v)=u-v,\varpi(\iota)=\sqrt{\iota}$ and*

$$\pi(\iota)=\begin{cases}\sqrt{\iota}, & \text{if } 0\le\iota\le1\\ \iota^2, & \text{if } \iota>1.\end{cases}$$

Then (π,ϖ,f) is a monotone triple.

Definition 1.7. [7] *Let $(\Lambda,\hat{d})$ be a b-metric-like space, and let $\alpha:\Lambda\times\Lambda\to[0,+\infty)$ be a function. Then, $\Upsilon:\Lambda\to\Lambda$ is said to be α-continuous on $(\Lambda,\hat{d})$ if*

$$\iota_n\to\iota \text{ and } \alpha\left(\iota_n,\iota_{n+1}\right)\ge1 \text{ for all } n\in\mathbb{N} \text{ implies } \Upsilon_{\iota_n}\to\Upsilon\iota.$$

2. Main results

In this section, we show some fixed point theorems for cyclic contraction on b-metric-like spaces, let $\pi\in\Pi,\varpi\in\amalg$ and $f\in\mathcal{C}$ such that
$^{(*)}$ $\pi(\iota)-f(\pi(\kappa),\varpi(\kappa))>0$ for all $\iota>0$ and $\kappa=\iota$ or $\kappa=0$.

Definition 2.1. *Let $(\Lambda,\hat{d})$ be a 0 - $\hat{d}$-complete b-metric-like space, $p\in\mathbb{N}$ and $\Gamma_1,\Gamma_2,\ldots,\Gamma_p$, be $\hat{d}$-closed nonempty subsets of Λ. Let $\Delta=\bigcup_{i=1}^p\Gamma_i$ and $\alpha:\Delta\times\Delta\to[0,+\infty)$ be a mapping. $(\Upsilon,\eta):\Delta\to\Delta$ is called a cyclic $\alpha-f\pi\varpi-$ contractive ordered pair of mappings if*
(1) $\Upsilon(\Gamma_j)\subseteq\Gamma_{j+1}$ and $\eta(\Gamma_j)\subseteq\Gamma_{j+1}$, for all $j=1,2,\ldots,p$, where $\Gamma_{p+1}=\Gamma_1$;
(2) for any $\iota\in\Gamma_i$ and $\kappa\in\Gamma_{i+1},(i=1,2,\ldots,p)$, where $\Gamma_{p+1}=\Gamma_1$ and

$$\max(\alpha\left(\iota,\Upsilon\kappa\right),\alpha\left(\iota,\eta\kappa\right))\ge1,$$

we have

$$\pi\left(\hat{d}\left(\Upsilon\iota,\eta\kappa\right)\right)\le f\left(\pi\left(M_{\Upsilon;\eta}(\iota,\kappa)\right),\varpi\left(M_{\Upsilon;\eta}(\iota,\kappa)\right)\right), \tag{1}$$

where $\pi\in\Pi,\varpi\in\amalg$ and $f\in\mathcal{C}$ such that the triple (π,ϖ,f) is monotone and satisfies the condition $^{(*)}$ and

$$M_{\Upsilon;\eta}(\iota,\kappa)=q\max\left(\hat{d}(\iota,\kappa),\hat{d}(\iota,\Upsilon\iota),\hat{d}(\kappa,\eta\kappa),\frac{\hat{d}(\iota,\eta\kappa)+\hat{d}(\kappa,\Upsilon\iota)}{2}\right), \tag{2}$$

for every $\iota$and κ in Λ and q is a constant so that

$$0<q<\frac{1}{2s}.$$

Remark 1. *If (Υ,η) was a cyclic $\alpha-f\pi\varpi-$ contractive ordered pair of mappings ,this does not imply necessarily that (η,Υ) is also a cyclic $\alpha-f\pi\varpi-$ contractive ordered pair of mappings.*

Remark 2. *If in Definition 2.1, $\Gamma_1=\Gamma_2=\cdots=\Gamma_p$, then we say (Υ,η) is an $\alpha-f\pi\varpi-$ contractive ordered pair of mappings.*

Lemma 2.2. *Every sequence $(x_n)_{n\in\mathbb{N}}$ of elements from a b-metric-like space (X,d) of constant s, having the property that there exists $0\le\gamma<1$ such that*

$$d\left(x_{n+1},x_n\right)\le\gamma d\left(x_n,x_{n-1}\right),$$

for every $n\in\mathbb{N}$, is 0-d-Cauchy.

Theorem 2.3. *Let $(\Lambda, \hat{d}, s)$ be a $0-\hat{d}$-complete b-metric-like space $(1 \leq s)$ and $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ be a mapping. Assume that $(\Upsilon; \eta) : \Lambda \to \Lambda$ is an $\alpha-$ $f\pi\varpi-$ contractive ordered pair of mappings and (Υ, η) satisfies the following assertions*
(i) Υ and η are α-admissible mappings,
(ii) $\alpha(\iota_0, \eta(\Upsilon\iota_0)) \geq 1$ for an element ι_0 in Λ,
(iii) Υ and η are α-continuous.
Then (Υ, η) admits a common fixed point in Λ. Moreover, if
(iv) $\alpha(\iota, \kappa) \geq 1$, whenever $\iota, \kappa \in Fix(\Upsilon) \cap Fix(\eta)$, then (Υ, η) admits a unique common fixed point.

Proof. Define Picard's sequence $\iota_{2n+1} = \Upsilon\iota_{2n}$, and $\iota_{2n+2} = \eta\iota_{2n+1}$ where ι_0 is the given point for which $\alpha(\iota_0, \eta(\Upsilon\iota_0)) = \alpha(\iota_0, \iota_2) \geq 1$. Since Υ is an α-admissible mapping, we get that $\alpha(\Upsilon\iota_0, \Upsilon\iota_2) = \alpha(\iota_1, \iota_3) \geq 1$. Again. Since η is an α-admissible mapping , it follows that $\alpha(\eta\iota_1, \eta\iota_3) = \alpha(\iota_2, \iota_4) \geq 1$. Continuing this process we have $\alpha(\iota_n, \iota_{n+2}) \geq 1$ for all $n \in \mathbb{N}$, and so, $max(\alpha(\iota_{2n}, \eta\iota_{2n+1}), \alpha(\iota_{2n}, \Upsilon\iota_{2n+1})) \geq 1$ for all $n \in \mathbb{N}$. Now, we show that the sequence $\hat{d}(\iota_n, \iota_{n+1})$ is decreasing. Set

$$\Gamma = q \max\left(\hat{d}(\iota_{2n+1}, \iota_{2n}), \hat{d}(\iota_{2n}, \Upsilon\iota_{2n}), \hat{d}(\iota_{2n+1}, \eta\iota_{2n+1}), \frac{\hat{d}(\iota_{2n}, \eta\iota_{2n+1}) + \hat{d}(\iota_{2n+1}, \Upsilon\iota_{2n})}{2}\right).$$

By (1) and (2) we get

$$\begin{aligned} \pi\left(\hat{d}(\iota_{2n+1}, \iota_{2n+2})\right) &= \pi\left(\hat{d}(\Upsilon\iota_{2n}, \eta\iota_{2n+1})\right) \\ &\leq f(\pi(\Gamma), \varpi(\Gamma)). \end{aligned} \tag{3}$$

On the other hand, we have

$$\hat{d}(\iota_n, \iota_n) \leq 2s\hat{d}(\iota_{n-1}, \iota_n),$$

and

$$\hat{d}(\iota_{n-1}, \iota_{n+1}) \leq s(\hat{d}(\iota_{n-1}, \iota_n) + \hat{d}(\iota_n, \iota_{n+1})),$$

for all $n \in \mathbb{N}$, therefore, from (3) we get

$$\pi\left(\hat{d}(\iota_{2n+1}, \iota_{2n+2})\right) \leq f(\pi(B), \varpi(B)),$$

where

$$B = q \max\left(\hat{d}(\iota_{2n+1}, \iota_{2n}), \hat{d}(\iota_{2n+1}, \eta\iota_{2n+1}), s\left(\frac{3}{2}\hat{d}(\iota_{2n+1}, \iota_{2n}) + \frac{1}{2}\hat{d}(\iota_{2n+2}, \iota_{2n+1})\right)\right),$$

and since (π, ϖ, f) is monotone

$$\hat{d}(\iota_{2n+1}, \iota_{2n+2}) \leq B \tag{4}$$

$$\max(\alpha(\iota_{2n+1}, \eta\iota_{2n+2}), \alpha(\iota_{2n+1}, \Upsilon\iota_{2n+2})) \geq 1.$$

Hence, it follows by a similar approach that

$$\hat{d}(\iota_{2n+2}, \iota_{2n+3}) \leq A, \tag{5}$$

where

$$A = q \max\left(\hat{d}(\iota_{2n+1}, \iota_{2n+2}), \hat{d}(\iota_{2n+2}, \Upsilon\iota_{2n+2}), s\left(\frac{3}{2}\hat{d}(\iota_{2n+1}, \iota_{2n+2}) + \frac{1}{2}\hat{d}(\iota_{2n+2}, \iota_{2n+3})\right)\right).$$

From (4)and (5) we find that

$$\hat{d}(\iota_n, \iota_{n+1}) \leq q \max\left(\hat{d}(\iota_{n-1}, \iota_n), \hat{d}(\iota_n, \iota_{n+1}), s\left(\frac{3}{2}\hat{d}(\iota_{n-1}, \iota_n) + \frac{1}{2}\hat{d}(\iota_n, \iota_{n+1})\right)\right).$$

For all $n \in \mathbb{N}$, thus

$$\begin{aligned} &\hat{d}(\iota_n, \iota_{n+1}) \leq qs\left(\frac{3}{2}\hat{d}(\iota_{n-1}, \iota_n) + \frac{1}{2}\hat{d}(\iota_n, \iota_{n+1})\right) \\ &\hat{d}(\iota_n, \iota_{n+1}) < qs\left(2\hat{d}(\iota_n, \iota_{n-1})\right) \\ &\hat{d}(\iota_n, \iota_{n+1}) < 2sq\hat{d}(\iota_n, \iota_{n-1}) \end{aligned}$$

(6) $$\hat{d}\left(\iota_n, \iota_{n+1}\right) < \gamma \hat{d}\left(\iota_n, \iota_{n-1}\right), \quad \gamma = 2qs < 1.$$

Using (6) and Lemma 2.2, we conclude that the sequence $\{\iota_n\}$ is a $0-\hat{d}-$ Cauchy sequence. Since $(\Lambda, \hat{d})$ is $0-$ $\hat{d}$-complete, there exists $\bar{\iota} \in \Lambda$ such that $\lim_{n\to+\infty} \iota_n = \bar{\iota}$.
Equivalently, we have

$$\hat{d}(\bar{\iota}, \bar{\iota}) = \lim_{n\to+\infty} \hat{d}\left(\bar{\iota}, \iota_n\right) = 0.$$

And using (iii), η is α-continuous. Then

$$\bar{\iota} = \lim_{n\to+\infty} \iota_{2n+2} = \lim_{n\to+\infty} \eta\iota_{2n+1} = \eta\bar{\iota},$$

i.e., $\bar{\iota}$ is a fixed point of η, and Υ is α-continuous. Then

$$\bar{\iota} = \lim_{n\to+\infty} \iota_{2n+1} = \lim_{n\to+\infty} \Upsilon\iota_{2n} = \Upsilon\bar{\iota},$$

i.e., $\bar{\iota}$ is a fixed point of Υ; thus, it is a common fixed point.

For the uniqueness of the common fixed point of the pair (Υ, η), suppose that $\bar{\iota}, \bar{\kappa}(\bar{\iota} \neq \bar{\kappa})$ are two fixed points of Υ.
Since $\bar{\iota}, \bar{\kappa} \in$ Fix $(\Upsilon)\cap$Fix(η), we have $\hat{d}(\bar{\iota}, \bar{\kappa}) > 0$ and $\alpha(\bar{\kappa}, \bar{\iota}) \geq 1$ (according to (iv)).
So, $\max(\alpha(\bar{\iota}, \Upsilon\bar{\kappa}); \alpha(\bar{\iota}, \eta\bar{\kappa})) \geq 1$, we have

$$\hat{d}(\bar{\iota}, \bar{\iota}) = \hat{d}(\bar{\kappa}, \bar{\kappa}) = 0.$$

Further, using (1)

$$\pi(\hat{d}(\Upsilon\bar{\iota}, \eta\bar{\kappa})) \leq f\left(\pi\left(\Gamma_{\bar{\iota},\bar{\kappa}}\right), \varpi\left(\Gamma_{\bar{\iota},\bar{\kappa}}\right)\right),$$

where

$$\begin{aligned}
\Gamma_{\bar{\iota},\bar{\kappa}} &= q\max\left\{\hat{d}(\bar{\iota}, \bar{\kappa}), \hat{d}\left(\bar{\iota}, \Upsilon_{\bar{\iota}}\right), \hat{d}(\bar{\kappa}, \eta\bar{\kappa}), \frac{\hat{d}(\bar{\iota}, \eta\bar{\kappa}) + \hat{d}\left(\bar{\kappa}, \Upsilon_{\bar{\iota}}\right)}{2}\right\} \\
\Gamma_{\bar{\iota},\bar{\kappa}} &= q\max\left\{\hat{d}(\bar{\iota}, \bar{\kappa}), \frac{\hat{d}(\bar{\iota}, \eta\bar{\kappa}) + \hat{d}\left(\bar{\kappa}, \Upsilon_{\bar{\iota}}\right)}{2}\right\} \\
\Gamma_{\bar{\iota},\bar{\kappa}} &= q\hat{d}(\bar{\iota}, \bar{\kappa}),
\end{aligned}$$

so

$$\pi(\hat{d}(\bar{\iota}, \bar{\kappa})) \leq f(\pi(q\hat{d}(\bar{\iota}, \bar{\kappa})), \varpi(q\hat{d}(\bar{\iota}, \bar{\kappa}))) \leq f(\pi(\hat{d}(\bar{\iota}, \bar{\kappa})), \varpi(\hat{d}(\bar{\iota}, \bar{\kappa}))),$$

that is $\hat{d}(\bar{\iota}, \bar{\kappa}) = 0$, which is a contradiction. This finishes the proof. □

Theorem 2.4. *Let $(\Lambda, \hat{d})$ be a 0-$\hat{d}$-complete b-metric-like space with a constant s. Let p be a positive integer, $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$ be nonempty $\hat{d}$-closed subsets of Λ, $\Delta = \bigcup_{i=1}^{p} \Gamma_i$ and $\alpha : \Lambda \times \Lambda \to [0, +\infty)$be a mapping.Assume that $(\Upsilon; \eta) : \Delta \to \Delta$ is a cyclic $\alpha-$ $f\pi\varpi-$ contractive ordered pair of mappings satisfying the following assertions:*
(i) Υ and η are α-admissible mappings,
(ii) $\alpha\left(\iota_0, \eta(\Upsilon\iota_0)\right) \geq 1$ for an element ι_0 in Δ,
(iii) Υ and η are α-continuous.
Then (Υ, η) admits a common fixed point in $\bigcap_{i=1}^{p} \Gamma_i$. Moreover, if
(iv) $\alpha(\iota, \kappa) \geq 1$, whenever $\iota, \kappa \in$ Fix $(\Upsilon)\cap$Fix(η), then (Υ, η) admits a unique common fixed point.

Proof. Define Picard's sequence $\iota_{2n+1} = \Upsilon\iota_{2n}$, and $\iota_{2n+2} = \eta\iota_{2n+1}$ where ι_0 is the given point in Δ for which $\alpha\left(\iota_0, \eta(\Upsilon\iota_0)\right) = \alpha\left(\iota_0, \iota_2\right) \geq 1$. Then there exists some i_0 such that $\iota_0 \in \Gamma_{i_0}.\Upsilon\left(\Gamma_{i_0}\right) \subseteq \Gamma_{i_0+1}$ implies that

$$\iota_1 = \Upsilon\iota_0 \in \Gamma_{i_0+1},$$

and so

$$\iota_2 = \eta\iota_1 \in \Gamma_{i_0+2}.$$

Since Υ is an α-admissible mapping, we get that $\alpha\left(\Upsilon\iota_0, \Upsilon\iota_2\right) = \alpha\left(\iota_1, \iota_3\right) \geq 1$ and $\iota_3 \in \Gamma_{i_0+3}$. Again. Since η is an α-admissible mapping , it follows that $\alpha\left(\eta\iota_1, \eta\iota_3\right) = \alpha\left(\iota_2, \iota_4\right) \geq 1$, and $\iota_4 \in \Gamma_{i_0+4}$.

Continuing this process we have $\alpha(\iota_n, \iota_{n+2}) \geq 1$ and there exists $i_n \in \{1, 2, \ldots, p\}$ such that $\iota_n \in \Gamma_{i_n}$ for all $n \in \mathbb{N}$, and so

$$\max(\alpha(\iota_{2n}, \eta\iota_{2n+1}), \alpha(\iota_{2n}, \Upsilon\iota_{2n+1})) \geq 1,$$

for all $n \in \mathbb{N}$. Set

$$\Gamma = q \max\left(\hat{d}(\iota_{2n+1}, \iota_{2n}), \hat{d}(\iota_{2n}, \Upsilon\iota_{2n}), \hat{d}(\iota_{2n+1}, \eta\iota_{2n+1}), \frac{\hat{d}(\iota_{2n}, \eta\iota_{2n+1}) + \hat{d}(\iota_{2n+1}, \Upsilon\iota_{2n})}{2}\right).$$

By (1) and (2) we get

$$\begin{aligned} \pi\left(\hat{d}(\iota_{2n+1}, \iota_{2n+2})\right) &= \pi\left(\hat{d}(\Upsilon\iota_{2n}, \eta\iota_{2n+1})\right) \\ &\leq f(\pi(\Gamma), \varpi(\Gamma)), \end{aligned} \tag{7}$$

by the same method as Theorem 2.3, we find that $\{\iota_n\}$ is a $0 - \hat{d}-$ Cauchy sequence. Since Δ is $\hat{d}$-closed in $(\Lambda, \hat{d})$, this means that there exists a unique $\bar{\iota} \in \Delta$ such that

$$\hat{d}(\bar{\iota}, \bar{\iota}) = \lim_{n\to+\infty} \hat{d}(\iota_n, \bar{\iota}) = \lim_{n,m\to+\infty} \hat{d}(\iota_n, \iota_m) = 0.$$

Further, as $\Upsilon(\Gamma_i) \subseteq \Gamma_{i+1}$, $\eta(\Gamma_j) \subseteq \Gamma_{j+1}$ and$\Gamma_{p+1} = \Gamma_1$, it follows that the sequence $\{\iota_n\}$ has infinitely many terms in each Γ_i for $i \in \{1, 2, \ldots, p\}$. Hence, we have the subsequences $\{\iota_{n_i}\}$ of $\{\iota_n\}$ where $\{\iota_{n_i}\} \subseteq \Gamma_i, i = 1, 2, \ldots, p$. It is clear that each ι_{n_i} converges to $\bar{\iota}$. It follows that $\Gamma = \cap_{i=1}^{p}\Gamma_i \neq \emptyset$, because it contains at least the element $\bar{\iota}$. Obviously, $(\Gamma, \hat{d})$ is a $0 - \hat{d}-$ complete b-metric-like space and $\Upsilon : \Gamma \to \Gamma$. Also, the restrictions $\Upsilon|_\Gamma$ and $\eta|_\Gamma$ of $\Upsilon$and η on Γ satisfy all conditions of Theorem2.3.
Hence, (Υ, η) admits a unique common fixed point $\bar{\iota}$ in Γ. This completes the proof of the theorem. □

Corollary 2.5. *Let $\left(\Lambda, \tilde{d}\right)$ be a $\tilde{d}-$complete metric-like space (see [2]). And $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ be a mapping. Assume that $(\Upsilon, \eta) : \Lambda \to \Lambda$ is a $\alpha - F\psi\varpi-$contractive ordered pair of mappings satisfying the following assertions*
(i) *Υ and η are $\alpha-$admissible mappings,*
(ii) *$\alpha(\iota_0, \eta(\Upsilon\iota_0)) \geq 1$ for an element ι_0 in Λ,*
(iii) *Υ and η are $\alpha-$continuous.*
Then Υ and η have common fixed point in Λ.
Moreover, if
(iv) *$\alpha(\iota, \kappa) \geq 1$, whenever $\iota, \kappa \in Fix(\Upsilon) \cap Fix(\eta)$, then Υ and η have a unique common fixed point in Λ.*

Proof. By setting $s = 1$ in Definition 2.1 and applying the Theorem 2.3, we obtain the result. □

Remark 3. *In Corollary 2.5, if we take $\Upsilon = \eta$ and then replace condition (ii) with*

$$\alpha(\iota_0, \Upsilon\iota_0) \geq 1,$$

for an element ι_0 in Λ,we will obtain a proof that Υ has a fixed point. And it is similar to the main result found in [7].

Example 2.6. *Let $\Lambda = [0, +\infty)$ and define the b-metric-like function $\hat{d} : \Lambda \times \Lambda \to [0, +\infty)$ by*

$$\hat{d}(\iota, \kappa) = (\iota + \kappa)^2,$$

this b-metric-like has a constant $s = 2$.
Define $\pi, \varpi : [0, +\infty) \to [0, +\infty)$ by

$$\pi(t) = \frac{t}{16}, \quad \varpi(t) = 1,$$

and the C-class function

$$f(s, t) = \frac{s}{1 + t}.$$

Let $\eta : \Lambda \to \Lambda$ *be the function defined as follows*

$$\eta(\iota) = \begin{cases} \frac{\iota}{4}, & \text{if } \iota \in [0, 0.25] \\ \iota, & \text{if } \iota > 0.25, \end{cases}$$

and define $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ *by*

$$\alpha(\iota, \kappa) = \begin{cases} 5, & \text{if } (\iota, \kappa) \in [0, 0.25]^2 \\ 0, & \text{otherwise.} \end{cases}$$

Let $\Upsilon : \Lambda \to \Lambda$ *be defined by*

$$\Upsilon(\iota) = \begin{cases} 0, & \text{if } \iota \in [0, 0.25] \\ \iota, & \text{if } \iota > 0.25. \end{cases}$$

Now we take

$$M_{\Upsilon;\eta}(\iota, \kappa) = q \cdot \max\left\{ \hat{d}(\iota, \kappa),\ \hat{d}(\iota, \Upsilon(\iota)),\ \hat{d}(\kappa, \eta(\kappa)),\ \frac{\hat{d}(\iota, \eta(\kappa)) + \hat{d}(\kappa, \Upsilon(\iota))}{2} \right\},$$

with $q = \frac{1}{8}$.

We verify that

$$\text{If } \max\left(\alpha(\iota, \Upsilon(\kappa)),\ \alpha(\iota, \eta(\kappa))\right) \geq 1,$$

then

$$\pi\left(\hat{d}(\Upsilon(\iota), \eta(\kappa))\right) \leq f\left(\pi\left(M_{\Upsilon;\eta}(\iota, \kappa)\right), \varpi\left(M_{\Upsilon;\eta}(\iota, \kappa)\right)\right).$$

Let $\iota, \kappa \in [0, 0.25]$. Then

$$\begin{aligned} \Upsilon(\iota) = 0, \eta(\kappa) &= \frac{\kappa}{4} \\ \hat{d}(\Upsilon(\iota), \eta(\kappa)) &= (\frac{\kappa}{4})^2 \leq 0.00390625 \\ M_{\Upsilon;\eta}(\iota, \kappa) &\leq \frac{1}{8} \cdot 0.25 = 0.03125 \\ \pi(M) &= \frac{0.03125}{16} = 0.001953125 \\ f(\pi(M), 1) &= \frac{0.001953125}{2} = 0.0009765625 \\ \pi(\hat{d}(\Upsilon(\iota), \eta(\kappa))) &= \frac{(\frac{\kappa}{4})^2}{16} \leq \frac{0.015625}{16} = 0.0009765625 \end{aligned}$$

Thus, $(\Upsilon; \eta)$ is an $\alpha-$ $f\pi\varpi-$ contractive ordered pair of mappings.
The conditions (i)-(iii) of Theorem 2.3 also hold true here, it follows that Υ and η have a common fixed point.

3. Conclusion

We presented and developed a new family of generalized cyclic $\alpha - f\pi\varpi-$contractions in b-metric-like spaces. With this new hybrid type of contraction, which brings together control functions, admissibility, and cyclic nature, we have provided new existence results for fixed points via several new fixed point theorems. These theorems new results take a general and extensible form from what is found in the literature. The approach presented here not only adds to the theoretical structure of fixed point theory in generalized metric-type spaces, but it also illuminates the flexibility which a b-metric-like framework has to include more general classes of mappings. These efforts pave the way for more work, or the beginning of that work in nonlinear analysis and in general mathematics.

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Sets of functions with the property (V). Density theorems

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Abstract. In this paper, some density theorems in the set $C(X; I)$ of all continuous functions defined on a compact X with values in the unit interval $I = [0, 1]$ are presented. We say that a subset F of $C(X; I)$ has the property (V) if, $1 - f$ and $f \cdot g$ belong to F for any f, g from F.

Von Neumann is the one who drew attention to this collections of functions in [4]. Moreover, he claims, without proof, a density theorem for such families of functions. A careful analysis of these sets and their properties was made by R.I. Jewett in [3]. In this paper, almost all the results from [3] are generalized. Thus we mentioned Theorems 4.1 and 4.6, which generalize Von Neumann's density Theorem and also Theorem 1 of [3]. We also mention Theorems 4.1 and 4.8 which connect the properties (V) and the Uryson property and allow to recover most of the results from [5] and [6].

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1. Introduction

If X is a Hausdorff compact space and $I = [0, 1]$, then we will denote by

$$C(X; I) = \{f : X \to I, \text{ continuous } \}.$$

Clearly, duality theory does not apply for the set $C(X; I)$, because this set is not a vector space.

Definition 1.1. *We say that a subset $F \subset C(X; I)$ has the property (V) if:*

$$\textit{(i) } 1 - f \in F, \forall f \in F;$$

$$\textit{(ii) } f \cdot g \in F, \forall f, g \in F.$$

The result stated by Von Neumann in [4], without proof, says that any subset $F \subset C(I^n; I)$ which has the property (V), contains the projections and at least a constant $c \in (0, 1)$ is dense in $C(I^n; I)$.This result was proved by R.I. Jewett in [3].

1.1. Preliminaries.

Definition 1.2. *We say that a subset $F \subset C(X; I)$ has the property (VN) if:*

$$\varphi \cdot f + (1 - \varphi) \cdot g \in F, \forall \varphi, f, g \in F.$$

Remark 1. *If the subset F has the property (VN) and contains the constant functions 0 and 1, then F has the property (V).*

Indeed, $1 - \varphi \in F, \forall \varphi \in F$, because $1 - \varphi = \varphi \cdot 0 + (1 - \varphi) \cdot 1$ and $\varphi \cdot f \in F, \forall \varphi, f \in F$, because $\varphi \cdot f = \varphi \cdot f + (1 - \varphi) \cdot 0$.

The definition of a family of functions that has the (VN) property and some density results for such families were studied by J.B. Prolla in [6].

Also, these families of functions were also studied in [5], where the connection between the property (VN) and the notion of Uryson families was shown.

It is clear that the closure of a set which has the property (V) is also a set which has the property (V) and any intersection of such sets has the property (V), also. Thus, any subset of $C(X; [0, 1])$ is contained in a smallest subset which has the property (V), respectively in a smallest closed subset with this property.

In the particular case where $X = I^n$, we will denote by $\mathcal{P}_n$ the smallest subset of the set $C(I^n; I)$ that has the property (V) and contains the n projections, i.e. the functions: $(x_1, \ldots, x_i, \ldots, x_n) \to x_i : I^n \to I, \forall i = \overline{i, n}$.

For instance $\mathcal{P}_1$ is the smallest subset of having the property (V) containing the identity $x \to x : I \to I$. The examples of elements of $\mathcal{P}_1$ are: $x; 1-x; x^m$; $1 - x^m; (1-x^m)^n; \ldots$.

Remark 2. *Let $F \subset C(X; [0,1])$ be a subset which has the property (V) and $p \in \mathcal{P}_n$. If we denote by $h : X \to [0,1]$ the function:*

$$h(x) = p[f_1(x), \ldots, f_n(x)], \forall x \in X,$$

then $h \in F, \forall f_1, \ldots, f_n \in F$.

Indeed, if we denote by $\mathcal{Q}$ the set of all functions $q : I^n \to [0,1]$ with the property that $q(f_1, \ldots, f_n) \in F, \forall f_1, \ldots, f_n \in F$, then the set $\mathcal{Q}$ has the property (V) and contains the projection functions, so $\mathcal{Q} \supset \mathcal{P}_n$.

The following Lemma plays a central role in the entire work of [3] and also in this work. A nice and simple proof of this lemma can be found in [5].

Lemma 1.3 (Jewett [3])**.** *Let $a, b \in \mathbb{R}$ be two real numbers such that $0 \leq a < b \leq 1$. Then, for any $\varepsilon > 0$ there are $m, n \in \mathbb{N}^*$ such that the polynomial function:*

$$p(x) = (1 - x^m)^n, x \in [0,1]$$

has the following properties:
$(i) p(x) > 1 - \varepsilon, \forall x \in [0, a]$;
$(ii) p(x) < \varepsilon, \forall x \in [b, 1]$.

Corollary 1.4. *(See [5], Theorem 2.1) Let $k \in \mathbb{N}^*, k \geq 2$ and let $a, b \in \mathbb{R}$ be two real numbers such that $0 \leq a < \frac{1}{k} < b \leq 1$. Then, for any $\varepsilon > 0$, there are $m \in \mathbb{N}^*$ such that the polynomial function:*

$$p(x) = (1 - x^m)^{k^m}, x \in [0,1]$$

has the following properties:
$(i) p(x) > 1 - \varepsilon, \forall x \in [0, a]$;
$(ii) p(x) < \varepsilon, \forall x \in [b, 1]$.

The statement follows from Lemma 1.3, if we choose $n = k^m$ and take into account that $k \cdot b > 1$.

2. Connection with Uryson families

Lemma 2.1. *For any point interior point $a = (a_1, \ldots, a_n) \in I^n$ and any $\varepsilon > 0, \delta > 0$, there exists a polynomial function $p \in \mathcal{P}_n$ with the properties:*

$$p(x) > (1-\varepsilon)^{2n}, \forall x = (x_1, \ldots, x_n) \in [a_1 - \delta, a_1 + \delta] \times \ldots \times [a_n - \delta, a_n + \delta]$$
$$p(x) < \varepsilon, \forall x = (x_1, \ldots, x_n) \in I^n \backslash [a_1 - 2\delta, a_1 + 2\delta] \times \ldots \times [a_n - 2\delta, a_n + 2\delta].$$

Proof. From Lemma 1.3, it follows that there are $2n$ polynomial functions $p_i, p_i' \in \mathcal{P}_1, i = \overline{1, n}$ such that:

(i) $p_i(t) > 1 - \varepsilon, \forall t \in [0, a_i + \delta]$;
(ii) $p_i(t) < \varepsilon, \forall t \in [a_i + 2\delta, 1]$;

and

(iii) $p_i'(t) > 1 - \varepsilon, \forall t \in [a_i - \delta, 1]$;
(iv) $p_i'(t) < \varepsilon, \forall t \in [0, a_i - 2\delta]$;

If we denote by: $A = [a_1 - \delta, a_1 + \delta] \times \ldots \times [a_n - \delta, a_n + \delta]$ and $B = [a_1 - 2\delta, a_1 + 2\delta] \times \ldots \times [a_n - 2\delta, a_n + 2\delta]$, then the product function: $p : I^n \to I$, defined by $p(x) = p(x_1, \ldots, x_n) = \prod_{i=1}^{n} p_i(x_i) \cdot p_i'(x_i), \forall x = (x_1, \ldots, x_n) \in I^n$ belongs to $\mathcal{P}_n$ and it has the properties:

$$p(x) > (1-\varepsilon)^{2n}, \forall x = (x_1, \ldots, x_n) \in A$$
$$p(x) < \varepsilon, \forall x = (x_1, \ldots, x_n) \in I^n \backslash B.$$

and so the proof is finished. ∎

The set of functions with Uryson property was introduced by I.Bucur in [5].

Definition 2.2. *We say that a family $U \subset C(X;I)$ has the Uryson property on X if for any two disjoint closed subset A, B of X and any $\varepsilon \in (0,1)$ there exists a function $u \in U$ such that:*

$$u(x) \geq 1-\varepsilon, \forall x \in A \text{ and } u(x) \leq \varepsilon, \forall x \in B.$$

The main result regarding Uryson families is Theorem 3.3 from [5] which establishes that if $U \subset C(X;I)$ has the Uryson property, then:

$$\overline{\text{co}}(U) = C(X;I).$$

Theorem 2.3. *The set $\mathcal{P}_n$ has the Uryson property.*

Proof. Let $A, B \subset I^n$ be two closed disjoint closed subsets such that $d(A,B) = 4r > 0$. For any point $a = (a_1, \ldots, a_n) \in A$ we have $D(a; 2r) \cap B = \emptyset$, where:

$$D(a;2r) = [a_1 - 2r, a_1 + 2r] \times \ldots \times [a_n - 2r, a_n + 2r].$$

Taking into account that the subset A is compact it results that there is a finite number of points $a_k = (a_{k1}, \ldots, a_{kn}) \in A, k = \overline{1,m}$ such that:

$$A \subset \bigcup_{k=1}^{m} D(a_k; r).$$

According to Lemma 2.1, for any $\varepsilon > 0$, there exists $p_k \in \mathcal{P}_k$ such that:

$$p_k > \left(1 - \frac{\varepsilon}{4nm}\right)^{2n} \text{ on } D(a_k;r) \text{ and } p_k < \left(\frac{\varepsilon}{4nm}\right) \text{ on } I^n \backslash D(a_k;2r) \supset B.$$

If we denote by $q_0 = (1-p_1)(1-p_2)\ldots(1-p_m)$, then $q_0 \in \mathcal{P}_n$ and, for any $a \in A$ there exists $k = \overline{1,m}$ such that $a \in D(a_k;r)$, whence it follows:

$$p_k(a) > \left(1 - \frac{\varepsilon}{4 \cdot n \cdot m}\right)^{2 \cdot n} \geq \left(1 - \frac{\varepsilon}{4 \cdot n}\right)^{2 \cdot n} \geq 1 - 2 \cdot n \cdot \frac{\varepsilon}{4 \cdot n} > 1 - \frac{\varepsilon}{2} > 1 - \varepsilon.$$

Further we have:

$$q_0(a) \leq 1 - p_k(a) < \varepsilon, \forall a \in A.$$

On the other hand, if $b \in B$, then $p_k(b) < \left(\frac{\varepsilon}{4mn}\right) \leq \frac{\varepsilon}{4m}$ whence it follows:

$$q_0(b) > \left(1 - \frac{\varepsilon}{4m}\right)^m \geq 1 - \frac{\varepsilon}{4} > 1 - \varepsilon, \forall b \in B.$$

□

3. Main results

Lemma 3.1 (Jewett [3])**.** *Let $A, B \subset I^n$ be two closed disjoint subsets. Then for any $\varepsilon > 0$ and any $p \in \mathcal{P}_n$, there are $q, \tilde{q} \in \mathcal{P}_n$ with the properties:*

(a) $q \geq p$, on I^n	*(a') $\tilde{q} \leq p$, on I^n*
(b) $q > 1-\varepsilon$, on A	*(b') $\tilde{q} < \varepsilon$, on A*
(c) $q < p + \varepsilon$, on B	*(c') $\tilde{q} > p - \varepsilon$, on B*

Proof. Let $A, B \subset I^n$ be two closed disjoint closed subsets.
According to Theorem 2.3 there exists a function $q_0 \in \mathcal{P}_n$ with the properties:

$$q_0 < \varepsilon \text{ on } A \text{ and } q_0 > 1 - \varepsilon \text{ on } B.$$

Let $p \in \mathcal{P}_n$ be arbitrary and let $q = 1 - (1-p) \cdot q_0$. Then we have:

$$q = 1 - (1-p) \cdot q_0 \geq 1 - (1-p) = p.$$

If $a = (a_1, \ldots, a_n) \in A$, then:

$$q(a) = 1 - (1 - p(a)) \cdot q_0(a) \geq 1 - q_0(a) > 1 - \varepsilon.$$

On the subset B we have:

$$q - p = 1 - q_0 + p \cdot q_0 - p = (1-q_0)(1-p) < 1 - q_0 < \varepsilon.$$

Similarly, for $1-p$ we deduce that there is $q_0' \in \mathcal{P}_n$ with the properties:

$$q' \geq 1 - p \text{ on } I^n$$
$$q' > 1 - \varepsilon \text{ on } A$$
$$q' < 1 - p + \varepsilon \text{ on } B.$$

Obviously, if we denote by $\tilde{q} = 1 - q'$, then we have:

$$\begin{array}{ll} \text{(a') } \tilde{q} \leq p, & \forall (x,y) \in I^n \\ \text{(b') } \tilde{q} < \varepsilon, & \forall (x,y) \in A \\ \text{(c') } \tilde{q} > p - \varepsilon, & \forall (x,y) \in B. \end{array}$$

□

Lemma 3.2 (Jewett [3])**.** *For any $a_k, b_k \in [0,1], k \in \{1,2,\ldots,n\}$ we have:*

$$\left| \prod_{k=1}^{n} a_k - \prod_{k=1}^{n} b_k \right| \leq \sum_{k=1}^{n} |a_k - b_k|.$$

Proof. We assume by mathematical induction that:

$$\left| a = \prod_{k=1}^{n-1} a_k - b = \prod_{k=1}^{n-1} b_k \right| \leq \sum_{k=1}^{n-1} |a_k - b_k|.$$

Further we have:

$$\begin{aligned} |a \cdot a_n - b \cdot b_n| &\leq |a \cdot a_n - b \cdot a_n| + |b \cdot a_n - b \cdot b_n| \\ &= |a - b| \cdot |a_n| + |b| \cdot |a_n - b_n| \\ &\leq |a - b| + |a_n - b_n| = \sum_{k=1}^{n} |a_k - b_k| \end{aligned}$$

□

Theorem 3.3. *Let $h : I^n \to I$ be a continuous function, $\varepsilon > 0$ be such that $\varepsilon < \frac{1}{4}$ and let $K_\varepsilon = \{x = (x_1, \ldots, x_n) \in I^n; \varepsilon \leq h(x) \leq 1 - \varepsilon\}$. If we additionally assume that there is a non-constant function $p \in \mathcal{P}_n$, such that $1 > p(x_1, \ldots, x_n) > 1 - \varepsilon, \forall (x_1, \ldots, x_n) \in K_\varepsilon$, then there exists a function $q \in \mathcal{P}_n$ such that:*

$$|h(x) - q(x)| < \varepsilon, \forall x \in I^n.$$

Proof. For any $k \in \mathbb{N}^*$ we will denote by:

$$\begin{array}{l} A_k = \left\{ x \in K_\varepsilon; p^k(x) \geq h(x) \right\}, \\ B_k = \left\{ x \in K_\varepsilon; p^k(x) \leq h(x) \right\}. \end{array}$$

Clearly we have: $A_1 = K_\varepsilon, A_k \supset A_{k+1}, B_k \supset K_\varepsilon \backslash A_k, A_{k+1} \cap B_k = \phi$ and
$B_k \subset B_{k+1}$.

Since $\bigcap\limits_{1}^{\infty} A_k = \emptyset$ and K_ε is a compact subset, it follows that there is
$n \in \mathbb{N}, n > 1$ such that $A_n = \emptyset$. Therefore:

$$K_\varepsilon = \bigcup_{k=1}^{n-1} (A_k \backslash A_{k+1}).$$

We notice that if $x \in K_\varepsilon$ then, there is $k = \overline{1, n-1}$ such that $x \in A_k \backslash A_{k+1}$ and so we have:

$$0 \leq p^k(x) - h(x) < p^k(x) - p^{k+1}(x) = p^k(x) \cdot (1 - p(x)) \leq 1 - p(x) < \varepsilon.$$

On the other hand, applying Lemma 3.1 for , there exists $q_k \in \mathcal{P}_n$ with the properties:

$$\begin{array}{l} q_k \geq p, \forall x \in I^n \\ q_k > 1 - \frac{\varepsilon}{n}, \forall x \in B_{k-1} \\ q_k < p + \frac{\varepsilon}{n}, \forall x \in A_k \end{array}$$

If we denote by: $q = q_1 \cdot q_2 \cdot \ldots \cdot q_n$, then $q \in \mathcal{P}_n$ and for any $x \in K_\varepsilon$ we have:

$$|q(x) - h(x)| < 4\varepsilon.$$

Indeed, let $x \in K_\varepsilon$ be an arbitrary point and let $k \in \mathcal{N}^*$ be such that
$x \in A_k \backslash A_{k+1}$. Then, we have:

$$\left|p^k - q\right| \leq \left|p^k - p^{k+1}\right| + \left|p^{k+1} - q\right| < \varepsilon + \left|p^{k+1} - \prod_{j=1}^{n} q_j\right| = \varepsilon + \left|p^{k+1} - \prod_{j=1}^{k+1} q_j + \prod_{j=1}^{k+1} q_j - \prod_{j=1}^{n} q_j\right|$$

$$\leq \varepsilon + \sum_{j=1}^{k} |p - q_j| + |p - q_{k+1}| + \prod_{j=1}^{k+1} q_j \left|1 - \prod_{j=k+2}^{n} q_j\right| \leq \varepsilon + \sum_{j=1}^{k} |p - q_j| + |p - q_{k+1}| + \left|1 - \prod_{j=k+2}^{n} q_j\right|$$

$$\leq \varepsilon + \sum_{j=1}^{k} |p - q_j| + |p - q_{k+1}| + \sum_{j=k+2}^{n} |1 - q_j| \leq \varepsilon + k \cdot \frac{\varepsilon}{n} + (1 - p) + (n - k - 1) \cdot \frac{\varepsilon}{n}$$

$$< \varepsilon + (n-1) \cdot \frac{\varepsilon}{n} + \varepsilon < 3 \cdot \varepsilon.$$

Therefore, if $x \in K_\varepsilon \Rightarrow |q(x) - h(x)| \leq \left|q(x) - p^k(x)\right| + \left|p^k(x) - h(x)\right| < 3 \cdot \varepsilon + \varepsilon = 4 \cdot \varepsilon$. Next we denote by:

$$C = \{x \in I^n; h(x) \geq 1 - \varepsilon\}$$

and by:

$$D = \{x \in I^n; h(x) \leq 1 - 2\varepsilon\}$$

From Lemma 3.1, it results that there exists $q' \in \mathcal{P}_n$ with the properties:

$$q' \geq q, \text{ on } I^n$$
$$q' > 1 - \varepsilon \text{ on } C$$
$$q' < q + \varepsilon \text{ on } D.$$

Let observe that if $h(x) \geq \varepsilon$, then $|q'(x) - h(x)| < 5 \cdot \varepsilon$.
Indeed:

$$\{x \in I^n; h(x) \geq \varepsilon\} = C \cup \{x \in I^n; \varepsilon \leq h(x) \leq 1 - 2 \cdot \varepsilon\} \cup \{x \in I^n; 1 - 2 \cdot \varepsilon \leq h(x) \leq 1 - \varepsilon\}$$

If $x \in C \Rightarrow 1 \geq q'(x) > 1 - \varepsilon$ and $1 \geq h(x) > 1 - \varepsilon \Rightarrow |q'(x) - h(x)| \leq \varepsilon$.
On the set $\{x \in I^n; \varepsilon \leq h(x) \leq 1 - 2\varepsilon\} \subset K_\varepsilon \cap D$ we have:

$$\left|q'(x) - h(x)\right| \leq \left|q'(x) - q(x)\right| + |q(x) - h(x)| < \varepsilon + 4 \cdot \varepsilon = 5 \cdot \varepsilon$$

On the set $\{x \in I^n; 1 - 2\varepsilon \leq h(x) \leq 1 - \varepsilon\} \subset K_\varepsilon$ we have:

$$-4 \cdot \varepsilon \leq q(x) - h(x) \leq q'(x) - h(x) \leq 1 - h(x) \leq 2 \cdot \varepsilon \leq 4 \cdot \varepsilon,$$

whence we deduce that:

$$|q'(x) - h(x)| \leq 4 \cdot \varepsilon.$$

Let $E = \{x \in I^n; h(x) \leq \varepsilon\}$ and $F = \{x \in I^n; h(x) \geq 2\varepsilon\}$. Applying again Lemma 3.1, for E, F and q' it results that there exists $\tilde{q} \in \mathcal{P}_n$ with the properties:

$$\tilde{q} \leq q', \text{ on } I^n$$
$$\tilde{q} < \varepsilon \text{ on } E$$
$$\tilde{q} > q' - \varepsilon \text{ on } F.$$

If $x \in E \Rightarrow |\tilde{q}(x) - h(x)| \leq \tilde{q}(x) + h(x) < 2 \cdot \varepsilon$.
If $x \in F \Rightarrow |\tilde{q}(x) - h(x)| \leq |\tilde{q}(x) - q'(x)| + |q'(x) - h(x)| < \varepsilon + 5 \cdot \varepsilon = 6 \cdot \varepsilon$.
If $\varepsilon \leq h(x) \leq 2\varepsilon$, then $\tilde{q}(x) \leq q'(x) < h(x) + 5 \cdot \varepsilon \leq 2 \cdot \varepsilon + 5 \cdot \varepsilon = 7 \cdot \varepsilon$ and so:

$$|\tilde{q}(x) - h(x)| \leq \tilde{q}(x) + h(x) < 7 \cdot \varepsilon + 2 \cdot \varepsilon = 9 \cdot \varepsilon.$$

Therefore we have:

$$|\tilde{q}(x) - h(x)| \leq 9 \cdot \varepsilon, \forall x \in I^n.$$

□

Theorem 3.4. *For any $\varepsilon > 0$ there exists a function $p \in \mathcal{P}_n$ such that:*

$$|x_1 \wedge ... \wedge x_n - q(x_1, ..., x_n)| < \varepsilon, \forall (x_1, ..., x_n) \in I^n.$$

Proof. The proof follows from Theorem 3.3, for $h(x_1, ..., x_n) = x_1 \wedge ... \wedge x_n$, $\forall (x_1, ..., x_n) \in I^n$. Indeed, if $(x_1, ..., x_n) \in K_\varepsilon = \{(x_1, ..., x_n); \varepsilon \leq x_1 \wedge ... \wedge x_n \leq 1 - \varepsilon\}$, then $0 < x_1 \cdot x_2 \cdot \ldots \cdot x_n < 1 - \varepsilon$. Obviously, there exists $m \in \mathbb{N}^*$ such that $x_1^m \cdot \ldots \cdot x_n^m < 1 - \varepsilon$, whence we deduce that the function $p(x_1, \ldots, x_n) =$
$= 1 - x_1^m \cdot \ldots \cdot x_n^m$ has the properties required in the hypothesis of the Theorem 3.3, that is $p \in \mathcal{P}_n$ and $1 > p(x_1, \ldots, x_n) > 1 - \varepsilon, \forall (x_1, \ldots, x_n) \in K_\varepsilon$.

Since $x_1 \vee ... \vee x_n = 1 - (1 - x_1) \wedge ... \wedge (1 - x_n)$, we deduce that the function

$$(x_1, \dots, x_n) \mapsto x_1 \wedge \dots \wedge x_n : I^n \to I$$

can also be uniformly approximated with a function of $\mathcal{P}_n$. □

Theorem 3.5 (Jewett [3]). *Any closed subset $F \subset C(X; [0,1])$ that has the property (V) is a lattice.*

Proof. According to Theorem 3.4, for any $n \in \mathbb{N}^*$, there exists $p_n \in \mathcal{P}_2$ such that:

$$|p_n(x, y) - x \wedge y| < \frac{1}{n}, \forall (x, y) \in [0, 1] \times [0, 1].$$

In particular, for any $f, g \in F$ and any $x \in X$ we have:

$$|p_n[f(x), g(x)] - f(x) \wedge g(x)| < \frac{1}{n}, \forall x \in X.$$

On the other hand, from Remark 2, it follows that the function $h_n : X \to [0, 1]$ defined by:

$$h_n(x) = p_n[f(x), g(x)], \forall x \in X,$$

belongs to F.
Since $h_n \xrightarrow[X]{u} f \wedge g$ and the subset F is closed, it results that $f \wedge g \in F$, so F is lattice. □

From Remark 1 and Theorem 3.4 it results immediately the following Prolla's result:

Corollary 3.6 (Theorem 4.12, [5]). *Let $M \subset C(X; [0,1])$ be an uniformly closed subset with the $(VN)-$ property, containing the constant functions 0 and 1, then M is a lattice.*

Theorem 3.7 (Von Neumann.). *If we assume that there exists a constant function $c \in \mathcal{P}_n, c \in (0, 1)$, then $\mathcal{P}_n$ is dense in $C(I^n, I)$.*

Proof. Let $h : I^n \to I$ be an arbitrary continuous function, let $\varepsilon \in \left(0, \frac{1}{4}\right)$ be also arbitrary and let $K_\varepsilon = \{x = (x_1, ..., x_n) \in I^n; \varepsilon \le h(x) \le 1 - \varepsilon\}$. If we denote by:

$$r(x) = c \cdot (1 - c \cdot x_1 \cdot \ldots \cdot x_n), \forall x = (x_1, \dots, x_n) \in K_\varepsilon,$$

where c is the function from the hypothesis, then $r \in \mathcal{P}_n$ and $0 < r(x) < 1$.
Clearly, if we choose $\varepsilon < 1 - M$, where $M = \sup\{r(x); \forall x \in K_\varepsilon\} < 1$, then:

$$r(x) < 1 - \varepsilon, \forall x \in K_\varepsilon.$$

Obviously, there is $m \in \mathbb{N}^*$ such that $r^m(x) < \varepsilon$. If we denote now by
$p(x) = 1 - r^m(x), \forall x \in I^n$, then $p \in \mathcal{P}_n$ and $1 > p(x) > 1 - \varepsilon, \forall x \in K_\varepsilon$.
The proof follows now from Theorem 3.3. □

Lemma 3.8. *For any $0 < a < b < 1, \lambda \in (0, 1)$ there exists a function $\varphi \in \mathcal{P}_1$ with the properties:*

$$|\varphi(a) - \lambda| < \varepsilon \text{ and } \varphi(x) > 1 - \varepsilon, \forall x \in [b, 1].$$

Proof. According to Lemma 1.3, there are $m, r \in \mathbb{N}^*$ such that the polynomial function:

$$p(x) = (1 - x^m)^r, x \in [0, 1]$$

has the following properties:
(i) $p(x) > 1 - \varepsilon, \forall x \in [0, a]$
(ii) $p(x) < \varepsilon, \forall x \in [b, 1]$.
Obviously, we can suppose that $\varepsilon < 1 - \lambda$. Then we have $p(a) > 1 - \varepsilon > \lambda$, and so, there is $k \in \mathbb{N}$ such that $p^k(a) \ge \lambda > p^{k+1}(a)$.
Since $p^k(a) - p^{k+1}(a) = p^k(a)[1 - p(a)] < 1 - p(a) < \varepsilon$, it follows that:

$$\left|p^k(a) - \lambda\right| < \varepsilon.$$

If we denote by $n = r \cdot k$, then then the function $\varphi'(x) = p^n(x)$ has the properties:

$$\left|\varphi'(a) - \lambda\right| < \varepsilon \text{ and } \varphi'(x) < \varepsilon, \forall x \in [b, 1].$$

Similarly, for a, b and $1 - \lambda$, there exists a function $\varphi'' \in \mathcal{P}_1$ such that:

$$\left|\varphi''(a) - (1 - \lambda)\right| < \varepsilon \text{ and } \varphi''(x) < \varepsilon, \forall x \in [b, 1].$$

If we denote by $\varphi = 1 - \varphi''$, then $\varphi \in \mathcal{P}_1$ and we have:

$$|\varphi(a) - \lambda| < \varepsilon \text{ and } \varphi(x) > 1 - \varepsilon, \forall x \in [b, 1].$$

□

Theorem 3.9. *Let $a, b \in \mathbb{R}, 0 < a < b < 1$ and let $r_1, r_2 \in (0,1)$ be two arbitrary numbers. Then for any $0 < \varepsilon < 1$ there exists function $q \in \mathcal{P}_1$ such that:*

$$|q(a) - r_1| < \varepsilon, |q(b) - r_2| < \varepsilon.$$

Proof. From Lemma 3.8 it results that there is a function $\varphi \in \mathcal{P}_1$ with the propertiesL

$$|\varphi(a) - r_1| < \varepsilon \text{ and } \varphi(x) > 1 - \varepsilon, \forall x \in [b, 1].$$

Through a slight change of the proof of Lemma 3.8 it can be shown that there is a function of $\psi \in \mathcal{P}_1$ with the properties:

$$|\psi(b) - r_2| < \varepsilon \text{ and } \psi(x) > 1 - \varepsilon, \forall x \in [0, a].$$

Indeed, it is sufficient to replace the function p with $q = 1 - p$ and a with b.
We note now that the function $q = \varphi \cdot \psi \in \mathcal{P}_1$ satisfies the requirements of the statement of the theorem. Indeed, we have:

$$r_1 - \varepsilon < \varphi(a) < r_1 + \varepsilon \text{ and } 1 - \varepsilon < \psi(a) < 1,$$

whence it is easily deduced that:

$$r_1 - 2\varepsilon < q(a) < r_1 + 2\varepsilon \text{ hence } |q(a) - r_1| < 2\varepsilon.$$

Similarly we have:

$$|q(b) - r_2| < 2\varepsilon.$$

□

Theorem 3.10. *Let X be an arbitrary set and let $F \subset C(X; I)$ be a subset which has the property (V), separates the points of X and contains at least function constant $c \in (0,1)$. Then for any $\varepsilon \in (0,1)$, any $u, v \in X, u \neq v$ and any $r_1, r_2 \in [0,1]$ there exists $H \in F$ such that:*

$$|H(u) - r_1| < \varepsilon, \quad |H(v) - r_2| < \varepsilon.$$

Proof. Let $u, v \in X, u \neq v$ be two arbitrary points. Since F separates the points of X it results that there exists $g \in F$ such that $g(u) \neq g(v)$. Let's suppose that $g(u) < g(v)$. Without restricting the generality we can assume that $g(u) + g(v) \neq 1$. For the moment we will show that there exists $h \in F$ such that: $h(u), h(v) \in (0,1)$ and $h(u) \neq h(v)$. Indeed, if we denote by $h_1 = c \cdot g$, then we have:

$$h_1(u) < h_1(v) < 1,$$

whence we deduce that $h_1(v) \in (0,1)$ and $h_1 \in F$. If $h_1(u) > 0$, the assertion is proved.
If $h_1(u) = 0$, then the function $h_1' = 1 - h_1 \in F$ and has the properties:

$$h_1'(v) \in (0,1) \text{ and } h_1'(u) = 1.$$

If we denote by $h_2 = c(1 - g)$, then we have:

$$h_2(v) < h_2(u) < 1,$$

so, $h_2(u) \in (0,1)$ and $h_2 \in F$. If $h_2(v) > 0$, the assertion is proved. Otherwise, we denote by $h_2' = 1 - h_2 \in F$ and we have:

$$h_2'(u) \in (0,1) \text{ and } h_2'(v) = 1.$$

It is obvious the function $h = h_1' \cdot h_2'$ has the properties:

$$h \in F, h(u), h(v) \in (0,1).$$

In addition we have:

$$h(u) \neq h(v).$$

Indeed, if we assume that $h(u) = h(v)$, then it results $g(u) + g(v) = 1$, which contradicts the above hypothesis. Let's suppose that $h(u) < h(v)$. Then, taking $a = h(u), b = h(v)$ and $r_i \in (0,1), i = 1,2$ in the Theorem 3.7 we deduce that there exists $q \in \mathcal{P}_1$ such that:

$$|q(a) - r_1| < \varepsilon, \quad |q(b) - r_2| < \varepsilon.$$

Clearly, the function $H = q \circ h \in F$ and:

$$|H(u) - r_1| < \varepsilon, |H(v) - r_2| < \varepsilon.$$

□

Theorem 3.11. *Let X be a compact Hausdorff space and let $F \subset C(X;I)$ be a closed subset which has the property (V), separates the points of X and contains at least function constant $c \in (0,1)$. Then:*

$$F = C(X,I).$$

Proof. According to Theorem 3.4,F is a lattice and therefore, according to Lemma 16.3 of [2], it is sufficient to show that any function $f \in C(X;I)$ can be approximated by a function from F at any two points $u, v \in X$. Let $f \in C(X;I)$ be arbitrary and let $u, v \in X, u \neq v$ be two arbitrary points. From Theorem [2] for $r_1 = f(u), r_2 = f(v)$ it follows that there exists $H \in F$ such that:

$$|H(u) - f(u)| < \varepsilon, |H(v) - f(v)| < \varepsilon,$$

and with this theorem is proved. □

Theorem 3.12 (Stone-Weierstrass)**.** *Let X be a compact Hausdorff space and let $A \subset C(X;\mathbb{R})$ be an algebra which separates the points of X and contains the constant functions. Then:*

$$\bar{A} = C(X;\mathbb{R}).$$

Proof. If we denote by $A_1 = \{a \in A; 0 \leq a \leq 1\}$, then A_1 fulfills the conditions of Theorem 3.11, so $\bar{A}_1 = C(X;I)$. Since for any $f \in C(X;\mathbb{R})$ the function $\frac{f+\|f\|}{\|f+\|f\|\|} \in \bar{A}_1$ it follows that $f \in \bar{A}$. □

Corollary 3.13 (Theorem 4.18 , [5])**.** *Let X be a compact Hausdorff space and let $F \subset C(X;[0,1])$ be a closed subset which has the property (VN), separates the points of X, contains the constant functions 0 and 1 at least one constant function $c \in (0,1)$. Then:*

$$F = C(X;I).$$

The statement follows from Remark 1 and Theorem 3.11.

Lemma 3.14. *Let $F \subset C(X;[0,1])$ be a closed subset with property (V), which separates the points of X. Let $x_0 \in X$ and U be an open neighborhood of x_0. Then there exists an open neighborhood V of $x_0, V \subset U$ such that for every $0 < \varepsilon < 1$ there is a $f \in F$ with the properties:*
1) $f(x) < \varepsilon$ if $x \notin U$,
2) $f(x) > 1 - \varepsilon$ if $x \in V$.

Proof. Let $K = X \backslash U$. Obviously, K is a compact subset. If $y \in K$, then $y \neq x_0$ and so there is a $f_y \in F$ such that $f_y(y) \neq f_y(x_0)$. We suppose that $f_y(y) < f_y(x_0)$. From Lemma 1.3 and $\varepsilon = \frac{1}{4}$, it results that there exists a polynomial of the form $p_y(x) = (1 - x^m)^n$ such that :

$$p_y(f_y(y)) > \frac{3}{4} \text{ and } p_y(f_y(x_0)) < \frac{1}{4}.$$

Let $W_y = \left\{x \in X; p_y(f_y(x)) > \frac{3}{4}\right\}$. Since F has property (V), it follows that $p_y(f_y) \in F$. Clearly, W_y is disjoint from x_0. By the compactness of K we infer that there are $y_i \in K, i = 1, 2, \ldots, k$ such that:

$$K \subset W_{y_1} \cup \ldots \cup W_{y_k}.$$

We shall denote by: $g_i = p_{y_i}(f_{y_i}), i = 1, \ldots, k$ and by:

$$h = g_1 \vee ... \vee g_k.$$

As F is lattice it results that $h \in F$. We notice that $h(x_0) < \frac{1}{4}$ and $h(x) > \frac{3}{4}, \forall x \in K$. Let $V = \left\{x \in X; h(x) < \frac{1}{4}\right\}$. Obviously, $x_0 \in V \subset U$ and V is an open neighborhood of x_0. If one denotes by:

$$q_n = (1 - h^n)^{3^n},$$

then $q_n \in F$. On the other hand, from Corollary 1.3 (for $a = \frac{1}{4} < \frac{1}{k} = \frac{1}{3} < b =$ $= \frac{3}{4}$) it results that $\{q_n\}_n$ converges to 1 on V and to 0 on K. Therefore for n sufficiently large $f = q_n > 1 - \varepsilon$ on V and $f < \varepsilon$ on $K = X \backslash U$. □

Theorem 3.15. *Let $F \subset C(X;[0,1])$ be a closed subset with the property (V), which separates the points of X. Then F is an Uryson family on X .*

Proof. Let A, B be two disjoint closed subsets of and A. If we denote with $U = X\backslash B$, then U is an open neighborhood of A. From Lemma 3.8 it results that for every $x \in A$ there exists an open neighborhood of $x, V_x \subset U$ and a function $f_x \in F$ such that:

$$\begin{matrix} f_x < \varepsilon \text{ on } V_x \\ f_x > 1-\varepsilon \text{ on } B \end{matrix}.$$

As A is a compact set and, $A \subset \bigcup\limits_{x\in A} V_x$, it follows that there are $x_1, \ldots, x_n \in A$ such that $A \subset \bigcup\limits_{i=1}^{n} V_{x_i}$. According again to Lemma 3.8 there is $f_i \in F$ with the properties:

$$f_i(x) < \frac{\varepsilon}{n}, \forall x \in V_{x_i} \text{ and } f_i(x) > 1 - \frac{\varepsilon}{n}, \forall x \in B.$$

If we denote by: $f(x) = f_1(x) \cdot f_2(x) \cdot \ldots \cdot f_n(x), x \in X$, then $f \in F$. Let $x \in A$ everyone. Then there exists $0 \leq j \leq n$ such that $x \in V_{x_j}$ and so $f(x) < \frac{\varepsilon}{n} < \varepsilon$. If $x \in B$, then $f(x) > \left(1 - \frac{\varepsilon}{n}\right)^n \geq 1 - n\frac{\varepsilon}{n} = 1 - \varepsilon$. □

Corollary 3.16 (Theorem 4.16 [5])**.** *Let $M \subset C(X; I)$ be a closed subset with (VN) property which contains the constant functions* 0 *and* 1 *and separates the points of X.*
Then M is a Uryson family on X.
The assertion follows from Theorem 3.12 and Remark 1.

Theorem 3.17 (Theorem 4.18 [5])**.** *Let $M \subset C(X; I)$ be a closed subset with (VN) property which contains the constant functions* 0 *and* 1 *separates the points of X.*
If M contains a constant function $0 < c < 1$ then $M = C(X; I)$.

The assertion follows from Theorem 3.11 and Remark 1.

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All data generated or analysed during this study are included in this published article [and its supplementary information files]

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Tree families with algebraic connectivity greater than or equal to $2\big(1-\cos(\frac{\pi}{9})\big)$

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Abstract. Classifying trees according to their algebraic connectivity values has been an interesting area of study. Yuan et al. classified trees such that $\alpha(T) \geq 2-\sqrt{3}$, Wang and Tan classified trees such that $\alpha(T) \geq \frac{5-\sqrt{21}}{2}$ where $\alpha(T)$ denotes algebraic connectivity of tree T. Later, Belay et al. found new tree classes such that $\alpha(T) \geq 2\big(1-\cos(\frac{\pi}{7})\big)$.
In this study, we give new limit points and find new tree classes with algebraic connectivity greater than or equal to $\alpha(T) \geq 2\big(1-\cos(\frac{\pi}{9})\big)$.

2020 Mathematics Subject Classifications: 05C50

Keywords: Trees, Laplacian matrix, algeabric connectivity.

1. Introduction

Let A be the adjacency matrix of a simple, undirected graph G with n vertices. The Laplacian matrix L of G is given by $L = D - A$, where D is the diagonal matrix of vertex degrees.

The second smallest eigenvalue of the Laplacian matrix of G is called algebraic connectivity and is often denoted by $\alpha(G)$ in the literature. From the Perron-Frobenius theorem, the algebraic connectivity of disconnected graphs is zero. For more detailed information about algebraic connectivity, [1, 4] can be examined.

Let T be a tree. Let $\alpha(T)$ represent the algebraic connectivity of the tree T. The importance of the problem of classifying trees according to algebraic connectivity was mentioned in [5]. In [12], Yuan et al. classified trees with $n \geq 15$ vertices as $\alpha(T) \geq 2-\sqrt{3}$. Extending this work, Wang and Tan introduced four new tree classes in [11] and proved the inequality $\frac{5-\sqrt{21}}{2} \leq \alpha(T) \leq 2-\sqrt{3}$ for the algebraic connectivity of these tree classes for $n \geq 45$. In [2], Belay et al. expanded the previous two studies and showed that the tree classes such that $\alpha(T) \geq (2\left(1-\cos\left(\frac{\pi}{7}\right)\right)$ for $n \geq 32$. In addition, for $n \geq 45$, they ordered these classes according to their algebraic connectivities.

The research of the limit points of the eigenvalues of graphs was started by Hoffman in [7]. In [6], Guo studied the limit points of the Laplacian eigenvalues of graphs. In addition, he also gave the two largest limit points for the algebraic connectivity of trees. In [9], Kirkland et al. studied the Perron value for path graphs and star graphs. Kirkland, who also used the Perron value, showed the four largest limit points for trees in [8]. These four limit points are $\frac{3-\sqrt{5}}{2}$, $2-\sqrt{3}$, $\frac{5-\sqrt{21}}{2}$ and $2\big(1-\cos(\frac{\pi}{7})\big)$.

The aim of our study is to find new tree classes according to algebraic connectivity values less than $2\left(1-\cos\left(\frac{\pi}{7}\right)\right)$ and to show that each of them has a limit point.

2. Preliminaries

A vertex v of a tree T is a cutpoint if $T-v$, the graph obtained by deleting v and all of its incident edges, is disconnected. In here, $T-v = T_1 \dot{\cup} T_2 \dot{\cup} \ldots \dot{\cup} T_k$, and we called T_i, $1 \leq i \leq k$ the brances of T at v. The bottleneck matrix M_i, for the branch T_i is the inverse of the principal submatrix of $L(T)$ on the rows and columns of the vertices of this branch. A branch T_j is said to be Perron branch of T at v if its bottleneck matrix M_j satisfies $\rho(M_j) = \max\{\rho(M_i) | \, i = 1, 2, \ldots k\}$, where $\rho(M_i)$ denotes the largest eigenvalue of M_i [2, 9].

Theorem 2.1. [12, 9] *Let M be the bottleneck matrix of a Perron branch of a tree T at a vertex v. Then $\alpha(T) \geq \frac{1}{\rho(M)}$ and equality holds if there are at least two Perron branches of T at v.*

Theorem 2.2. [12] *If T' is a subtree of tree T, then $\alpha(T') \geq \alpha(T)$.*

For a connected graph G, $v \in V(G)$, and $l \geq k \geq 1$, let $G_{k,l}$ be the graph obtained from G by attaching two new paths of length k and l, respectively, at v. We say that the graph $G_{l-1,k+1}$ is obtained from $G_{k,l}$ by grafting an edge.

Theorem 2.3. [10] *Let G be a tree on $n \geq 2$ vertices and let v be a vertex of G. If $l \geq k \geq 1$, then $\alpha(G_{l-1,k+1}) \geq \alpha(G_{k,l})$.*

The following six tree classes were introduced and examined by Yuan et al. in [12].

Definition 2.4. [12] *Let n, k, p and q be nonnegative integers with $3k + 2p + q = n - 1$. Let $T(k, p, q)$ be the tree of order n which contains a vertex v such that $T(k, p, q) - v_0 = kK_{1,2}\dot{\cup}pK_{1,1}\dot{\cup}qK_1$. The following classes of trees are defined:*
(1) $C_1 = \{T(0, 0, n - 1)\}$
(2) $C_2 = \{T(0, 1, n - 3)\}$
(3) $C_3 = \{T(0, p, q) : p \geq 2\}$
(4) $C_4 = \{T(1, 0, n - 4)\}$
(5) $C_5 = \{T(1, p, q) : p \geq 1\}$
(6) $C_6 = \{T(k, p, q) : k \geq 2\}$.

The tree obtained from the path P_{d-1}, $d \geq 3$, connecting s vertices to one pendant vertex and t to the other is indicated by $B(s, t, d)$.

$T(n, b, h_1, h_2, h_3, ..., h_b)$ tree of order n is obtained from the star graphs, $K_{1,b}, K_{1,h_1}, \dots K_{1,h_b}$, by identifying the pendant vertices of $K_{1,b}$ and the centers of $K_{1,h_1}, \dots K_{1,h_b}$, respectively, where $h_1 \geq h_2 \geq \cdots h_b \geq 0$, $h_1 \geq h_2 > 0$ and $1 + b + h_1 + \cdots + h_b = n$ [2, 11].

To expand the above definition, Wang and Tan [11] proposed four new classes of trees according to their algebraic connectivities.

Definition 2.5. [11] *Let $T(n, b, h_1, h_2, h_3, ..., h_b)$ be a tree of order n and diameter 4. The following classes of trees are defined:*
(1) ${C_1}' = \{B(3, n - 5, 3)\}$
(2) ${C_2}' = \{T(n, b, 3, 1, h_3, ..., h_b)\}$;
(3) ${C_3}' = \{T(n, b, 3, 2, h_3, ..., h_b)\}$;
(4) ${C_4}' = \{T(n, b, 3, 3, h_3, ..., h_b)\}$.

The four classes given in the above definition are subsumed by the following classes given by Belay et al.

Definition 2.6. [2] *Let $T(i, j, k, p, q)$ be a tree of order n containing a vertex v_0 such that $T(i, j, k, p, q) - v_0 = iP_3\dot{\cup}jK_{1,3}\dot{\cup}kK_{1,2}\dot{\cup}pK_{1,1}\dot{\cup}qK_1$. The following classe of trees are defined:*
(1) $C_7 = \{T(0, 1, k, p, q)\}$
(2) $C_8 = \{T(0, j, k, p, q)\}$
(3) $C_9 = \{T(1, j, k, p, q)\}$
(4) $C_{10} = \{T(i, j, k, p, q) : i \geq 2\}$.

The following conclusions expressed by Belay et al. will be used in our main results.

Theorem 2.7. [2] *Let T be a tree of order $n \geq 32$ and $T \notin \cup_{i=1}^{10} C_i$, then $\alpha(T) < 2\left(1 - \cos\left(\frac{\pi}{7}\right)\right)$.*

Theorem 2.8. [2] *Let T be a tree of order $n \geq 45$, then $2(1 - \cos\left(\frac{\pi}{7}\right)) \leq \alpha(T) < \frac{5-\sqrt{21}}{2}$ if and only if $T \in C_9 \cup C_{10}$. Furthermore, if $T \in C_{10}$, then $\alpha(T) = 2(1 - \cos\left(\frac{\pi}{7}\right))$.*

Lemma 2.9. [2] *Let $T \in \cup_{i=1}^{10} C_i$ be a tree of order n. Then $\alpha(T) \geq 2(1 - \cos\left(\frac{\pi}{7}\right))$.*

Lemma 2.10. [2] *Let T_i be a tree of order $n \geq 45$ in class C_i, $1 \leq i \leq 10$. Then,*
$\alpha(T_1) > \alpha(T_2) > \alpha(T_3) > \alpha(T_4) > \alpha(T_5) > \alpha(T_6) > \alpha(T_7) > \alpha(T_8) > \alpha(T_9) > \alpha(T_{10})$.

3. New families of trees

Let us begin by determining the diameters of the trees we will examine in this section.

Lemma 3.1. [3] *Let T be a tree on n vertices with diameter $diam(T)$. Then*

$$\alpha(T) \leq 2\left(1 - \cos\left(\frac{\pi}{(diam(T) + 1)}\right)\right).$$

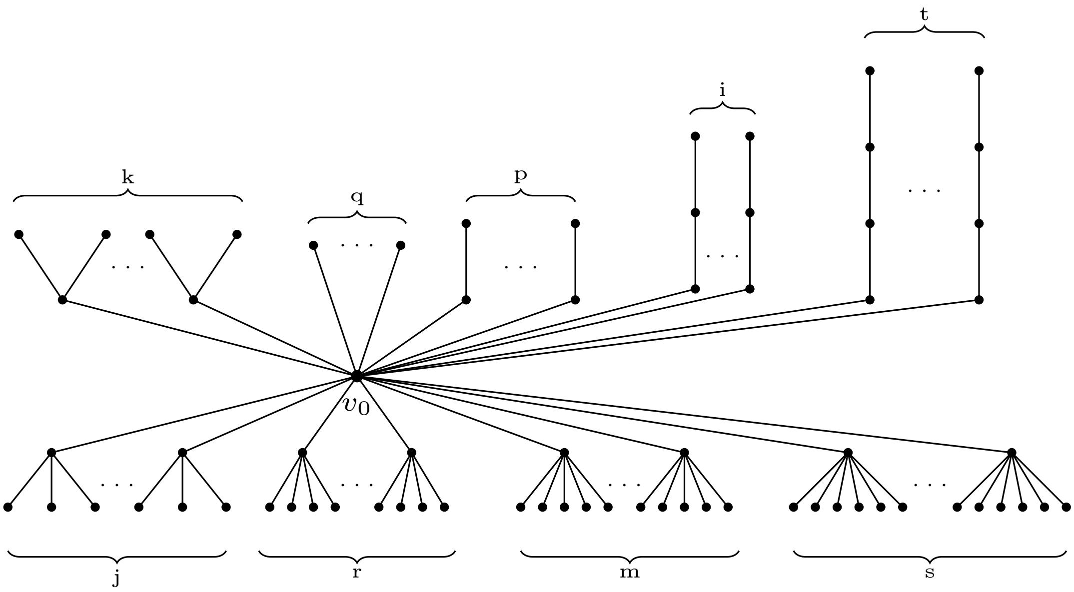

FIGURE 1. $T(i,j,k,p,q,r,m,s,t)$

From above lemma, if $diam(T) \geq 9$, then $\alpha(T) < 2(1-\cos\left(\frac{\pi}{9}\right))$. For this reason, from now on we should to consider trees with diameters of at most eight.

In the following, we define eight new classes of trees and study their algebraic connectivities.

Definition 3.2. *Let $T(i,j,k,p,q,r,m,s,t)$ be a tree of order n that contains a vertex v_0 such that $T(i,j,k,p,q,r,m,n,t) - v_0 = iP_3 \dot\cup jK_{1,3} \dot\cup kK_{1,2} \dot\cup pK_{1,1} \dot\cup qK_1 \dot\cup rK_{1,4} \dot\cup mK_{1,5} \dot\cup sK_{1,6} \dot\cup tP_4$ (Figure 1).The following classes of trees are defined:*

(1) $C_{11} = \{T(i,j,k,p,q,1,0,0,0)\}$
(2) $C_{12} = \{T(i,j,k,p,q,r,0,0,0) : r \geq 2\}$
(3) $C_{13} = \{T(i,j,k,p,q,r,1,0,0)\}$
(4) $C_{14} = \{T(i,j,k,p,q,r,m,0,0) : m \geq 2\}$
(5) $C_{15} = \{T(i,j,k,p,q,r,m,1,0)\}$
(6) $C_{16} = \{T(i,j,k,p,q,r,m,s,0) : s \geq 2\}$
(7) $C_{17} = \{T(i,j,k,p,q,r,m,s,1)\}$
(8) $C_{18} = \{T(i,j,k,p,q,r,m,s,t) : t \geq 2\}$.

The following lemma is required to determine the limit points.

Lemma 3.3. [9] *Let T be an unweighted tree and let m be a vertex of T. If the branch B of T at vertex m contains k vertices, then*

$$\frac{1}{2\left(1-\cos\left(\frac{\pi}{2k+1}\right)\right)} \geq \rho(B) \geq \frac{k+1+\sqrt{k^2-2k-3}}{2}$$

where $\rho(B)$ is the Perron value of a B branches.

Using the above lemma, the four vertex path graph and the k vertex star graph ($2 \leq k \leq 7$), four new limit points are obtained. Then the second smallest quadruple group limit points are $3-2\sqrt{2}$, $\frac{2}{7+3\sqrt{5}}$, $4-\sqrt{15}$, $2\left(1-\cos(\frac{\pi}{9})\right)$, respectively.

Let us state here that it is sufficient for the proofs that any tree with appropriate order, not in class C_i, $1 \leq i \leq 18$, has a subtree with algebraic connectivity smaller than $2\left(1-\cos(\frac{\pi}{9})\right) \cong 0,120614$.

Lemma 3.4. *If T be a tree of order $n \geq 106$, $diam(T) = 3$ and $T \notin \cup_{i=1}^{18} C_i$, then $\alpha(T) < 2(1-\cos\left(\frac{\pi}{9}\right))$.*

Proof. Let $T = B(s, n-2-s, 3)$ be a tree of order $n \geq 106$. Since $T \notin \cup_{i=1}^{18} C_i$, $s \geq 7$. As [5] proved, $\alpha(B(s,t,3)) < \alpha(B(s-1,t+1,3))$ if $s < t$, so $\alpha(B(s,n-2-s,3)) \leq \alpha(B(7,n-9,3))$.

Then, as $n \geq 106$, by Theorem 2.2, we conclude that $\alpha(B(s,n-2-s,3)) \leq \alpha(B(7,n-9,3)) \leq \alpha(B(7,97,3)) \cong 0,120593$. □

Lemma 3.5. *If T be a tree of order $n \geq 106$, $diam(T) = 4$ and $T \notin \cup_{i=1}^{18} C_i$, then $\alpha(T) < 2\big(1 - \cos(\frac{\pi}{9})\big)$.*

Proof. Let $T(n, b, h_1, h_2, h_3, ..., h_b)$ be a tree with diameter 4 and $T \notin \cup_{i=1}^{18} C_i$. Then, $h_1 \geq 7$. Because it is sufficient to examine this tree structure from Theorem 2.3.
Thus, $\alpha((106, 97, 7, 1, 0, ..., 0)) \cong 0,120550 < 2(1 - \cos\left(\frac{\pi}{9}\right)) \cong 0,120614$ is valid. □

Lemma 3.6. *Let $T(i,j,k,p,q,r,0,0,0), T(i,j,k,p,q,r,m,0,0), T(i,j,k,p,q,r,m,s,0)$ and $T(i,j,k,p,q,r,m,s,t)$ with $r,m,s,t \geq 2$ be an arbitrary tree in C_{12}, C_{14}, C_{16} and C_{18}, respectively. Then,*

$\alpha(T(i,j,k,p,q,r,0,0,0)) = 3 - 2\sqrt{2}$
$\alpha(T(i,j,k,p,q,r,m,0,0)) = \frac{2}{7+3\sqrt{5}}$
$\alpha(T(i,j,k,p,q,r,m,s,0)) = 4 - \sqrt{15}$
$\alpha(T(i,j,k,p,q,r,m,s,t)) = 2\big(1 - \cos(\frac{\pi}{9})\big)$.

Proof. Let v be the vertex of the tree $T(i,j,k,p,q,r,0,0,0)$. Then we get the Perron value $\rho(M) = 3 + 2\sqrt{2}$ and using Theorem 2.1 we get $\alpha(T(i,j,k,p,q,r,0,0,0)) = 3 - 2\sqrt{2}$. The other three trees are shown in the same way. □

Theorem 3.7. *Let T be a tree of order $n \geq 106$ and $T \notin \cup_{i=1}^{18} C_i$, then $\alpha(T) < 2\big(1 - \cos(\frac{\pi}{9})\big)$.*

Proof. From Lemma 3.1, if $diam(T) \geq 9$, then $\alpha(T) < 2\big(1 - \cos(\frac{\pi}{9})\big)$. It is shown that $\alpha(T) < 2\big(1 - \cos(\frac{\pi}{9})\big)$ in Lemma 3.4 and Lemma 3.5 when diameter is 3 and 4. For trees with a diameter of 5 to 8, the calculation can be done similarly. □

Theorem 3.8. *If T be a tree of order $n \geq 106$, then $2\big(1 - \cos(\frac{\pi}{9})\big) \leq \alpha(T) < 2\big(1 - \cos(\frac{\pi}{7})\big)$ if and only if $T \in \cup_{i=11}^{18} C_i$. Furthermore, if $T \in C_{18}$, then $\alpha(T) = 2\big(1 - \cos(\frac{\pi}{9})\big)$.*

Proof. From Theorem 3.7, $T \in \cup_{i=1}^{18} C_i$ and Theorem 2.7 $T \notin \cup_{i=1}^{10} C_i$. Thus $T \in \cup_{i=11}^{18} C_i$. Let T be a tree of order $n \geq 106$ and $T \in \cup_{i=11}^{18} C_i$. From Theorem 2.7, $\alpha(T) < 2\big(1 - \cos(\frac{\pi}{7})\big)$. With this, from Theorem 2.1 and Lemma 3.6, we get $\alpha(T) \geq 2\big(1 - \cos(\frac{\pi}{9})\big)$ and if $T \in C_{18}$, $\alpha(T) = \frac{1}{\rho(M)} = 2\big(1 - \cos(\frac{\pi}{9})\big)$. □

Lemma 3.9. *Let T_i be a tree of order $n \geq 106$ in class C_i, $11 \leq i \leq 18$. Then,*
$\alpha(T_{11}) > \alpha(T_{12}) > \alpha(T_{13}) > \alpha(T_{14}) > \alpha(T_{15}) > \alpha(T_{16}) > \alpha(T_{17}) > \alpha(T_{18})$.

Proof. It is clear from Lemma 3.6 and Theorem 2.2. □

Corollary 3.10. *Let $T \in \cup_{i=1}^{18} C_i$ be a tree of order n. Then, $\alpha(T) \geq 2\big(1 - \cos(\frac{\pi}{9})\big)$.*

Proof. Similar to Belay's proof, let $T(i,j,k,p,q,r,m,s,t) \in \cup_{i=1}^{18} C_i$. If $n \geq 106$, Theorem 3.7, $\alpha(T) \geq 2\big(1 - \cos(\frac{\pi}{9})\big)$. If $n < 106$, $T(i,j,k,p,q+106-n,r,m,s,t)$ is a tree of order 106 and from Theorem 3.7, $\alpha(T(i,j,k,p,q+106-n,r,m,s,t)) \geq 2\big(1 - \cos(\frac{\pi}{9})\big)$. From Theorem 2.2 $\alpha(T(i,j,k,p,q+106-n,r,m,s,t)) \geq \alpha(T(i,j,k,p,q+106-n,r,m,s,t)) \geq 2\big(1 - \cos(\frac{\pi}{9})\big)$. □

4. Conclusion

In this work, we extended the studies of trees classified by their algebraic connectivity. Previous studies investigated trees with certain algebraic connectivity values and most recently trees whose algebraic connectivity is greater than or equal to $2\big(1 - \cos(\frac{\pi}{7})\big)$ have been classified. We introduce new classes of trees whose algebraic connectivity is greater than or equal to $2\big(1 - \cos(\frac{\pi}{9})\big)$ which naturally include all previously defined classes. In addition, we identify new limit points for the algebraic connectivity of trees. As a direction for further study, it would be interesting to investigate whether similar techniques can be applied to even smaller algebraic connectivity values or to general graph families beyond trees.

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Fuzzy soft element and its application to decision-making

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Abstract. This paper proposes the novel concept of fuzzy soft elements, thereby extending the traditional notion of fuzzy soft points through the more flexible assignment of alternatives to attributes. Consequently, this concept offers a new vantage point for understanding fuzzy soft set operations. Through this, after some of the properties of fuzzy soft set operations are given, a decision-making example is presented that demonstrates the practical application.

2020 Mathematics Subject Classifications: 03E72, 03E20, 91B06

Keywords: Fuzzy soft set, Fuzzy soft element, Decision-making

1. Introduction

Both fuzzy sets [19] and soft sets [11], developed to handle uncertainty, possess their own unique advantages. Soft sets provide a flexible model for uncertainty using parameters, without the need for a membership function. Fuzzy soft sets [8], on the other hand, combine these two structures; they bring together the concept of graded membership from fuzzy sets with the flexible parameters of soft sets to create a richer framework for modelling complex and uncertain systems.

There are various approaches to performing set operations on fuzzy soft sets, inspired by methods from fuzzy set and soft set theories. Additionally, different interpretations exist regarding the members of a fuzzy soft set [1, 8, 10, 18]. These interpretations are typically expressed through the membership degrees of an element (or alternative) in universe of discourse with respect to a specific parameter (or attribute), and both theoretical and practical applications are developed accordingly. This study proposes a novel approach to the definition of fuzzy soft element, based on the notion of soft element introduced by Das and Samanta [4]. For further studies related to the soft element, see [5–7, 15–17]. Essentially, the proposed model provides a way to determine the sensitivity of alternatives for each descriptive attribute. Therefore, it facilitates achieving more accurate results in decision-making problems within this framework by enabling different evaluation strategies. Topology, metric structures and various applications can be developed using the basic method discussed in this study. Using the basic method discussed in this paper, studies on topology, metric structures and some applications can be developed in a manner similar to those in [2, 3, 12].

In this paper, the necessary basic definitions and notations are first provided, after which the notion of fuzzy soft element is introduced and illustrated with an example. Then, the set operations in fuzzy soft sets are redefined based on this notion, and some of their properties are shown by revealing its relations with previous definitions. Finally, to demonstrate its practical applicability, a decision-making method based on the fuzzy soft element is proposed. Here, a decision is modelled not as a single element of the universe of discourse but as a fuzzy soft element, which provides an alternative for each descriptive attribute.

1.1. Preliminaries.

In this section, the following definitions and notations are based on [13, 20] for fuzzy sets, [4, 11, 15] for soft sets and [1, 8, 14] for fuzzy soft sets.

Definition 1.1. *A fuzzy set $A = \{u^{\mu_A(u)} : u \in U\}$ on a universe of discourse U is defined by a membership function $\mu_A : U \to [0, 1]$, with the membership value $\mu_A(u)$ indicating the degree to which $u \in U$ belongs to the fuzzy set A. The set of all fuzzy sets on U is denoted by $F(U)$.*

A fuzzy point x^t in U is a fuzzy set with membership function

$$\mu_{x^t}(u) = \begin{cases} t \in (0,1] & \text{when } u = x, \\ 0 & \text{otherwise.} \end{cases}$$

A fuzzy point x^t is said to be member of a fuzzy set A, denoted by $x^t \in A$, iff $\mu_{x^t}(u) \leq \mu_A(u)$ for each $u \in U$. Also, the set of all fuzzy points of a fuzzy set A is represeted by $FP(A)$.

Definition 1.2. *Let $A, B \in F(U)$ be two fuzzy sets. Then,*

- *$A = \emptyset$ and $A = U$ iff $\mu_A = 0$ and $\mu_A = 1$ for each $u \in U$, respectively.*
- *$A \subseteq B$ iff $\mu_A(u) \leq \mu_B(u)$ for each $u \in U$.*
- *$A = B$ iff $\mu_A(u) = \mu_B(u)$ for each $u \in U$.*
- *$C = A \vee B$ iff $\mu_C(u) = \mu_A(u) \vee \mu_B(u)$, i.e. $\mu_C(u) = \max\{\mu_A(u), \mu_B(u)\}$ for each $u \in U$.*
- *$D = A \wedge B$ iff $\mu_D(u) = \mu_A(u) \wedge \mu_B(u)$, i.e. $\mu_D(u) = \min\{\mu_A(u), \mu_B(u)\}$ for each $u \in U$.*
- *$E = A^C$ iff $\mu_E(u) = 1 - \mu_A(u)$ for each $u \in U$.*

Definition 1.3. *A pair (G, P) is called a soft set on U with a parameters set P, where $G : P \to P(U)$ is a mapping with $P(U)$ being the power set of U. The set of all soft set on U with a parameters set P is denoted by $S_P(U)$ or simply $S(U)$.*

A function $\varepsilon : P \to U$ is called a soft element of U and ε is said to be member of (G, P) if $\varepsilon(\alpha) \in G(\alpha)$ for each $\alpha \in P$. The class of soft elements of (G, P) is denoted by $SE(G, P)$. Also, the soft elements are denoted by $\tilde{x}$ and the soft elements such that $\tilde{x} = c$ for all $\alpha \in P$ and for a constant $c \in U$ are denoted by $\bar{c}$.

Definition 1.4. *A pair (g, P) is called a fuzzy soft set on U with a parameters set P, where $g : P \to F(U)$. The set of all fuzzy soft sets on U with a parameters set P is denoted by $FS_P(U)$ or simply $FS(U)$.*

A special type of fuzzy soft set $(g, P) \in FS(U)$ is called fuzzy soft point, denoted by P_x^λ, such that for a fixed $x \in U$ and $\lambda \in (0, 1]$, $\mu_{g(\alpha)}(u) = \lambda$ if $u = x$, and $\mu_{g(\alpha)}(u) = 0$ if $u \neq x$ for $\alpha \in P$. Also, for a fuzzy soft set $(g, P) \in FS(U)$, a fuzzy soft point P_x^λ is a member of (g, P) if $\lambda \leq \mu_{g(\alpha)}(u)$ for each $\alpha \in P$.

Definition 1.5. *The fuzzy soft set (g, P) is said to be a null fuzzy soft set if $g(\alpha) = \emptyset$ and an absolute fuzzy soft set if $g(\alpha) = U$ for each $\alpha \in P$, denoted by $\tilde{\mathbf{\Phi}}$ and $\tilde{\mathbf{U}}$, respectively.*

Definition 1.6. *The union and intersection of $(g, P), (h, P) \in FS(U)$ are fuzzy soft sets defined as*

$$(g, P)\tilde{\vee}(h, P) = \{(\alpha, g(\alpha) \vee h(\alpha)) : \forall \alpha \in P,\ g(\alpha), h(\alpha) \in F(U)\} \tag{1}$$

and

$$(g, P)\tilde{\wedge}(h, P) = \{(\alpha, g(\alpha) \wedge h(\alpha)) : \forall \alpha \in P,\ g(\alpha), h(\alpha) \in F(U)\}, \tag{2}$$

respectively. The complement of (g, P) is denoted by $(g, P)^C$, where $(g, P)^C(\alpha) = g(\alpha)^C$ for each $\alpha \in P$.

Example 1.7. *Let $P = \{\alpha, \beta, \gamma\}$ and $U = \{u, v, w\}$. The fuzzy soft sets*

$$(g_1, P) = \{(\alpha, \{u^{0.1}, v^{0.7}, w^{0.5}\}), (\beta, \{u^{0.8}, v^{0.2}, w^{0.6}\}), (\gamma, \{u^{0.9}, v^{0.8}, w^{0.1}\})\},$$
$$(g_2, P) = \{(\alpha, \{u^{0.5}, w^{0.5}\}), (\beta, \{u^{0.1}, v^{0.2}, w^{0.3}\}), (\gamma, \{v^{0.5}\})\},$$
$$(g_1, P)\tilde{\vee}(g_2, P) = \{(\alpha, \{u^{0.5}, v^{0.7}, w^{0.5}\}), (\beta, \{u^{0.8}, v^{0.2}, w^{0.6}\}), (\gamma, \{u^{0.9}, v^{0.8}, w^{0.1}\})\},$$
$$(g_1, P)\tilde{\wedge}(g_2, P) = \{(\alpha, \{u^{0.1}, w^{0.5}\}), (\beta, \{u^{0.1}, v^{0.2}, w^{0.3}\}), (\gamma, \{v^{0.5}\})\},$$
$$(g_1, P)^C = \{(\alpha, \{u^{0.9}, v^{0.3}, w^{0.5}\}), (\beta, \{u^{0.2}, v^{0.8}, w^{0.4}\}), (\gamma, \{u^{0.1}v^{0.2}, w^{0.9}\})\}$$

are represented in tabular form as in Table 1 and 2.

(g_1, P)	α	β	γ	(g_2, P)	α	β	γ
u	0.1	0.8	0.9	u	0.5	0.1	0
v	0.7	0.2	0.8	v	0	0.2	0.5
w	0.5	0.6	0.1	w	0.5	0.3	0

Table 1. The tabular representation of the fuzzy soft sets (g_1, P) and (g_2, P)

$(g_1,P)\tilde{\vee}(g_2,P)$	α	β	γ	$(g_1,P)\tilde{\wedge}(g_2,P)$	α	β	γ	$(g_1,P)^C$	α	β	γ
u	0.5	0.8	0.9	u	0.1	0.1	0	u	0.9	0.2	0.1
v	0.7	0.2	0.8	v	0	0.2	0.5	v	0.3	0.8	0.2
w	0.5	0.6	0.1	w	0.5	0.3	0	w	0.5	0.4	0.9

TABLE 2. The tabular representation of the union, intersection and complement of fuzzy soft sets (g_1,P) and (g_2,P)

It will be used solely g for a fuzzy soft set hereafter at its place (g,P) for simplicity.

2. Main results

Definition 2.1. *Let U be a universe and P be a parameters set. A fuzzy soft element $\tilde{\mathbf{x}}$ is a fuzzy soft set for which $\tilde{\mathbf{x}}: P \to FP(U)$ and a fuzzy soft element $\tilde{\mathbf{x}}$ is said to be a member of a fuzzy soft set $g \in FS(U)$, denoted by $\tilde{\mathbf{x}}\tilde{\in}g$, iff $\tilde{\mathbf{x}}(\alpha) \in g(\alpha)$ i.e. $\mu_{\tilde{\mathbf{x}}(\alpha)}(u) \le \mu_{g(\alpha)}(u)$ for each $\alpha \in P$ and $u \in U$. The class of fuzzy soft elements is denoted by $FSE(g)$. Also, a fuzzy soft element $\tilde{\mathbf{x}}\tilde{\in}g$ is called a constant fuzzy soft element of g if there exists a $u \in U$ such that for each $\alpha \in P$, $\mu_{\tilde{\mathbf{x}}(\alpha)}(u) = 1$ and $\mu_{\tilde{\mathbf{x}}(\alpha)}(u') = 0$ for each $u' \in U - \{u\}$. The class of all constant fuzzy soft element of g is denoted by $\mathfrak{c}_g$. A constant fuzzy soft element can be considered the most desirable, competent or unique object that fulfils all attributes.*

Example 2.2. *Suppose that a technology company wants to distribute IT support requests to technicians. Let*

$$U = \{u = \textit{technician1},\ v = \textit{technician2},\ w = \textit{technician3},\ x = \textit{technician4}\}$$

be a set of technicians and

$$P = \{\alpha = \textit{security breach},\ \beta = \textit{network issues},\ \gamma = \textit{software bugs}\}$$

be a set of support requests.

Each technician is specialised to a certain extent in different requests, which can be represented by a fuzzy soft set such as

$$g = \{(\alpha, \{u^{0.7}, v^1, w^{0.2}, x^{0.4}\}), (\beta, \{u^1, v^1, w^{0.3}, x^{0.5}\}), (\gamma, \{v^1, w^1\})\},$$

where $\mu_{g(\alpha)}(u)$ indicates the degree of service capacity in each support request for each technician.

g	α	β	γ
u	0.7	1	0
v	1	1	1
w	0.2	0.3	1
x	0.4	0.5	0

TABLE 3. The tabular representation of g

Each fuzzy soft element of g represents a scenario in which IT support requests are distributed to technicians according to prioritisation, workload balance, etc. Then,

$$\mathfrak{c}_{\tilde{\mathbf{U}}} = \begin{Bmatrix} \tilde{\mathbf{x}}_1 = \{(\alpha,u^1),(\beta,u^1),(\gamma,u^1)\}, & \tilde{\mathbf{x}}_3 = \{(\alpha,w^1),(\beta,w^1),(\gamma,w^1)\}\}, \\ \tilde{\mathbf{x}}_2 = \{(\alpha,v^1),(\beta,v^1),(\gamma,v^1)\}, & \tilde{\mathbf{x}}_4 = \{(\alpha,x^1),(\beta,x^1),(\gamma,x^1)\} \end{Bmatrix},$$
$$\mathfrak{c}_g = \{\tilde{\mathbf{x}}_2 = \{(\alpha,v^1),(\beta,v^1),(\gamma,v^1)\}\}$$

and some of the fuzzy soft elements of g and their tabular forms can be given as follows.

$$\tilde{\mathbf{x}}_5 = \{(\alpha, v^{0.6}), (\beta, u^{0.3}), (\gamma, w^{0.1})\},$$
$$\tilde{\mathbf{x}}_6 = \{(\alpha, u^{0.3}), (\beta, x^{0.5}), (\gamma, v^{0.2})\},$$
$$\tilde{\mathbf{x}}_7 = \{(\alpha, x^{0.2}), (\beta, w^{0.2}), (\gamma, w^{0.6})\}.$$

$\tilde{\mathbf{x}}_5$	α	β	γ
u	0	0.3	0
v	0.6	0	0
w	0	0	0.1
x	0	0	0

$\tilde{\mathbf{x}}_6$	α	β	γ
u	0.3	0	0
v	0	0	0.2
w	0	0	0
x	0	0.5	0

$\tilde{\mathbf{x}}_7$	α	β	γ
u	0	0	0
v	0	0	0
w	0	0.2	0.6
x	0.2	0	0

TABLE 4. The tabular representation of the fuzzy soft elements $\tilde{\mathbf{x}}_5$, $\tilde{\mathbf{x}}_6$ and $\tilde{\mathbf{x}}_7$

Definition 2.3. *Let $\mathfrak{b}$ be a class of fuzzy soft elements of $\tilde{\mathbf{U}}$. The fuzzy soft set $FSS(\mathfrak{b})$ produced by the class of fuzzy soft elements $\mathfrak{b}$ is defined by*

$$g = FSS(\mathfrak{b}) = \{(\alpha, g(\alpha)) : \forall \alpha \in P,\ g(\alpha) = \bigvee_{\tilde{\mathbf{x}} \in \mathfrak{b}} \tilde{\mathbf{x}}(\alpha)\}.$$

Example 2.4. *From Example 2.2, suppose that $\mathfrak{b} = \{\tilde{\mathbf{x}}_5, \tilde{\mathbf{x}}_6, \tilde{\mathbf{x}}_7\}$ is a class of fuzzy soft elements of $\tilde{\mathbf{U}}$. Then, the fuzzy soft set produced by $\mathfrak{b}$ is obtained as*

$$h = FSS(\mathfrak{b}) = \{(\alpha, \{u^{0.3}, v^{0.6}, x^{0.2}\}), (\beta, \{u^{0.3}, w^{0.2}, x^{0.5}\}), (\gamma, \{v^{0.2}, w^{0.6}\})\}.$$

h	α	β	γ
u	0.3	0.3	0
v	0.6	0	0.2
w	0	0.2	0.6
x	0.2	0.5	0

TABLE 5. The tabular representation of the fuzzy soft set produced by $\mathfrak{b}$

It is clear that $h \tilde{\subseteq} g$. Also, notice that $\mathfrak{b} \subset FSE(FSS(\mathfrak{b}))$, i.e. $\mathfrak{b}$ and $FSE(FSS(\mathfrak{b}))$ are not the same.

Definition 2.5. *Let $g, h \in FS(U)$ be two fuzzy soft sets. The fuzzy soft sets*

$$g \sqcup h = FSS\left(FSE(g) \cup FSE(h)\right)$$

and

$$g \sqcap h = FSS\left(FSE(g) \cap FSE(h)\right)$$

are called ξ-union and ξ-intersection of g and h, respectively.

The ξ-complement of g is denoted by $g^{\complement} = FSS\left(FSE(g^C)\right)$.

Example 2.6. *From Example 1.7, suppose that*

$$g_3 = \{(\alpha, \{v^{0.4}\}), (\beta, \{u^{0.4}, v^{0.1}, w^{0.5}\}), (\gamma, U)\}$$

is a fuzzy soft set in $FS(U)$. Then,

$$\begin{aligned} g_1 \sqcup g_2 &= g_1 \tilde{\vee} g_2 \\ &= \{(\alpha, \{u^{0.5}, v^{0.7}, w^{0.5}\}), (\beta, \{u^{0.8}, v^{0.2}, w^{0.6}\}), (\gamma, \{u^{0.9}, v^{0.8}, w^{0.1}\})\}, \\ g_1 \sqcap g_2 &= g_1 \tilde{\wedge} g_2 \\ &= \{(\alpha, \{u^{0.1}, w^{0.5}\}), (\beta, \{u^{0.1}, v^{0.2}, w^{0.3}\}), (\gamma, \{v^{0.5}\})\} \end{aligned}$$

and

$$\begin{aligned} g_1^{\complement} &= g_1^C \\ &= \{(\alpha, \{u^{0.9}, v^{0.3}, w^{0.5}\}), (\beta, \{u^{0.2}, v^{0.8}, w^{0.4}\}), (\gamma, \{u^{0.1} v^{0.2}, w^{0.9}\})\}. \end{aligned}$$

But,

$$\begin{aligned} g_2 \sqcap g_3 &= \tilde{\mathbf{\Phi}}, \\ g_2 \tilde{\wedge} g_3 &= \{(\alpha, \emptyset), (\beta, \{u^{0.1}, v^{0.1}, w^{0.3}\}), (\gamma, \{v^{0.5}\})\} \end{aligned}$$

and

$$\begin{aligned} g_3^{\complement} &= \tilde{\mathbf{\Phi}}, \\ g_3^C &= \{(\alpha, \{u^1, v^{0.6}, w^1\}), (\beta, \{u^{0.6}, v^{0.9}, w^{0.5}\}), (\gamma, \emptyset)\}. \end{aligned}$$

So, $g_2 \sqcap g_3 \neq g_2 \tilde{\wedge} g_3$ and $g_3^{\complement} \neq g_3^C$.

Remark 1.
- *The ξ-union of g and h is the same as in Definition 1.6* (1).
- *The ξ-intersection of g and h are the same as as in Definition 1.6* (2) *if for each $\alpha \in P$ and for each same $u \in U$, $\mu_{g(\alpha)}(u) \neq 0$ and $\mu_{h(\alpha)}(u) \neq 0$ or $\mu_{g(\alpha)}(u) = \mu_{h(\alpha)}(u) = 0$ simultaneously. However, if for at least one parameter $\alpha \in P$ and for each same $u \in U$, $\mu_{g(\alpha)}(u) \neq 0$ and $\mu_{h(\alpha)}(u) = 0$ or vice versa, then they are different. In other words; if for each $\alpha \in P$, $g(\alpha) \wedge h(\alpha) \neq \emptyset$, then they mean the same. But, if for at least one parameter $\alpha \in P$, $g(\alpha) \wedge h(\alpha) = \emptyset$, then they correspond to different.*
- *The ξ-complement of g is the same as in Definition 1.6 if for each $\alpha \in P$ and for each $u \in U$, $\mu_{g(\alpha)}(u) \neq 1$. However, if for at least one parameter $\alpha \in U$ and for each $u \in U$, $\mu_{g(\alpha)}(u) = 1$, then they are different. In other words; if for each $\alpha \in P$, $g(\alpha) \neq U$, then they mean the same. But, if for at least one parameter $\alpha \in P$, $g(\alpha) = U$, then they correspond to different.*

Considering Remark 1, the class of fuzzy soft sets, denoted $FS(\tilde{\mathbf{U}})$, is identified for which either $g(\alpha) \neq \emptyset$ for each $\alpha \in P$ or $g(\alpha) = \emptyset$ for each $\alpha \in P$. Then, the following proposition is given to show the relations of Definition 1.6 and 2.5.

Proposition 2.7. *Let $g, h \in FS(\tilde{\mathbf{U}})$.*

(1) $g \vee h = g \sqcup h$.
(2) $g \sqcap h \tilde{\subseteq} g \tilde{\wedge} h$ *and if* $g \tilde{\wedge} h \in FS(\tilde{\mathbf{U}})$, *then* $g \sqcap h = g \tilde{\wedge} h$.
(3) $g^{\complement} \tilde{\subseteq} g^{C}$ *and if* $g^{C} \in FS(\tilde{\mathbf{U}})$, *then* $g^{\complement} = g^{C}$.
(4) $g \sqcup g^{\complement} \tilde{\subseteq} \tilde{\mathbf{U}}$ *and if* $g^{C} \in FS(\tilde{\mathbf{U}})$, *then* $g \sqcup g^{\complement} = \tilde{\mathbf{U}}$.
(5) $g \sqcap g^{\complement} = \tilde{\mathbf{\Phi}}$.
(6) $(g \sqcup h)^{C} \tilde{\supseteq} g^{C} \sqcap h^{C}$ *and* $(g \sqcap h)^{C} \tilde{\supseteq} g^{C} \sqcup h^{C}$.
(7) If $g \vee h$, $g^{C} \wedge h^{C}$, g^{C}, $h^{C} \in FS(\tilde{\mathbf{U}})$, *then* $(g \sqcup h)^{\complement} = g^{\complement} \sqcap h^{\complement}$ *and* $(g \sqcap h)^{\complement} = g^{\complement} \sqcup h^{\complement}$.
(8) If $g_i = FSS(\mathfrak{b}_i)$, $i \in I$, *then* $\bigsqcup_{i \in I} g_i = FSS\left(\bigcup_{i \in I} \mathfrak{b}_i\right)$ *and* $\bigsqcap_{i \in I} g_i \tilde{\supseteq} FSS\left(\bigcap_{i \in I} \mathfrak{b}_i\right)$.

Proof. The proof of $4, 5, 6, 8$ follows from $1, 2, 3$.

1. Since
$$(g \sqcup h)(\alpha) = \bigvee_{\tilde{\mathbf{x}} \in FSE(g) \cup FSE(h)} \tilde{\mathbf{x}}(\alpha) = \bigvee_{\tilde{\mathbf{x}} \in FSE(g)} \tilde{\mathbf{x}}(\alpha) \vee \bigvee_{\tilde{\mathbf{x}} \in FSE(h)} \tilde{\mathbf{x}}(\alpha) = g(\alpha) \vee h(\alpha)$$
for each $\alpha \in P$, then $g \sqcup h = g \tilde{\vee} h$.
2. If $g(\alpha) \wedge h(\alpha) = \emptyset$ or $g(\alpha) \wedge h(\alpha) \neq \emptyset$ for each $\alpha \in P$, then $g \tilde{\wedge} h \in FS(\tilde{\mathbf{U}})$ and so $g \tilde{\wedge} h = g \sqcap h$. If $g(\alpha) \wedge h(\alpha) \neq \emptyset$ for some $\alpha \in P$ and $g(\alpha) \wedge h(\alpha) = \emptyset$ for others, then $g \tilde{\wedge} h \neq \mathbf{\Phi}$ and $g \tilde{\wedge} h \notin FS(\tilde{\mathbf{U}})$ but $g \sqcap h = \mathbf{\Phi}$. Hence, $g \sqcap h \tilde{\subseteq} g \tilde{\wedge} h$.
3. If $g^{C}(\alpha) = \emptyset$ or $g^{C}(\alpha) \neq \emptyset$ for each $\alpha \in P$, then $g^{C} \in FS(\tilde{\mathbf{U}})$ and so $g^{\complement} = g^{C}$. If $g^{C} \neq \emptyset$ for some $\alpha \in P$ and $g^{C} = \emptyset$ for others, then $g^{C} \neq \mathbf{\Phi}$ and $g^{C} \notin FS(\tilde{\mathbf{U}})$ but $g^{\complement} = \mathbf{\Phi}$. Hence, $g^{\complement} \tilde{\subseteq} g^{C}$.
7.
$$\begin{aligned}
(g \sqcup h)^{\complement} &= FSS\{\tilde{\mathbf{x}} \tilde{\in} \tilde{\mathbf{U}} : \tilde{\mathbf{x}} \tilde{\in} (g \sqcup h)^{C}\} \\
&= FSS\{\tilde{\mathbf{x}} \tilde{\in} \tilde{\mathbf{U}} : \tilde{\mathbf{x}} \tilde{\in} (g \tilde{\vee} h)^{C}\} \\
&= FSS\{\tilde{\mathbf{x}} \tilde{\in} \tilde{\mathbf{U}} : \tilde{\mathbf{x}} \tilde{\in} g^{C} \tilde{\wedge} h^{C}\} \\
&= FSS\{\tilde{\mathbf{x}} \tilde{\in} \tilde{\mathbf{U}} : \tilde{\mathbf{x}} \tilde{\in} g^{C} \sqcap h^{C}\} \\
&= FSS\{\tilde{\mathbf{x}} \tilde{\in} \tilde{\mathbf{U}} : \tilde{\mathbf{x}} \tilde{\in} g^{C} \text{ and } \tilde{\mathbf{x}} \tilde{\in} h^{C}\} \\
&= g^{\complement} \sqcap h^{\complement}
\end{aligned}$$

□

2.1. **Decision-making application: Capability-workload equilibrium.** On the basis of fuzzy soft elements and within the context of Example 2.2, a decision-making method is proposed regarding technicians and support requests in order to balance capability and workload. Therefore, the following algorithm is given:

Algorithm Determining the technician assignments

Step 1. Input the set of technicians $U = \{u_1, u_2, \ldots, u_n\}$, the set of support requests as $P = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ and the weights $\omega_i \in [0,1]$, $i = 1, \ldots, m$, of parameters to represent the importance of each request.

Step 2. Input the fuzzy soft set $g \in FS_P(U)$ to represent the capabilities of technicians in response to the support requests.

Step 3. Input specialization threshold T for prioritize technicians with higher expertise, specialization bonus B for specialization incentive and workload balance factor ξ to control workload and capability balancing.

Step 4. Input the set of fuzzy soft elements $\{\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \ldots \tilde{\mathbf{x}}_k\} \subset FSE(g)$ to represent the intensity of assignment of request α to technician u.

Step 5. Calculate weighted capability score (WCS) for the assignment $\tilde{\mathbf{x}}_k$ with

$$WCS_k = \sum_{j=1}^{n}\sum_{i=1}^{m} \mu_{\tilde{\mathbf{x}}_k(\alpha_i)}(u_j) \cdot \mu_{g(\alpha_i)}(u_j) \cdot \omega_i \cdot (1 + \delta_{ij}),$$

where

$$\delta_{ij} = \begin{cases} B, & \text{if } \mu_{g(\alpha_i)}(u_j) \geq T, \\ 0, & \text{if } \mu_{g(\alpha_i)}(u_j) < T. \end{cases}$$

Step 6. Calculate weighted workload imbalance (WWI) for the assignment $\tilde{\mathbf{x}}_k$ with

$$WWI_k = \max_{j \in J} L_{k_j} - \min_{j \in J} L_{k_j},$$

where

$$L_{k_j} = \sum_{i=1}^{m} \mu_{\tilde{\mathbf{x}}_k(\alpha_i)}(u_j) \cdot \omega_i$$

is the workload of technician u_j in the assignment $\tilde{\mathbf{x}}_k$.

Step 7. Calculate final score for the assignment $\tilde{\mathbf{x}}_k$ with

$$S_k = WCS_k - \xi \cdot WWI_k.$$

Step 8. Rank the assignments according to their scores S_k and choose the highest one.

This algorithm evaluates the capability and availability of each technician while considering the urgency and complexity of support requests, ultimately facilitating a more efficient allocation of resources and enhancing overall service quality. By implementing this method, technician performance can be optimized and response times to support inquiries can be improved.

From Example 2.2, let $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \gamma$ and $u_1 = u$, $u_2 = v$, $u_3 = w$, $u_4 = x$. Assume that the specialization threshold $T = 0.65$, the specialization bonus $B = 0.3$ and the workload balance factor $\xi = 0.4$. In addition, let the company give the weights for each request as $\omega_1 = 0.7$, $\omega_2 = 0.5$ and $\omega_3 = 0.3$. Then, the sample calculations for $\tilde{\mathbf{x}}_6$ are given as follows:

$$\begin{aligned} L_{6_1} &= \mu_{\tilde{\mathbf{x}}_6(\alpha_1)}(u_1) \cdot \omega_1 = 0.3 \cdot 0.7 = 0.21, & L_{6_3} &= 0, \\ L_{6_2} &= \mu_{\tilde{\mathbf{x}}_6(\alpha_3)}(u_2) \cdot \omega_3 = 0.2 \cdot 0.3 = 0.06, & L_{6_4} &= \mu_{\tilde{\mathbf{x}}_6(\alpha_2)}(u_4) \cdot \omega_2 = 0.5 \cdot 0.5 = 0.25. \end{aligned}$$

So,

$$WWI_6 = 0.25 - 0 = 0.25.$$

Then,

$$\left.\begin{aligned} \mu_{g(\alpha_1)}(u_1) = 0.7 \geq 0.65, \\ \mu_{g(\alpha_2)}(u_4) = 0.5 < 0.65, \\ \mu_{g(\alpha_3)}(u_2) = 1 \geq 0.65. \end{aligned}\right\} \Rightarrow \begin{aligned} \delta_{11} = 0.3, \\ \delta_{24} = 0, \\ \delta_{32} = 0.3. \end{aligned}$$

So,

$$\left.\begin{aligned} \mu_{\tilde{\mathbf{x}}_6(\alpha_1)}(u_1) \cdot \mu_{g(\alpha_1)}(u_1) \cdot \omega_1 \cdot (1 + \delta_{11}) = 0.3 \cdot 0.7 \cdot 0.7 \cdot 1.3 = 0.1911, \\ \mu_{\tilde{\mathbf{x}}_6(\alpha_2)}(u_4) \cdot \mu_{g(\alpha_2)}(u_4) \cdot \omega_2 \cdot (1 + \delta_{24}) = 0.5 \cdot 0.5 \cdot 0.5 \cdot 1 = 0.125, \\ \mu_{\tilde{\mathbf{x}}_6(\alpha_3)}(u_2) \cdot \mu_{g(\alpha_3)}(u_2) \cdot \omega_3 \cdot (1 + \delta_{32}) = 0.2 \cdot 1 \cdot 0.3 \cdot 1.3 = 0.078. \end{aligned}\right\} \Rightarrow WCS_6 = 0.3941.$$

Hence, the score of $\tilde{\mathbf{x}}_6$ is

$$S_6 = WCS_6 - \xi \cdot WWI_6 = 0.3941 - 0.4 \cdot 0.25 = 0.2941.$$

Thus, it is similarly obtained that $S_5 = 0.6120$ and $S_7 = 0,2080$. As a result, $\tilde{\mathbf{x}}_5$ is chosen for the assignments.

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New fixed point theorem for rational F-contraction

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Abstract. In this paper, by considering Wardowski's technique, we present a fixed-point result for rational type F-contraction mappings on a space equipped with two metrics. The use of a two-metric space allows for the simultaneous evaluation of different distance measures, which is important because in some problems a single metric may not fully capture the underlying structure of the space. By employing two metrics, more flexible and comprehensive analyses become possible. Motivated by potential applications in areas such as integral equations and fractal analysis, where complex contraction conditions naturally arise, this study aims to extend the theoretical framework and provide tools that could be useful in these and related fields.

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1. Introduction and Preliminaries

In [18], Wardowski introduced the notion of an F-contraction. Every F-contraction mapping on a complete metric space was shown to have a unique fixed point, with successive approximations converging to that point. In particular, if the function F is taken as $\mathsf{F}(x) = \ln x$ for $x \in (0, \infty)$, then the Banach contraction is recovered as a special case of an F-contraction. A considerable number of works have addressed this topic; see [1, 2, 3, 9, 16, 17]. The most prominent results focus on the applications of F-contractions to differential equations, integral equations, homotopy theory, and related areas. We now present the definition of an F-contraction, along with some of its properties and related results.

Let F denote the class of all functions $\mathsf{F} : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}$ which satisfy the following conditions:

(F1) F is non-decreasing;
(F2) for any sequence $\{\alpha_n\} \subset (0, \infty)$, $\alpha_n \to 0$ as $n \to \infty$ if and only if $\mathsf{F}(\alpha_n) \to -\infty$;
(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^k \mathsf{F}(\alpha) = 0$.

Using this function class, Wardowski [18] defined $\mathsf{F}-$contraction such as:

Definition 1.1 ([18]). *Let (Y, σ) be a metric space and $\mathsf{T} : \mathsf{Y} \to \mathsf{Y}$ be a mapping. Given $\mathsf{F} \in \mathtt{F}$, we say that T is $F-$contraction, if there exists $\tau > 0$ such that*

$$\varpi, \upsilon \in \mathsf{Y}, \sigma(\mathsf{T}\varpi, \mathsf{T}\upsilon) > 0 \Longrightarrow \tau + \mathsf{F}(\sigma(\mathsf{T}\varpi, \mathsf{T}\upsilon)) \leq \mathsf{F}(\sigma(\varpi, \upsilon)). \quad (1)$$

For example, $\mathsf{F}_1 : (0, \infty) \to \mathbb{R}$, $\mathsf{F}_1 = \ln \alpha + \alpha$ and $\mathsf{F}_2 : (0, \infty) \to \mathbb{R}$, $\mathsf{F}_2 = \ln \alpha$ is an of F.

Without assuming any particular structure on the space, Wardowski established the following result:

Theorem 1.2 ([18]). *Let (Y, σ) be a complete metric space and let $\mathsf{T} : \mathsf{Y} \to \mathsf{Y}$ be an $\mathsf{F}-$contraction. Then T has a unique fixed point in X.*

In a different approach, Jleli et al. proposed in [6] a family of functions H: $[0, \infty)^3 \to [0, \infty)$ subject to the following conditions:

$(H1)$ $\max\{\varsigma, \tau\} \leq \mathtt{H}(\varsigma, \tau, \phi)$ for all $\varsigma, \tau, \phi \in [0, \infty)$;
$(H2)$ $\mathtt{H}(0, 0, 0) = 0$;
$(H3)$ $\mathtt{H}$ is continuous.

Several examples of functions in the family H are presented as follows.

(i) $\mathtt{H}(\varsigma, \tau, \phi) = \varsigma + \tau + \phi$ for all $\varsigma, \tau, \phi \in [0, +\infty)$;

(ii) $\mathtt{H}(\varsigma,\tau,\phi)=\max\{\varsigma,\tau\}+\phi$ for all $\varsigma,\tau,\phi\in[0,+\infty)$;
(iii) $\mathtt{H}(\varsigma,\tau,\phi)=\varsigma+\tau+\varsigma\tau+\phi$ for all $\varsigma,\tau,\phi\in[0,+\infty)$.

Using a function $\mathtt{H}\in\mathrm{H}$, the authors of [6] introduced the following notion of $(\mathtt{H},\phi)-$contraction.

Definition 1.3 ([6]). *Let (Y,σ) be a metric space, $\phi:\mathsf{Y}\to[0,\infty)$ be a given function, and $\mathtt{H}\in\mathrm{H}$, Then $\mathsf{T}:\mathsf{Y}\to\mathsf{Y}$ is called a $(\mathtt{H},\phi)-$contraction with respect to the metric σ if and only if*

$$\mathtt{H}(\sigma(\mathsf{T}\varpi,\mathsf{T}\upsilon),\phi(\mathsf{T}\varpi),\phi(\mathsf{T}\upsilon))\leq k\mathtt{H}(\sigma(\varpi,\upsilon),\phi(\varpi),\phi(\upsilon)) \text{ for all } \varpi,\upsilon\in\mathsf{Y},$$

for some constant $k\in(0,1)$.

Lemma 1.4 ([15]). *Let (Y,σ) be a metric space and let $\mathsf{T}:\mathsf{Y}\to\mathsf{Y}$ be an $\mathsf{F}-\mathtt{H}-$contraction with respect to the functions $\mathsf{F}\in\mathcal{F}$, $\mathtt{H}\in\mathrm{H}$, $\phi:\mathsf{Y}\to[0,+\infty)$, and the real number $\tau>0$. If $\{\varpi_n\}$ is sequence of Picard starting at $\varpi_0\in\mathsf{Y}$, then*

$$\lim_{n\to\infty}\mathtt{H}(\sigma(\varpi_{n-1},\varpi_n),\phi(\varpi_{n-1}),\phi(\varpi_n))=0,$$

and hence

$$\lim_{n\to\infty}\sigma(\varpi_{n-1},\varpi_n)=0 \text{ and } \lim_{n\to\infty}\phi(\varpi_n)=0.$$

2. Main Results

In this section, we first recall the definition of a rational type $\mathsf{F}-\mathtt{H}$-contraction. Then, we prove a fixed point theorem based on this definition. Finally, we present an example to support our result.

Definition 2.1 ([4]). *Let (Y,σ) be a metric space and $\mathsf{T}:\mathsf{Y}\to\mathsf{Y}$ be a mapping. T is called a rational type $\mathsf{F}-\mathtt{H}-$contraction if there exists $\mathsf{F}\in\mathcal{F}$, $\mathtt{H}\in\mathrm{H}$, a real number $\tau>0$, and $\phi:\mathsf{Y}\to[0,\infty)$ s.t.*

$$\tau+\mathsf{F}(\mathtt{H}(\sigma(\mathsf{T}\varpi,\mathsf{T}\upsilon),\phi(\mathsf{T}\varpi),\phi(\mathsf{T}\upsilon)))\leq\mathsf{F}(\mathtt{H}(M(\varpi,\upsilon),\phi(\varpi),\phi(\upsilon))) \tag{2}$$

for all ϖ,$\upsilon\in\mathsf{Y}$ with $\mathtt{H}(\sigma(\mathsf{T}\varpi,\mathsf{T}\upsilon),\phi(\mathsf{T}\varpi),\phi(\mathsf{T}\upsilon))>0$
where

$$M(\varpi,\upsilon)=\max\left\{\sigma(\varpi,\upsilon),\frac{\sigma(\varpi,\mathsf{T}\varpi)[1+\sigma(\upsilon,\mathsf{T}\upsilon)]}{1+\sigma(\mathsf{T}\varpi,\mathsf{T}\upsilon)}\right\}.$$

Theorem 2.2. *Let Y be a set equipped with two metrics ρ and σ, where (Y,ρ) is complete. $\mathsf{T}:\mathsf{Y}\to\mathsf{Y}$ be a rational type $\mathsf{F}-\mathtt{H}-$contraction with respect to σ. $\mathsf{F}\in\mathcal{F}$, $\mathtt{H}\in\mathrm{H}$, a real number $\tau>0$ and $\phi:\mathsf{Y}\to[0,\infty)$ such that 2 holds for all ϖ,$\upsilon\in\mathsf{Y}$ with $\mathtt{H}(\sigma(\mathsf{T}\varpi,\mathsf{T}\upsilon),\phi(\mathsf{T}\varpi),\phi(\mathsf{T}\upsilon))>0$. Suppose that there exist $c>0$ such that*

$$\rho(\varpi,\upsilon)\leq c.\sigma(\varpi,\upsilon), \text{ for each } \varpi,\upsilon\in\mathsf{Y} \tag{3}$$

Then T has a fixed point ς such that $\phi(\varsigma)=0$.

Proof. Starting from an arbitrary point $\varpi_0\in Y$, we define a sequence $\{\varpi_n\}$ and prove that it converges to a fixed point of the given mapping. If $\varpi_1\in\mathsf{T}\varpi_1$ then ϖ_1 is a fixed point of T and so the proof is completed. Let $\varpi_1\neq\mathsf{T}\varpi_1$. Then $\sigma(\varpi_1,\mathsf{T}\varpi_1)>0$. On the other hand, from (F1) and

$$\mathtt{H}(\sigma(\varpi_1,\mathsf{T}\varpi_1),\phi(\varpi_1),\phi(\mathsf{T}\varpi_1))\leq\mathtt{H}(\sigma(\mathsf{T}\varpi_0,\mathsf{T}\varpi_1),\phi(\mathsf{T}\varpi_0),\phi(\mathsf{T}\varpi_1))$$

we get

$$\mathsf{F}(\mathtt{H}(\sigma(\varpi_1,\mathsf{T}\varpi_1),\phi(\varpi_1),\phi(\mathsf{T}\varpi_1)))\leq\mathsf{F}(\mathtt{H}(\sigma(\mathsf{T}\varpi_0,\mathsf{T}\varpi_1),\phi(\mathsf{T}\varpi_0),\phi(\mathsf{T}\varpi_1))).$$

Since T be a rational type $\mathsf{F}-\mathtt{H}-$contraction with respect to metric σ, we obtain that

$$\begin{aligned} & \mathsf{F}\left(\mathtt{H}\left(\sigma\left(\varpi_1, \mathsf{T}\varpi_1\right), \phi\left(\varpi_1\right), \phi\left(\mathsf{T}\varpi_1\right)\right)\right) \\ \leq\ & \mathsf{F}\left(\mathtt{H}\left(\sigma\left(\mathsf{T}\varpi_0, \mathsf{T}\varpi_1\right), \phi\left(\mathsf{T}\varpi_0\right), \phi\left(\mathsf{T}\varpi_1\right)\right)\right) \\ \leq\ & \mathsf{F}\left(\mathtt{H}\left(M\left(\varpi_0, \varpi_1\right), \phi\left(\varpi_0\right), \phi\left(\varpi_1\right)\right)\right) \\ \leq\ & \mathsf{F}\left(\mathtt{H}\left(\max\left\{\sigma\left(\varpi_0, \varpi_1\right), \frac{\sigma\left(\varpi_0, \mathsf{T}\varpi_0\right)\left[1+\sigma\left(\varpi_1, \mathsf{T}\varpi_1\right)\right]}{1+\sigma\left(\mathsf{T}\varpi_0, \mathsf{T}\varpi_1\right)}\right\}, \phi\left(\varpi_0\right), \phi\left(\varpi_1\right)\right)\right)-\tau. \end{aligned}$$

Indeed, it follows from Lemma 1.4 that

$$H(\sigma(\varpi_{k-1}, \varpi_k), \varphi(\varpi_{k-1}), \varphi(\varpi_k)) = 0.$$

Moreover, according to the condition (H1) satisfied by the function H, we have $\varphi(\varsigma) = 0$. Therefore, we can assume that $\varpi_{k-1} \notin \varpi_n$ for every $n \in \mathbb{N}$.

In this step, we show that $\{\varpi_n\}$ is a Cauchy. By Lemma 1.4, we say that

$$0 < \mathrm{h}_{n-1} = \mathtt{H}\left(\sigma\left(\varpi_{n-1}, \varpi_n\right), \phi\left(\varpi_{n-1}\right), \phi\left(\varpi_n\right)\right) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

There exist $k \in (0,1)$ such that $\mathrm{h}_n^k \mathsf{F}\left(\mathtt{H}_n\right) \rightarrow 0$ as $n \rightarrow \infty$ the condition (F3) of F. Using (2) with $\varpi = \varpi_{n-1}$ and $\upsilon = \varpi_n$ we get

$$\begin{aligned} & \mathsf{F}\left(\mathtt{H}\left(\sigma\left(\varpi_n, \varpi_{n+1}\right), \phi\left(\varpi_n\right), \phi\left(\varpi_{n+1}\right)\right)\right) \\ \leq\ & \mathsf{F}\left(\mathtt{H}\left(M\left(\varpi_{n-1}, \varpi_n\right), \phi\left(\varpi_{n-1}\right), \phi\left(\varpi_n\right)\right)\right)-\tau \\ \leq\ & \mathsf{F}\left(\mathtt{H}\left(\max\left\{\sigma\left(\varpi_{n-1}, \varpi_n\right), \frac{\sigma\left(\varpi_{n-1}, \mathsf{T}\varpi_{n-1}\left[1+\sigma\left(\varpi_n, \mathsf{T}\varpi_n\right)\right]\right)}{1+\sigma\left(\mathsf{T}\varpi_{n-1}, \mathsf{T}\varpi_n\right)}\right\}, \phi\left(\varpi_{n-1}\right), \phi\left(\varpi_n\right)\right)\right)-\tau \\ \leq\ & \mathsf{F}\left(\mathtt{H}\left(\sigma\left(\varpi_{n-1}, \varpi_n\right), \phi\left(\varpi_{n-1}\right), \phi\left(\varpi_n\right)\right)\right)-\tau \\ & \vdots \\ \leq\ & \mathsf{F}\left(\mathtt{H}\left(\sigma\left(\varpi_0, \varpi_1\right), \phi\left(\varpi_0\right), \phi\left(\varpi_1\right)\right)\right)-n\tau \end{aligned}$$

for all $n \in \mathbb{N}$; that is

$$\mathsf{F}\left(\mathrm{h}_n\right) \leq \mathsf{F}\left(\mathrm{h}_{n-1}\right)-\tau \leq \cdots \leq \mathsf{F}\left(\mathrm{h}_0\right)-n\tau \text{ for all } n \in \mathbb{N}.$$

From

$$0 = \lim_{n\rightarrow+\infty} \mathrm{h}_n^k \mathsf{F}\left(\mathrm{h}_n\right) \leq \lim_{n\rightarrow+\infty} \mathrm{h}_n^k \left(\mathsf{F}\left(\mathrm{h}_0\right)-n\tau\right) \leq 0,$$

we deduce that

$$\lim_{n\rightarrow+\infty} \mathrm{h}_n^k n = 0.$$

This provides that $\sum_{n=1}^{\infty} \mathrm{h}_n$ the convergent. According to condition (H1) satisfied by the function H, also the series $\sum_{n=1}^{\infty} \sigma\left(\varpi_n, \varpi_{n+1}\right)$ is convergent. Thus $\{\varpi_n\}$ is a Cauchy sequence in (Y, σ).

From (3) the sequence $\{\varpi_n\}$ is a Cauchy in (Y, ρ) too. Given that (Y, ρ) is a complete metric space, there exist $\varpi \in \mathsf{Y}$ with $\rho\left(\varpi_n, \varpi\right) \rightarrow 0$ as $n \rightarrow \infty$.

Now since (Y, ρ) is complete, there exists $\varsigma \in \mathsf{Y}$ such that

$$\lim_{n\rightarrow+\infty} \varpi_n = 0.$$

By (2) and (3), With the assumption that ϕ is lower semi-continuous, we have

$$0 \leq \phi\left(\varsigma\right) \leq \liminf_{n\rightarrow+\infty} \phi\left(\varpi_n\right) = 0;$$

that is, $\phi\left(\varsigma\right) = 0$. Now show that ς is a fixed point. If there exists a subsequence $\{\varpi_{n_k}\}$ of $\{\varpi_n\}$ such that $\varpi_{n_k} = \varsigma$ or $\mathsf{T}\varpi_{n_k} = \mathsf{T}\varsigma$ for all $k \in \mathbb{N}$, then ς, is a fixed point. If this is not the case, we can assume that $\varpi_n \neq \varsigma$ and $\mathsf{T}\varpi_{n_k} \neq \mathsf{T}\varsigma$ for all $n \in \mathbb{N}$. So using (2) with $\varpi = \varpi_n$ and $\upsilon = \varsigma$, we deduce that

$$\tau + \mathsf{F}\left(\mathtt{H}\left(\rho\left(\mathsf{T}\varpi_n, \mathsf{T}\varsigma\right), \phi\left(\mathsf{T}\varpi_n\right), \phi\left(\mathsf{T}\varsigma\right)\right)\right) \leq \mathsf{F}\left(\mathtt{H}\left(M\left(\varpi_n, \varsigma\right), \phi\left(\varpi_n\right), \phi\left(\varsigma\right)\right)\right).$$

Since $\tau > 0$, we obtain

$$\mathtt{H}\left(\rho\left(\mathsf{T}\varpi_n, \mathsf{T}\varsigma\right), \phi\left(\mathsf{T}\varpi_n\right), \phi\left(\mathsf{T}\varsigma\right)\right) \leq \mathtt{H}\left(\rho\left(\varpi_n, \varsigma\right), \phi\left(\varpi_n\right), \phi\left(\varsigma\right)\right) \text{ for all } n \in \mathbb{N}$$

and so

$$\begin{array}{rcl}\rho\left(\varsigma, \mathsf{T}\varsigma\right) & \leq & \rho\left(\varsigma, \mathsf{Y}\right) + \rho\left(\mathsf{T}\varpi_n, \mathsf{T}\varsigma\right) \\ & \leq & \rho\left(\varsigma, \varpi_{n+1}\right) + \mathtt{H}\left(\rho\left(\mathsf{T}\varpi_n, \mathsf{T}\varsigma\right), \phi\left(\mathsf{T}\varpi_n\right), \phi\left(\mathsf{T}\varsigma\right)\right) \\ & \leq & \rho\left(\varsigma, \varpi_{n+1}\right) + \mathsf{F}\left(\mathtt{H}\left(\rho\left(\mathsf{T}\varpi_n, \mathsf{T}\varsigma\right), \phi\left(\mathsf{T}\varpi_n\right), \phi\left(\mathsf{T}\varsigma\right)\right)\right) \\ & < & \rho\left(\varsigma, \varpi_{n+1}\right) + \mathsf{F}\left(\mathtt{H}\left(M\left(\varpi_n, \varsigma\right), \phi\left(\varpi_n\right), \phi\left(\varsigma\right)\right)\right) \\ & < & \rho\left(\varsigma, \varpi_{n+1}\right) + \mathsf{F}\left(\mathtt{H}\left(\max\left\{\rho\left(\varpi_n, \varsigma\right), \dfrac{\rho\left(\varpi_n, \mathsf{T}\varpi_n\right)\left[1+\rho\left(\varsigma, \mathsf{T}\varsigma\right)\right]}{1+\rho\left(\mathsf{T}\varpi_n, \mathsf{T}\varsigma\right)}\right\}, \phi\left(\varpi_n\right), \phi\left(\varsigma\right)\right)\right) \\ & \leq & \rho\left(\varsigma, \varpi_{n+1}\right) + \mathsf{F}\left(\mathtt{H}\left(\max\left\{\rho\left(\varpi_n, \varsigma\right), \dfrac{\rho\left(\varpi_n, \varpi_{n+1}\right)\left[1+\rho\left(\varsigma, \mathsf{T}\varsigma\right)\right]}{1+\rho\left(\varpi_{n+1}, \mathsf{T}\varsigma\right)}\right\}, \phi\left(\varpi_n\right), \phi\left(\varsigma\right)\right)\right)\end{array}$$

for all $n \in \mathbb{N}$.

By taking the limit as n approaches infinity in the above computations, and using that $\mathtt{H}$ is continuous in $(0,0,0)$, we deduce that $\rho\left(\varsigma, \mathsf{T}\varsigma\right) \leq \mathtt{H}(0,0,0) = 0$; that is $\varsigma = \mathsf{T}\varsigma$. ☐

Example 2.3. *Let* $\mathsf{Y} = \left\{\varpi_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\right\}$, $\rho\left(\varpi, \upsilon\right) = \left|\varpi - \upsilon\right|$ *and*

$$\sigma\left(\varpi, \upsilon\right) = \left\{\begin{array}{ccc} 0 & , & \varpi = \upsilon \\ 1 + \left|\varpi - \upsilon\right| & , & \varpi \neq \upsilon \end{array}\right.$$

then (Y, ρ) *is complete metric space. Define a map* $\mathsf{T} : \mathsf{Y} \to \mathsf{Y}$

$$\mathsf{T}\varpi = \left\{\begin{array}{ccl} \varpi_1 & , & \varpi = \varpi_1 \\ \varpi_{n-1} & , & \varpi = \varpi_n,\ n \geq 2 \end{array}\right.$$

Clearly, when $\mathsf{F}\left(\alpha\right) = \ln \alpha$, T *is not an* F*-contraction.* T *is a rational* $\mathsf{F} - \mathrm{H}-$*contraction with respect to the functions* $\mathsf{F} \in \mathcal{F}$ *defined* $\mathsf{F}\left(\alpha\right) = \ln \alpha + \alpha$ *for* $\alpha \geq 0$, $\mathit{H} \in \mathrm{H}$ *defined by* $\mathit{H}(\varsigma, \tau, \phi) = \max\left(\varsigma, \tau\right) + \phi$ *for all* $\varsigma, \tau, \phi \in (0, \infty)$*, the real number* $\tau = 1$ *and an associated lower semi-continuous function* $\phi : \mathsf{Y} \to (0, \infty)$, $\phi\left(t\right) = t$, *for* $t \in \mathsf{Y}$*. The contractive condition of Teorem 2.2 is equivalent to the following:*

$$\mathtt{H}\left(\sigma\left(\mathsf{T}\varpi, \mathsf{T}\upsilon\right), \phi\left(\mathsf{T}\varpi\right), \phi\left(\mathsf{T}\upsilon\right)\right) > 0,$$

$$\frac{\mathtt{H}(\sigma(\mathsf{T}\varpi,\mathsf{T}\upsilon),\phi(\mathsf{T}\varpi),\phi(\mathsf{T}\upsilon))}{\mathtt{H}(M(\varpi,\upsilon),\phi(\varpi),\phi(\upsilon))} e^{\mathtt{H}(\sigma(\mathsf{T}\varpi,\mathsf{T}\upsilon),\phi(\mathsf{T}\varpi),\phi(\mathsf{T}\upsilon)) - \mathtt{H}(M(\varpi,\upsilon),\phi(\varpi),\phi(\upsilon))} \leq e^{-1}.$$

First of all, observe that for every $m, n \in \mathbb{N}$

$$\mathtt{H}\left(\sigma\left(\mathsf{T}\varpi, \mathsf{T}\upsilon\right), \phi\left(\mathsf{T}\varpi\right), \phi\left(\mathsf{T}\upsilon\right)\right) > 0 \Leftrightarrow \left(m > 2 \text{ and } n = 1\right) \text{ or } \left(m > n > 1\right).$$

Therefore, the following two cases need to be considered:
Case 1: When $m > 2$ *and* $n = 1$*, it holds that*

$$\frac{\mathtt{H}\left(\sigma\left(\mathsf{T}\varpi, \mathsf{T}\upsilon\right), \phi\left(\mathsf{T}\varpi\right), \phi\left(\mathsf{T}\upsilon\right)\right)}{\mathtt{H}\left(M\left(\varpi, \upsilon\right), \phi\left(\varpi\right), \phi\left(\upsilon\right)\right)} e^{\mathtt{H}(\sigma(\mathsf{T}\varpi,\mathsf{T}\upsilon),\phi(\mathsf{T}\varpi),\phi(\mathsf{T}\upsilon)) - \mathtt{H}(M(\varpi,\upsilon),\phi(\varpi),\phi(\upsilon))} \leq e^{-m} \leq e^{-1}.$$

Case 2. When $m > n > 1$*, it holds that*

$$\frac{\mathtt{H}\left(\sigma\left(\mathsf{T}\varpi, \mathsf{T}\upsilon\right), \phi\left(\mathsf{T}\varpi\right), \phi\left(\mathsf{T}\upsilon\right)\right)}{\mathtt{H}\left(M\left(\varpi, \upsilon\right), \phi\left(\varpi\right), \phi\left(\upsilon\right)\right)} e^{\mathtt{H}(\sigma(\mathsf{T}\varpi,\mathsf{T}\upsilon),\phi(\mathsf{T}\varpi),\phi(\mathsf{T}\upsilon)) - \mathtt{H}(M(\varpi,\upsilon),\phi(\varpi),\phi(\upsilon))} \leq e^{-2(m+n)} \leq e^{-1}.$$

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On the complete controllability of fractional-order integrodifferential system involving generalized conformable derivatives

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ABSTRACT. Controllability concepts have been essential across various disciplines, including engineering, robotics, optimal control, and applied mathematics. Following Kalman's definition, controllability is characterized by the ability to steer the solution of a dynamical system from any initial state to a desired target at a given finite time. In this paper, the complete controllability of fractional integrodifferential systems involving generalized conformable derivatives is investigated. A set of sufficient conditions for the complete controllability of fractional-order integrodifferential semilinear systems is established, assuming that the associated linear part is completely controllable. The results obtained are based on the use of advanced mathematical tools, including the generalized exponential operator, the generalized conformable fractional calculus, and Krasnoselskii's fixed point theorem. A concrete example is provided at the end to illustrate the practical effectiveness of the theoretical findings.

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1. INTRODUCTION

In recent decades, Fractional Calculus, or non-integer order derivative, has been a topic of interest since Leibniz's time and has attracted the attention of mathematicians, physicists, and engineers. In 1695, [1] l'Hopital asked what it means to take the derivative of order p if $p \in (0, 1]$. Since then, the definition of fractional derivative has been one of the most important topics for many researchers [2, 3, 4, 5]. In fact, when working with traditional calculus and its uses, a major benefit is that both the derivative and the integral have clear physical and geometric interpretations. However, the physical and geometric significance of fractional operators has not yet been firmly established [6]. This continues to be an active area of research, with numerous attempts to solve these open problems, but a consensus on their fundamental meaning is still lacking. Consequently, several definitions for fractional derivatives have been proposed, such as the Riemann-Liouville, Hadamard, Erdelyi-kober, Grunwald-letnikov, Marchaud and Riesz definitions [7]. Furthermore, these definitions coincide with the classical definition of the first derivatives if $p = 1$.
For the first time, Khalil et al. [8] introduced a new fractional integral and derivative called " The Conformable Fractional Derivative" (CFD). This definition is a natural extension of the classical derivative because it is based on the same limit definition and follows the same core rules (like product and chain rules). Furthermore, Sarikaya, et al.[24] generalized the conformable fractional derivative and integral, with numerous effects, like the product rule, quotient rule, chain rule, linearity, Rolle's Theorem, and the Mean Value Theorem, for solving real-world problems, particularly in engineering and applied sciences. Unlike the non-local, memory-dependent Riemann-Liouville and Caputo derivatives, the GCFD provides a local alternative. It acts like a classical derivative but on a locally stretched or compressed time scale, making it ideal for phenomena sensitive to immediate power-law changes rather than the system's full history. This local nature is confirmed by its successful application in modeling systems without long-range memory[22, 23]. In nonlinear control theory, the GCFD provides a framework for analyzing system dynamics. This analysis is essential for synthesizing effective control inputs that steer a system along a desired trajectory between states.
The concept of controllability was first defined by Kalman [10] in 1960 as the ability to move a system from any starting point to any desired point within a finite time. Later, when

analyzing systems in infinite-dimensional spaces, this concept was divided into the distinct notions of complete and approximate controllability.
Wei [11] studied the necessary and sufficient conditions for the controllability of fractional control systems with control delays by providing solutions for these systems. Jneid[12] obtained a sufficient conditions for approximate controllability of semilinear integro-differential fractional control systems with nonlocal conditions. Jneid and Awadalla [13] established a set of practical controllability conditions for conformable fractional derivative semi-linear fractional in finite-dimensional control systems. The present methodology is based on the iterative method, the Gramian matrix, and the exponential matrix of conformable fractional derivatives. Azouz [14] used the rank and the conformable Gram criterion to investigate the controllability of differential linear and nonlinear systems with the general conformable fractional derivative. Balachandran and Anthoni [15] discussed the controllability of second-order semi-linear differential systems in Banach spaces. Park and Han [16] utilized the Schauder fixed-point theorem to develop sufficient conditions for the approximate controllability of second-order integrodifferential systems in Banach spaces. The controllability of second-order semi-linear Volterra integrodifferential systems in Banach spaces was studied by Balachandran et al.[17], and delay integrodifferential systems were investigated by them [18]. Balachandran et al. [19] examined sufficient conditions for controllability and provided controllability of second-order nonlinear integrodifferential systems in Banach spaces. Using the Schaefer fixed-point theorem and strongly continuous cosine families of bounded linear operators, results are established. Jneid [20, 21] studied the partial controllability of semilinear fractional control systems in a conformable sense.
Despite a thorough literature review, no prior studies on the complete controllability of integro-differential semilinear control systems with general conformable fractional derivative were found. This identified gap motivates the present work, which investigates this property for these kinds of systems in finite-dimensional spaces.

This paper is structured as follows: Section 2 outlines the essential definitions and preliminary facts related to the linear case. Section 3 establishes sufficient conditions for semilinear integrodifferential systems governed by the generalized conformable fractional derivative. Section 4 provides an illustrative example to validate the primary findings. Finally, the main results and conclusions are summarized in Section 5.

2. Preliminaries

This section outlines the essential definitions and theorems that form the basis of our work. For more details see [14, 24].

Definition 2.1. For all $s \in J$ with $J = [0, b]$. The general conformable fractional derivative "$GCFD$" of order $p \in (0, 1]$ of a function $g : (0, \infty) \to \mathbb{R}$ for all $s \in J$ with $J = [0, b]$ is defined by:

$$ {}^{GC}D^p_\varphi g(s) = \lim_{\epsilon \to 0} \frac{g(s + \epsilon\varphi(s, p)) - g(s)}{\epsilon} \tag{1} $$

where $\varphi(s, p)$ is a continuous nonnegative function of order $p \in (0, 1]$ that depends on p and satisfies

$$ \begin{cases} \varphi(s, 1) = 1, \\ \text{If } p \neq q, \text{ then } \varphi(\cdot, p) \neq \varphi(\cdot, q), \quad \text{where } p, q \in (0, 1]. \end{cases} \tag{2} $$

Whenever we determine a function $\varphi(s, p)$, we get a CFD definition form. Let k, b be constant numbers, $k + b = 1$, $\beta > 0$, $a \neq 0$, $h(p) \neq 0$, $0 < p \leq 1$, and $\psi(s, p)$ is a monotonous function of s such that $\psi(s, 1) = a$.

From elementary simple functions, we get various $\varphi(s, p)$ such that:

- **Linear**: $\varphi(s, p) = pk + b$, particularly $\varphi(s, p) = p$,
- **Power**: $\varphi(s, p) = p^\beta$, particularly $\varphi(s, p) = p^2$,
- **Exponent**: $\varphi(s, p) = a^{(1-p)h(p)}$, particularly $\varphi(s, p) = s^{(1-p)h(p)}$,
- **Logarithm**: $\varphi(s, p) = \log_a \psi(s, p)$,
- **Trigonometric**: $\varphi(s, p) = \sin\left(\frac{\pi}{2}p\right)$ or $\varphi(s, p) = \tan\left(\frac{\pi}{2}p\right)$.

Remark 1.

$$^{GC}D^p_\varphi g(0) := \lim_{s\to 0^+} {}^{GC}D^p_\varphi g(s) \tag{3}$$

Theorem 2.2. Let $0 < p \leq 1$ and the functions $g, h : (0, \infty) \to \mathbb{R}$ be p-differentiable at $s \in J$. Then:

(1) $^{GC}D^p_\varphi(ag + bh)(s) = a\,{}^{GC}D^p_\varphi g(s) + b\,{}^{GC}D^p_\varphi h(s)$ for all $a, b \in \mathbb{R}$.
(2) $^{GC}D^p_\varphi(k) = 0$ for all constant functions.
(3) $^{GC}D^p_\varphi(gh)(s) = g(s)\,{}^{GC}D^p_\varphi h(s) + h(s)\,{}^{GC}D^p_\varphi g(s)$.
(4) $^{GC}D^p_\varphi\left(\frac{g}{h}\right)(s) = \frac{g(s)\,{}^{GC}D^p_\varphi h(s) - h(s)\,{}^{GC}D^p_\varphi g(s)}{h^2(s)}$.
(5) **Chain rule**: If g is differentiable, then $^{GC}D^p_\varphi(g \circ h)(s) = g'(h(s))\,{}^{GC}D^p_\varphi h(s)$.
(6) If g is differentiable, then $^{GC}D^p_\varphi g(s) = \varphi(s, p)\,\frac{dg}{ds}(s)$.
(7) If g is differentiable, and $^{GC}D^p_\varphi(g) = g$, then

$$g(s) = ke^{\int \frac{1}{\varphi(s,p)}ds}$$

where k is any positive constant.

Definition 2.3. Let $s \in J$ and g be a function defined on $(0, \infty)$. Then the general conformable fractional integral "$GCFI$" is defined by:

$$^{GC}I^p_\varphi g(s) = \int_0^s \frac{g(t)}{\varphi(t, p)}dt \tag{4}$$

Theorem 2.4. Let $s \in J$, $p \in (0, 1]$, and g be a continuous function defined on $(0, \infty)$ such that $^{GC}I^p_\varphi g(s)$ exists. Then

$$^{GC}D^p_\varphi\left({}^{GC}I^p_\varphi\, g(s)\right) = g(s). \tag{5}$$

Theorem 2.5. Let $s \in J$, $p \in (0, 1]$, and $g : (0, \infty) \to \mathbb{R}$ be differentiable. Then,

$$^{GC}I^p_\varphi\left({}^{GC}D^p_\varphi(g(s)\right) = g(s) - g(0). \tag{6}$$

Definition 2.6. Let A be an operator. Then the generalized exponential operator is defined as:

$$^{GC}E^p_\varphi(A, s, 0) = e^{\left(A\int_0^s \frac{1}{\varphi(t,p)}dt\right)} \tag{7}$$

Theorem 2.7. *The following relations are satisfied:*

(1) $^{GC}E^p_\varphi(A, s, 0) \cdot {}^{GC}E^p_\varphi(B, s, 0) = {}^{GC}E^p_\varphi(A + B, s, 0)$
(2) $^{GC}E^p_\varphi(A\,B, s, 0) = \left[{}^{GC}E^p_\varphi(A, s, 0)\right]^B$
(3) $^{GC}E^p_\varphi(A, s, t) = {}^{GC}E^p_\varphi(A, s, 0)\ .\ {}^{GC}E^p_\varphi(-A, t, 0)$
(4) $^{GC}D^p_\varphi\left[{}^{GC}E^p_\varphi(A, s, 0)\right] = A\ \times\ {}^{GC}E^p_\varphi(A, s, 0)$
(5) $^{GC}I^p_\varphi\left[{}^{GC}E^p_\varphi(A, s, 0)\right] = \int_0^s \frac{{}^{GC}E^p_\varphi(A,t,0)}{\varphi(t,p)}dt$
(6) $^{GC}D^p_\varphi\ \left[{}^{GC}I^p_\varphi\left({}^{GC}E^p_\varphi(A, s, 0)\right)\right] = E^p_\varphi(A, s, 0)$
(7) $^{GC}I^p_\varphi\ \left[{}^{GC}D^p_\varphi\left({}^{GC}E^p_\varphi(A, s, 0)\right)\right] = E^p_\varphi(A, s, 0) - E^p_\varphi(A, 0, 0)$

Proof:

(1)

$$\begin{aligned}
^{GC}E^p_\varphi(A, s, 0) \cdot {}^{GC}E^p_\varphi(B, s, 0) &= e^{A\int_0^s \frac{1}{\varphi(t,p)}dt} \cdot e^{B\int_0^s \frac{1}{\varphi(t,p)}dt} \\
&= e^{(A+B)\int_0^s \frac{1}{\varphi(t,p)}dt} \\
&= {}^{GC}E^p_\varphi(A + B, s, 0)
\end{aligned}$$

(2)

$$\begin{aligned}
^{GC}E^p_\varphi(A\,B, s, 0) &= e^{(AB)\int_0^s \frac{1}{\varphi(t,p)}dt} \\
&= \left[e^{A\int_0^s \frac{1}{\varphi(t,p)}dt}\right]^B \\
&= \left[{}^{GC}E^p_\varphi(A, s, 0)\right]^B
\end{aligned}$$

(3)

$$\begin{aligned} {}^{GC}E^p_\varphi(A,s,t) &= e^{A\int_t^s \frac{1}{\varphi(\tau,p)}d\tau} \\ &= \frac{e^{A\int_0^s \frac{1}{\varphi(\tau,p)}d\tau}}{e^{A\int_0^t \frac{1}{\varphi(\tau,p)}d\tau}} \\ &= e^{A\int_0^s \frac{1}{\varphi(\tau,p)}d\tau}.e^{-A\int_0^t \frac{1}{\varphi(\tau,p)}d\tau} \\ &= {}^{GC}E^p_\varphi(A,s,0).{}^{GC}E^p_\varphi(-A,t,0) \end{aligned}$$

(4)

$$\begin{aligned} {}^{GC}D^p_\varphi\left[{}^{GC}E^p_\varphi(A,s,0)\right] &= {}^{GC}D^p_\varphi\left[e^{A\int_0^s \frac{1}{\varphi(t,p)}dt}\right] \\ &= A\cdot e^{A\int_0^s \frac{1}{\varphi(t,p)}dt} \\ &= A\cdot {}^{GC}E^p_\varphi(A,s,0) \end{aligned}$$

(5)

$$\begin{aligned} {}^{GC}I^p_\varphi\left[{}^{GC}E^p_\varphi(A,s,0)\right] &= {}^{GC}I^p_\varphi\left[e^{A\int_0^s \frac{1}{\varphi(t,p)}dt}\right] \\ &= \int_0^s \frac{e^{A\int_0^t \frac{1}{\varphi(\tau,p)}d\tau}}{\varphi(t,p)}dt \\ &= \int_0^s \frac{{}^{GC}E^p_\varphi(A,t,0)}{\varphi(t,p)}dt \end{aligned}$$

(6)

$$\begin{aligned} {}^{GC}D^p_\varphi\left[{}^{GC}I^p_\varphi\left({}^{GC}E^p_\varphi(A,s,0)\right)\right] &= {}^{GC}D^p_\varphi\int_0^s \frac{{}^{GC}E^p_\varphi(A,t,0)}{\varphi(t,p)}dt \\ &= \varphi(s,p)\cdot\frac{{}^{GC}E^p_\varphi(A,s,0)}{\varphi(s,p)} \\ &= {}^{GC}E^p_\varphi(A,s,0) \end{aligned}$$

(7)

$$\begin{aligned} {}^{GC}I^p_\varphi\left[{}^{GC}D^p_\varphi\left({}^{GC}E^p_\varphi(A,s,0)\right)\right] &= \int_0^s \frac{{}^{GC}D^p_\varphi\left({}^{GC}E^p_\varphi(A,t,0)\right)}{\varphi(t,p)}dt \\ &= \int_0^s \frac{1}{\varphi(t,p)}\cdot\varphi(t,p)\frac{d}{dt}{}^{GC}E^p_\varphi(A,t,0)\,dt \\ &= {}^{GC}E^p_\varphi(A,s,0) - {}^{GC}E^p_\varphi(A,0,0) \end{aligned}$$

Consider the following linear system:

$$\left\{\begin{array}{ll} {}^{GC}D^p_\varphi z(s) = Az(s) + Bu(s), & s\in J \\ z(0) = z_0 \in Z \end{array}\right. \tag{8}$$

where:

- ${}^{GC}D^p_\varphi z(s)$ is the general conformable fractional derivative of order $p \in (0,1]$
- $A : \mathcal{D}(A) \subset \mathbb{R}^n \to \mathbb{R}^n$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{As}, s \in J\}$ on $\mathbb{R}^n$.
- $B : U \to \mathbb{R}^n$ is a bounded linear control operator.
- $z(s)$ and $u(s) \in \mathbb{R}^n$ are the state and control vectors respectively .

Theorem 2.8. *The mild solution of equation* (8) *is given by:*

$$z(s) = {}^{GC}E^p_\varphi(A,s,0)z_0 + \int_0^s \frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(A,s,t)Bu(t)\,dt \tag{9}$$

where:

$${}^{GC}E^p_\varphi(A,s,t) = e^{\left(A\int_t^s \frac{1}{\varphi(\tau,p)}d\tau\right)} \tag{10}$$

Proof. Since ${}^{GC}E^p_\varphi(-A,s,0)\neq 0$, we can multiply both sides of (8) by ${}^{GC}E^p_\varphi(-A,s,0)$ and apply the fractional integration operator to obtain the solution.

$$^{GC}D^p_\varphi\left[{}^{GC}E^p_\varphi(-A,s,0)z(s)\right]={}^{GC}E^p_\varphi(-A,s,0)Bu(s)$$

$$^{GC}I^p_\varphi\left\{{}^{GC}D^p_\varphi\left[{}^{GC}E^p_\varphi(-A,s,0)z(s)\right]\right\}={}^{GC}I^p_\varphi\left\{{}^{GC}E^p_\varphi(-A,s,0)Bu(s)\right\}$$

$$^{GC}E^p_\varphi(-A,s,0)z(s)-{}^{GC}E^p_\varphi(-A,0,0)z(0)=\int_0^s\frac{{}^{GC}E^p_\varphi(-A,t,0)}{\varphi(t,p)}Bu(t)dt \tag{11}$$

Multiplying both sides of equation (11) by ${}^{GC}E^p_\varphi(A,s,0)$ yields:

$$z(s)={}^{GC}E^p_\varphi(A,s,0)z_0+\int_0^s\frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(A,s,t)Bu(t)dt$$

Verification of Solution:

$$\begin{aligned}
{}^{GC}D^p_\varphi z(s)&={}^{GC}D^p_\varphi\left[{}^{GC}E^p_\varphi(A,s,0)z_0\right]+{}^{GC}D^p_\varphi\left[\int_0^s\frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(A,s,t)Bu(t)dt\right]\\
&=A^{GC}E^p_\varphi(A,s,0)z_0+\frac{d}{ds}\left(\int_0^s\frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(A,s,t)Bu(t)dt\right)\cdot\varphi(s,p)\\
&=A^{GC}E^p_\varphi(A,s,0)z_0+A\int_0^s\frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(A,s,t)Bu(t)dt+Bu(s)\\
&=Az(s)+Bu(s)
\end{aligned}$$

□

Definition 2.9. Let $z_0,z_1\in\mathbb{R}^n$. If there exists a control $u(s)$ such that the solution $z(s)$ of (8) satisfies $z(0)=z_0$ and $z(b)=z_1$, then the system is said to be completely controllable on $[0,b]$.

The controllability Gram matrix W is defined as:

$$W=\int_0^b\frac{1}{\varphi(t,p)}\left[{}^{GC}E^p_\varphi(A,b,t)B\right]\left[{}^{GC}E^p_\varphi(A,b,t)B\right]^*dt \tag{12}$$

where $*$ denotes the matrix transpose.

Theorem 2.10. *Let $b>0$. The controllability Gram matrix W defined by* (12) *is positive definite if and only if the control system* (8) *is controllable on* $[0,b]$.

Proof. (⇒) Assume W is positive definite and invertible. Define the control function:

$$u(s)=B^{*\,GC}E^p_\varphi(-A^*,s,0)\,{}^{GC}E^p_\varphi(A^*,b,0)W^{-1}\left[z_1-{}^{GC}E^p_\varphi(A,b,0)z_0\right] \tag{13}$$

Substituting into the solution (9) at $s=b$:

$$\begin{aligned}
z(b)&={}^{GC}E^p_\varphi(A,b,0)z_0+\int_0^b\frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(A,b,t)B\,B^{*GC}E^p_\varphi(-A^*,t,0)dt\\
&\times{}^{GC}E^p_\varphi(A^*,b,0)W^{-1}\left[z_1-{}^{GC}E^p_\varphi(A,b,0)z_0\right]\\
&={}^{GC}E^p_\varphi(A,b,0)z_0+WW^{-1}\left[z_1-{}^{GC}E^p_\varphi(A,b,0)z_0\right]\\
&=z_1
\end{aligned}$$

(⇐) Assuming the Gram matrix W is not positive definite, there exists $y\neq 0$ such that:

$$y^*Wy=\int_0^b\frac{1}{\varphi(t,p)}\left\|y^{*GC}E^p_\varphi(A,b,t)B\right\|^2dt=0$$

$$\Rightarrow y^{*GC}E^p_\varphi(A,b,t)B\equiv 0\quad\forall t\in[0,b]$$

Choose initial condition $z_0=\left[{}^{GC}E^p_\varphi(A,b,0)\right]^{-1}y$. For any control $u(t)$, the solution at $t=b$ satisfies:

$$\begin{aligned}
z(b)&={}^{GC}E^p_\varphi(A,b,0)z_0+\int_0^b\frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(A,b,t)Bu(t)dt\\
&=y+\int_0^b\frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(A,b,t)Bu(t)dt
\end{aligned}$$

Mulpily by $y*$

$$y^*y + \int_0^b \frac{1}{\varphi(t,p)} \underbrace{y^* \, {}^{GC}E_\varphi^p(A,b,t)B}_{=0 \text{ by earlier result}} u(t)dt = 0 \Rightarrow y^*y = 0$$

This contradicts $y \neq 0$, proving the Gram matrix W must be positive definite for controllability.

□

Theorem 2.11. *The system* (8) *is controllable on* J *if and only if the controllability matrix*

$$\phi_c = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \tag{14}$$

has full rank, i.e., rank$(\phi_c) = n$*, where* n *is the dimension of the state vector* $z(t)$*.*

Proof. ($\Rightarrow$) Assume the system is not controllable. By theorem (2.10), there exists $y \neq 0$ such that:

$$y^* \left[{}^{GC}E_\varphi^p(A,b,s)\, B\right] = 0, \quad \forall s \in [0,b] \tag{15}$$

Evaluating at $s = b$ yields:

$$y^*B = 0$$

Differentiating (15) with respect to s and using $\varphi(s,p) \neq 0$ gives:

$$y^*A \left[{}^{GC}E_\varphi^p(A,b,s)\, B\right] = 0$$

Setting $s = b$ again produces:

$$y^*AB = 0$$

Repeating this process $n-1$ times generates the full set of conditions:

$$y^* \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = 0$$

This contradicts the full rank assumption of ϕ_c, proving necessity.

[Proof of Necessity ($\Leftarrow$)] Assume rank$(\phi_c) < n$. Then $\exists y \neq 0$ such that:

$$y^*\phi_c = y^* \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0 \tag{16}$$

$$\Rightarrow y^*A^iB = 0 \quad \text{for } i = 0,1,\ldots,n-1$$

By the Cayley-Hamilton theorem, this implies:

$$y^*e^{A\alpha}B = 0, \quad \forall \alpha = \int_0^s \frac{1}{\varphi(t,p)}dt \in \mathbb{R}.$$

For the fractional system, we obtain:

$$\begin{aligned} y^* \, {}^{GC}E_\varphi^p(A,b,s)\, B &= y^* e^{A\int_0^b \frac{1}{\varphi(t,p)}dt} e^{-A\int_0^s \frac{1}{\varphi(t,p)}dt} B \\ &= y^* e^{A\int_s^b \frac{1}{\varphi(t,p)}dt} B = 0 \quad \forall s \in J \end{aligned} \tag{17}$$

This leads to:

$$y^*Wy = \int_0^b \frac{1}{\varphi(t,p)} \left\| y^* \, {}^{GC}E_\varphi^p(A,b,t)B \right\|^2 dt = 0$$

$$\Rightarrow W \text{ is not positive definite}$$

Therefore, the system is not controllable when rank$(\phi_c) < n$. □

3. Controllability Results

Consider the following integrodifferential system [25]:

$$\begin{cases} {}^{GC}D_\varphi^p z(s) = Az(s) + B \;\; u_\varphi^p(s) + g(s, z(s), \int_0^s \; h(s,r,z(r))dr), \;\; s \in J \\ z(0) = z_0 \end{cases} \tag{18}$$

where:

- ${}^{GC}D_\varphi^p z(s)$ is the general conformable fractional derivative of order $p \in (0,1]$
- $\varphi : J \times (0,1] \to \mathbb{R}^+$ is a nonnegative continuous function generalizing both standard fractional and conformable derivatives
- $z(s)$ and $u(s) \in \mathbb{R}^n$ are the state and control vectors respectively
- $A : \mathcal{D}(A) \subset \mathbb{R}^n \to \mathbb{R}^n$ is the infinitesimal generator of a strongly continuous semigroup $\{e^{As}, s \in J\}$ on $\mathbb{R}^n$

- $B : U \to \mathbb{R}^n$ is a bounded linear control operator.
- $g : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $h : J \times J \times \mathbb{R}^n \to \mathbb{R}^n$ are nonlinear functions
- The solution space is $\hat{R} = C(J, \mathbb{R}^n)$ (continuous functions from J to $\mathbb{R}^n$)

Definition 3.1. *Define the nonlinear map of* ${}^{GC}\mathcal{P}^p_\varphi : \hat{R} \to \hat{R}$ *by:*

$$\begin{aligned} {}^{GC}\mathcal{P}^p_\varphi(z(s)) &= {}^{GC}E^p_\varphi(A, s, 0)z_0 + \int_0^s \frac{1}{\varphi(t,p)} {}^{GC}E^p_\varphi(A, s, t) \\ &\times \Big[g\Big(t, z(t), \int_0^t h(t, r, z(r))dr\Big) + B\,{}^{GC}u^p_\varphi(t)\Big]dt \end{aligned} \tag{19}$$

where:

- ${}^{GC}u^p_\varphi(s) \in L^2(J, U)$ *is the admissible control process*
- *The control function is given by:*

$$\begin{aligned} {}^{GC}u^p_\varphi(s) &= \mathcal{Q}^{-1}\Big[z_1 - {}^{GC}E^p_\varphi(A, b, 0)z_0 - \int_0^b \frac{1}{\varphi(t,p)} {}^{GC}E^p_\varphi(A, b, t) \\ &\times g\Big(t, z(t), \int_0^t h(t, r, z(r))dr\Big)dt\Big](s), \quad s \in J \end{aligned} \tag{20}$$

- $\mathcal{Q} : L^2(J, \mathbb{R}^n) \to \mathbb{R}^n$ *is an operator defined by:*

$$\mathcal{Q} = \int_0^s \frac{1}{\varphi(t,p)} {}^{GC}E^p_\varphi(A, s, t)\, B\, dt \tag{21}$$

Lemma 3.2. The mapping ${}^{GC}\mathcal{P}^p_\varphi(z(s))$ constitutes a mild solution of system (18).

Proof. Since ${}^{GC}E^p_\varphi(-A, s, 0) \neq 0.$, we can multiply equation (18) by ${}^{GC}E^p_\varphi(-A, s, 0)$, Then we obtain the following:

$$\begin{aligned} &{}^{GC}E^p_\varphi(-A, s, 0){}^{GC}D^p_\varphi z(s) - A{}^{GC}E^p_\varphi(-A, s, 0)z(s) = {}^{GC}E^p_\varphi(-A, s, 0)B\,{}^{GC}u^p_\varphi(s) \\ &+ {}^{GC}E^p_\varphi(-A, s, 0)g\Big(s, z(s), \int_0^s h(s, r, z(r))dr\Big) \end{aligned}$$

This can be rewritten as:

$$\begin{aligned} {}^{GC}D^p_\varphi\left[{}^{GC}E^p_\varphi(-A, s, 0)z(s)\right] &= {}^{GC}E^p_\varphi(-A, s, 0)B\,{}^{GC}u^p_\varphi(s) \\ &+ {}^{GC}E^p_\varphi(-A, s, 0)g\Big(s, z(s), \int_0^s h(s, r, z(r))dr\Big) \end{aligned}$$

Applying the fractional integral operator ${}^{GC}I^p_\varphi$, we get:

$$\begin{aligned} &{}^{GC}E^p_\varphi(-A, s, 0)z(s) - E^p_\varphi(-A, 0, 0)z(0) \\ &= \int_0^s \frac{1}{\varphi(t,p)}\, {}^{GC}E^p_\varphi(-A, t, 0)\; B\,{}^{GC}u^p_\varphi(t)dt \\ &+ \int_0^s \frac{1}{\varphi(t,p)} {}^{GC}E^p_\varphi(-A, t, 0)\; g\Big(t, z(t), \int_0^t h(t, r, z(r))dr\Big)dt \end{aligned} \tag{22}$$

Multiplying both sides of equation (22) by ${}^{GC}E^p_\varphi(A, s, 0)$, we finally obtain

$$\begin{aligned} \mathcal{P}^p_\varphi\, z(s) &= {}^{GC}E^p_\varphi(A, s, 0)\, z_0 + \int_0^s \frac{1}{\varphi(t,p)} {}^{GC}E^p_\varphi(A, s, t) \\ &\times \Big[B\, u(t) + g\Big(t, z(t), \int_0^t h(t, r, z(r))dr\Big)\Big]dt \end{aligned}$$

which confirms that ${}^{GC}P^p_\varphi z(s)$ is a mild solution of (18).

□

Now, let introduce the following assumptions:

$(\mathcal{A}_1)$: Let $\frac{1}{\varphi}$ be a square integrable function and $\mathcal{Q} : L^2(J, \mathbb{R}^n) \to \mathbb{R}^n$ be an operator such that is defined by the following equation:

$$\mathcal{Q}\,{}^{GC}u^p_\varphi(t) = \int_0^s \frac{1}{\varphi(t,p)} {}^{GC}E^p_\varphi(A, s, t)B\,{}^{GC}u^p_\varphi(t)dt. \tag{23}$$

$\mathcal{Q}^{-1} \in L^2(J, \mathbb{R}^n) \mid \ker \mathcal{Q}$ is an inverse operator of $\mathcal{Q}$.

($\mathcal{A}_2$): Let $g : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function, and for all $z_i, y_i \in \mathbb{R}^n$, $s \in J$, there exists $L_g(s) \in \hat{R}$ a non-negative function such that,

$$\|g(s, z_1, y_1) - g(s, z_2, y_2)\| \leq L_g(s)\Big(\|z_1 - z_2\| + \|y_1 - y_2\|\Big) \tag{24}$$

($\mathcal{A}_3$): Let $h : J \times J \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function, and for all $z_i \in \mathbb{R}^n$, $s \in J$, there exists $L_h(s,t) \in C(J \times J, \mathbb{R}^n)$ a non-negative function such that:

$$\Big\|\int_0^s h(s, r, z_1)dr - \int_0^s h(s, r, z_2)dr\Big\| \leq \int_0^s L_h(s, r)\|z_1 - z_2\|dr \leq L_h(s)\|z_1 - z_2\| \tag{25}$$

($\mathcal{A}_2$) + ($\mathcal{A}_3$): There exists non-negative functions $L_g(s) \in \hat{R}$ and $L_h(s,t) \in C(J \times J, \mathbb{R}^n)$ such that, for all $z_i, y_i \in \mathbb{R}^n$ and $s \in J$:

$$\begin{aligned}
&\Big\|g\big(s, z_1, \int_0^s h(s, r, z_1)\, dr\big) - g\big(s, z_2, \int_0^s h(s, r, z_2)\, dr\big)\Big\| \\
&\leq L_g(s)\Big(\|z_1 - z_2\| + \Big\|\int_0^s h(s, r, z_1)\, dr - \int_0^s h(s, r, z_2)dr\Big\|\Big) \\
&\leq L_g(s)\Big(\|z_1 - z_2\| + \int_0^s L_h(s, r)\|z_1 - z_2\|dr\Big) \\
&\leq L_g(s)\, \|z_1 - z_2\|\Big(1 + L_h(s)\Big)
\end{aligned} \tag{26}$$

We consider the following.

- $\mathcal{N}_A = \|A\|$
- $H = \big\|\mathcal{Q}^{-1}\big\|_{\mathbf{L}(\mathbb{R}^n, L^2(J,\mathbb{R}^n)|\ker \mathcal{Q})}$
- $S_g = \sup_{s\in J} \|g\left(s, 0, \int_0^s h(s, r, 0)dr\right)\|$
- $H_1 = {}^{GC}E^p_\varphi(\mathcal{N}_A, s, 0)\|z_0\| + \frac{S_g}{\mathcal{N}_A}\ {}^{GC}E^p_\varphi(\mathcal{N}_A, b, 0)\Big({}^{GC}E^p_\varphi(\mathcal{N}_A, b, 0) - 1\Big)$
- $H_2 = \frac{1}{\mathcal{N}_A}\ {}^{GC}E^p_\varphi(\mathcal{N}_A, b, 0)\Big({}^{GC}E^p_\varphi(\mathcal{N}_A, b, 0) - 1\Big)\|L_g\|_{\hat{R}}$
- $H_3 = \frac{1}{\mathcal{N}_A}\ {}^{GC}E^p_\varphi(\mathcal{N}_A, b, 0)\Big({}^{GC}E^p_\varphi(\mathcal{N}_A, b, 0) - 1\Big)\|L_g\|_{\hat{R}}\ \|L_h\|_{\hat{R}}$
- $H_4 = H\ {}^{GC}E^p_\varphi(\mathcal{N}_A, b, 0)\Big(\int_0^b \frac{1}{\varphi^2(t,p)}\ {}^{GC}E^p_\varphi(2\mathcal{N}_A, t, 0)dt\Big)^{\frac{1}{2}}$

Theorem 3.3. Suppose that (23) and (26) are fulfilled. If the following condition is satisfied

$$H_2\left[1 + H_4\|B\|\right] < 1, \tag{27}$$

then the system (18) is completely controllable on J.

Proof. There will be multiple steps involved in providing the proof.
Step 1: Showing that $\mathcal{P}(z_1(s)) + \mathcal{P}(z_2(s)) \in B_r$. Let B_r be a closed, bounded, and convex subset of $\hat{R}$ defined by

$$B_r = \{z \in \hat{R}, \quad \|z\|_C \leq r\} \tag{28}$$

Our aim is to prove that for any $z_1, z_2 \in B_r$, it hols that $\big(\mathcal{P}(z_1(s)) + \mathcal{P}(z_2(s)\big) \in B_r$ According to assumptions (23),(24) and (26) and using the fact $\|e^{As}\| \leq e^{\|As\|}$, it follows that for any $z \in B_r$:

$$\begin{aligned}
\|{}^{GC}u^p_\varphi(s)\|_{\mathbf{L}(Z,L^2(J,U)|ker\mathcal{Q})} &\leq \|\mathcal{Q}^{-1}\|_{L(Z,L^2(J,Z)|ker\mathcal{Q})} \\
&\times \Big\|z_1 - {}^{GC}E^p_\varphi(A, b, 0)z_o - \int_0^b \frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(A, b, 0){}^{GC}E^p_\varphi(-A, t, 0)\Big\| \\
&\times \Big\|g\big(t, z(t), \int_0^t h(t, r, z(r))dr\big)dt\Big\|
\end{aligned}$$

Applying the growth conditions on g, we obtain

$$\begin{aligned}
\|{}^{GC}u^p_\varphi(s)\| &\leq H\,\|z_1\| + H\ {}^{GC}E^p_\varphi(\mathcal{N}_A, b, 0)\|z_0\| \\
&+ H\ {}^{GC}E^p_\varphi(\mathcal{N}_A, b, 0)\int_0^b \frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(\mathcal{N}_A, t, 0) \\
&\Big\|g\big(t, z(t), \int_0^t h(t, r, z(r))dr\big) - g\big(t, 0, \int_0^t h(t, r, 0)dr\big) + g\big(t, 0, \int_0^t h(t, r, 0)dr\big)\Big\|dt
\end{aligned}$$

By the Lipschitz continuity of g and h, this further implies

$$
\begin{aligned}
\|{}^{GC}u^p_\varphi(s)\| \le\; & H\,\|z_1\| + H\ {}^{GC}E^p_\varphi(\mathcal{N}_A,b,0)\|z_0\| \\
& +\ H\ {}^{GC}E^p_\varphi(\mathcal{N}_A,b,0)\int_0^b \frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(\mathcal{N}_A,t,0) \\
& \times L_g(t)\Big(\|z_1\| + \int_0^t\ \|h(t,r,z(r)) -\ h(t,r,0))dr\|\Big)dt \\
& + H\ {}^{GC}E^p_\varphi(\mathcal{N}_A,b,0)\int_0^b \frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(\mathcal{N}_A,t,0)\left\|g\Big(t,0,\int_0^t\ h(t,r,0)dr\Big)\right\|dt \\
\le\; & H\,\|z_1\| + H\ {}^{GC}E^p_\varphi(\mathcal{N}_A,b,0)\|z_0\| \\
& +\ H\ {}^{GC}E^p_\varphi(\mathcal{N}_A,b,0)\int_0^b \frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(\mathcal{N}_A,t,0) \\
& \times L_g(t)\Big(\|z\| + \int_0^t L_h(t,r)\|z\|dr\Big)dt \\
& + H\ {}^{GC}E^p_\varphi(\mathcal{N}_A,b,0)\int_0^b \frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(\mathcal{N}_A,t,0)\left\|g\Big(t,0,\int_0^t\ h(t,r,0)dr\Big)\right\|dt
\end{aligned}
$$

Performing the integration and simplifying gives

$$
\begin{aligned}
\|{}^{GC}u^p_\varphi(s)\| \le\; & H\|z_1\| + H^{GC}E^p_\varphi(\mathcal{N}_A,b,0)\|z_0\| + \frac{H}{\mathcal{N}_A}{}^{GC}E^p_\varphi(\mathcal{N}_A,b,0)\big({}^{GC}E^p_\varphi(\mathcal{N}_A,b,0)-1\big) \\
& \times\Big[\|L_g\|_{\hat{R}}\ \|z\|\ \big(1+\|L_h\|_{\hat{R}}\big)+S_g\Big]
\end{aligned}
$$

Thus,

$$
\|{}^{GC}u^p_\varphi(s)\|\ \le H\Big(\|z_1\| +\ H_1 + r\,H_2(\,1+\ H_3)\Big) \tag{29}
$$

Next, decompose the operator ${}^{GC}\mathcal{P}^p_\varphi = {}^{GC}\mathcal{P}_1 + {}^{GC}\mathcal{P}_2$, where for all $s \in J$

$$
{}^{GC}\mathcal{P}_1(z(s)) = {}^{GC}E^p_\varphi(A,s,0)z_0 + \int_0^s \frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(A,s,0){}^{GC}E^p_\varphi(-A,t,0)\ B\ {}^{GC}u^p_\varphi(t)dt, \tag{30}
$$

$$
{}^{GC}\mathcal{P}_2(z(s)) = \int_0^s\ \frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(A,s,0)\ {}^{GC}E^p_\varphi(-A,t,0)\ \ g\Big(t,z(t),\int_0^t h(t,r,z(r))dr\Big)dt. \tag{31}
$$

Hence,

$$
\begin{aligned}
\|{}^{GC}\mathcal{P}_1(z(s))\ + {}^{GC}\mathcal{P}_2(z(s))\| \le\; & {}^{GC}E^p_\varphi(\mathcal{N}_A,s,0)\|z_0\| \\
& + {}^{GC}E^p_\varphi(\mathcal{N}_A,s,0)\int_0^s \frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(\mathcal{N}_A,t,0) \\
& \times\Big(\|B\|\ \|{}^{GC}u^p_\varphi(t)\|\ +\ \left\|g\Big(t,z(t),\int_0^t h(t,r,z(r))dr\Big)\right\|\Big)dt
\end{aligned}
$$

According to assumptions (23),(24) and (26), we obtain the following.

$$\|^{GC}\mathcal{P}_1(z(s)) + {}^{GC}\mathcal{P}_2(z(s))\| \leq {}^{GC}E^p_\varphi(\mathcal{N}_A,s,0)\|z_0\| + {}^{GC}E^p_\varphi(\mathcal{N}_A,s,0)\int_0^s \frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(\mathcal{N}_A,t,0) \times \Big(L_g(t)\ \|z\|\big(1+L_h(t)\big) + \Big\|g\big(t,0,\int_0^t h(t,r,0)dr\big)\Big\|\Big)dt + {}^{GC}E^p_\varphi(\mathcal{N}_A,b,0)\Big(\int_0^s \frac{1}{\varphi^2(t,p)}{}^{GC}E^p_\varphi(2\mathcal{N}_A,t,0)\,dt\Big)^{\frac{1}{2}}\ \|B\| \times \|^{GC}u^p_\varphi\|_{L^2(J,Z)}$$

$$\leq {}^{GC}E^p_\varphi(\mathcal{N}_A,s,0)\|z_0\| + \frac{1}{\mathcal{N}_A}\ {}^{GC}E^p_\varphi(\mathcal{N}_A,s,0)\big({}^{GC}E^p_\varphi(\mathcal{N}_A,s,0)-1\big) \times \Big(\|L_g\|_{\hat{R}}\ \|z\|\big(1+\|L_h\|_{\hat{R}}\big)+\mathcal{S}_g\Big) + {}^{GC}E^p_\varphi(\mathcal{N}_A,b,0)\Big(\int_0^s \frac{1}{\varphi^2(t,p)}{}^{GC}E^p_\varphi(2\mathcal{N}_A,t,0)\,dt\Big)^{\frac{1}{2}}\|B\| \times \|^{GC}u^p_\varphi\|_{L^2(J,Z)}$$

Substituting (29) and simplifying the constants, we conclude that

(32)

$$\|^{GC}\mathcal{P}_1(z(s)) + {}^{GC}\mathcal{P}_2(z(s))\| \leq H_1 + H\,H_4\|B\|\,(H_1+\|z_1\|) + rH_2\big(1+H_3\big)(1+H\,H_4\|B\|) \leq r$$

(33) $$\text{for}\quad r = \frac{H_1 + H\,H_4\|B\|\,(H_1+\|z_1\|)}{1 - rH_2\big(1+H_3\big)(1+H\,H_4\|B\|)}$$

(34) $$\text{Therefore,}\quad \Big({}^{GC}\mathcal{P}_1(z(s)) + {}^{GC}\mathcal{P}_2(z(s)\Big) \in B_r$$

Step 2: Controllability using the contraction mapping We have to prove that ${}^{GC}\mathcal{P}_1$ is a contraction mapping.

According to assumption (26), it follows that for any $z,y \in B_r$:

$$\big\|^{GC}u^p_\varphi(z(s)) - {}^{GC}u^p_\varphi(y(s))\big\| \leq \|\mathcal{Q}^{-1}\| \times \Big|\Big|\int_0^b \frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(A,b,0){}^{GC}E^p_\varphi(-A,t,0) \times \Big[g\big(t,z(t),\int_0^t h(t,r,z(r))dr\big) - g\big(t,y(t),\int_0^t h(t,r,y(r))dr\big)\Big]dt\Big|\Big|$$

Using the Lipschitz continuity of g and h, it follows that

(35) $$\big\|^{GC}u^p_\varphi(z(s)) - {}^{GC}u^p_\varphi(y(s))\big\| \leq \frac{H}{\mathcal{N}_A}\ {}^{GC}E^p_\varphi(\mathcal{N}_A,b,0)\big({}^{GC}E^p_\varphi(\mathcal{N}_A,b,0)-1\big) \times \Big(L_g(t)\ \|z(t)-y(t)\|\ \big(1+L_h(t)\big)\Big) \leq H\,H_2\big(1+H_3\big)\,\|z-y\|_{\hat{R}}$$

Therefore,

$$\big\|^{GC}\mathcal{P}_1(z(s)) - {}^{GC}\mathcal{P}_1(y(s))\big\| \leq {}^{GC}E^p_\varphi(\mathcal{N}_A,s,0)\int_0^s \frac{1}{\varphi(t,p)}{}^{GC}E^p_\varphi(\mathcal{N}_A,t,0)\ \|B\| \times \big\|^{GC}u^p_\varphi\ (z(t)) - {}^{GC}u^p_\varphi\ (y(t))\big\|_{L^2(J,Z)}\ dt$$

$$\leq {}^{GC}E^p_\varphi(\mathcal{N}_A,s,0)\Big(\int_0^s \frac{1}{\varphi^2(t,p)}{}^{GC}E^p_\varphi(2\mathcal{N}_A,t,0)\,dt\Big)^{\frac{1}{2}}\ \|B\|\,\big\|^{GC}u^p_\varphi\ (z) - {}^{GC}u^p_\varphi\ (y$$

$$\leq H_2\ H_4\ \|B\|\ \big(1+H_3\big)\ \|z-y\|_{\hat{R}}$$

(36) $$\text{Thus,}\quad \big\|^{GC}\mathcal{P}_1(z(s)) - {}^{GC}\mathcal{P}_1(y(s))\big\| \leq K\|z-y\|,$$

(37) $$k = H\ H_2\ H_4\ \|B\|\big(1+H_3\big)$$

We conclude that ${}^{GC}\mathcal{P}_1$ is a contraction mapping.

Step 3: Show that ${}^{GC}\mathcal{P}_2$ is a continuous and compact operator.

First, ${}^{GC}\mathcal{P}^p_\varphi$ is continuous since for all $s \in J$, ${}^{GC}E^p_\varphi(A, s_2, 0)$ is strongly continuous in J, g and h are continuous in J. Second, we have to prove that ${}^{GC}\mathcal{P}^p_\varphi(\mathcal{B}_r) \subset \hat{R}$ is bounded for each $z \in \mathcal{B}_r$ and $s \in J$.

$$\begin{aligned}\left\|{}^{GC}\mathcal{P}_2(z(s))\right\| \leq\ & {}^{GC}E^p_\varphi(\mathcal{N}_A, s, 0) \int_0^s \frac{1}{\varphi(t,p)} {}^{GC}E^p_\varphi(\mathcal{N}_A, t, 0) \\ & \times \left\|g(t, z(t), \int_0^t h(t,r,z(r))dr) - g(t, 0, \int_0^t h(t,r,0)dr)\right\|dt \\ & + {}^{GC}E^p_\varphi(\mathcal{N}_A, s, 0) \int_0^s \frac{1}{\varphi(t,p)} {}^{GC}E^p_\varphi(\mathcal{N}_A, t, 0) \\ & \times \left\|g(t, 0, \int_0^t h(t,r,0)dr\right\|dt\end{aligned}$$

By assumption (26), it follows that

$$\begin{aligned}\left\|{}^{GC}\mathcal{P}_2(z(s))\right\| \leq\ & \frac{1}{\mathcal{N}_A}\ {}^{GC}E^p_\varphi(\mathcal{N}_A, b, 0)\big({}^{GC}E^p_\varphi(\mathcal{N}_A, b, 0) - 1\big) \\ & \times \Big(\|L_g\|_{\hat{R}}\ \|z\|(1 + \|L_h)\|_{\hat{R}} + \mathcal{S}_g\Big)\end{aligned}$$

(38) Hence, $\left\|{}^{GC}\mathcal{P}_2(z(s))\right\| \leq H_1 + r\,H_2\big(1 + H_3\big) - {}^{GC}E^p_\varphi(\mathcal{N}_A, s, 0)\|z_0\|$

So, ${}^{GC}\mathcal{P}_2(\mathcal{B}_r)$ is bounded. Third, we have to prove that ${}^{GC}\mathcal{P}_2$ is equicontinuous. Let $s_1, s_2 \in J$. According to (26), we have:

$$\begin{aligned}\|{}^{GC}\mathcal{P}_2(z(s_2)) - {}^{GC}\mathcal{P}_2(z(s_1))\| \leq\ & {}^{GC}E^p_\varphi(\mathcal{N}_A, s_2, 0) \int_0^{s_2} \frac{1}{\varphi(t,p)} {}^{GC}E^p_\varphi(\mathcal{N}_A, t, 0) \\ & \times \left\|g\Big(t, z(t), \int_0^{s_2} h(t,r,z(r))dr\Big)dt\right\|ds_2 \\ & - {}^{GC}E^p_\varphi(\mathcal{N}_A, s_1, 0) \int_0^{s_1} \frac{1}{\varphi(t,p)} {}^{GC}E^p_\varphi(\mathcal{N}_A, t, 0) \\ & \times \left\|g\Big(t, z(t), \int_0^{s_1} h(t,r,z(r))dr\Big)dt\right\|ds_1 \\ & \leq \mathcal{G}_1 + \mathcal{G}_2\end{aligned}$$

where

(39)
$$\begin{aligned}\mathcal{G}_1 =\ & \|{}^{GC}E^p_\varphi(A, s_2, 0) - {}^{GC}E^p_\varphi(A, s_1, 0)\| \int_{s_1}^{s_2} \frac{1}{\varphi(t,p)} {}^{GC}E^p_\varphi(\mathcal{N}_A, t, 0) \\ & \times \left\|g(t, z(t), \int_0^t h(t,r,z(r))dr\right\| dt\end{aligned}$$

(40)
$$\mathcal{G}_2 = {}^{GC}E^p_\varphi(\mathcal{N}_A, s_2, 0) \int_{s_1}^{s_2} \frac{1}{\varphi(t,p)} {}^{GC}E^p_\varphi(\mathcal{N}_A, t, 0) \times \left\|g(t, z(t), \int_0^t h(t,r,z(r))dr\right\| dt$$

$$\begin{aligned}\mathcal{G}_1 \leq\ & \left\|{}^{GC}E^p_\varphi(A, s_2, 0) - {}^{GC}E^p_\varphi(A, s_1, 0)\right\| \int_{s_1}^{s_2} \frac{1}{\varphi(t,p)} {}^{GC}E^p_\varphi(\mathcal{N}_A, t, 0) \\ & \times \left\|g(t, z(t), \int_0^t h(t,r,z(r))dr) - g(t, 0, \int_0^t h(t,r,0)dr)\right\|dt + \left\|{}^{GC}E^p_\varphi(A, s_2, 0)\right. \\ & \left. - {}^{GC}E^p_\varphi(A, s_1, 0)\right\| \int_{s_1}^{s_2} \frac{1}{\varphi(t,p)} {}^{GC}E^p_\varphi(\mathcal{N}_A, t, 0) \left\|g(t, 0, \int_0^t h(t,r,0)dr\right\|dt\end{aligned}$$

By applying the Lipschitz condition, we obtain

$$\begin{aligned}\mathcal{G}_1 \leq \frac{1}{\mathcal{N}_A}\left|\left|{}^{GC}E^p_\varphi(\mathcal{N}_A, s_2, 0) - {}^{GC}E^p_\varphi(\mathcal{N}_A, s_1, 0)\right|\right| \left({}^{GC}E^p_\varphi(\mathcal{N}_A, s_2, 0) - 1\right) \\ \times \left(r \left\|L_g\right\|_{\hat{R}} \left(1 + \|L_h)\|_{\hat{R}} + S_g\right)\end{aligned} \tag{41}$$

Therefore, $\mathcal{G}_1$ **tend to** 0 **as** s_1 **tend to** s_2

$$\begin{aligned}\mathcal{G}_2 \leq\ & {}^{GC}E^p_\varphi(\mathcal{N}_A, s_2, 0) \int_{s_1}^{s_2} \frac{1}{\varphi(t,p)} {}^{GC}E^p_\varphi(\mathcal{N}_A, t, 0) \\ & \times \left|\left|g(t, z(t), \int_0^t h(t, r, z(r))dr) - g(t, 0, \int_0^t h(t, r, 0)dr)\right|\right|dt \\ & + {}^{GC}E^p_\varphi(\mathcal{N}_A, s_2, 0) \int_{s_1}^{s_2} \frac{1}{\varphi(t,p)} {}^{GC}E^p_\varphi(\mathcal{N}_A, t, 0) \left|\left|g(t, 0, \int_0^t h(t, r, 0)dr\right|\right|dt\end{aligned}$$

By applying the Lipschitz condition, we obtain

$$\begin{aligned}\mathcal{G}_2 \leq \frac{1}{\mathcal{N}_A}\ {}^{GC}E^p_\varphi(\mathcal{N}_A, s_2, 0) \left({}^{GC}E^p_\varphi(\mathcal{N}_A, s_2, 0) - {}^{GC}E^p_\varphi(\mathcal{N}_A, s_1, 0)\right) \\ \times \left(r \left\|L_g\right\|_{\hat{R}} \left(1 + \|L_h\|_{\hat{R}}\right) + S_g\right)\end{aligned} \tag{42}$$

Hence, $\mathcal{G}_2$ tends to 0 as s_1 tends to s_2. Therefore, $\mathcal{P}_2(\mathcal{B}_r)$ is equicontinuous. Since $\mathcal{P}_2(\mathcal{B}_r)$ is bounded and equicontinuous, it follows that ${}^{GC}\mathcal{P}_2$ is compact. Thus, by Krasnoselskii's fixed point theorem, ${}^{GC}\mathcal{P}$ has a fixed point $z \in \mathcal{B}_r$, which is a mild solution of system (18) satisfying $z(b) = z_1$. Therefore, the system (18) is completely controllable □

4. Application

Example 4.1. Consider the system (18) with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = 10^5 \times \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix},$$

$$\varphi(s, p) = \tfrac{1}{2}(s + 1) + p, \quad \text{with } p = \tfrac{1}{2} \text{ and } b = 1.$$

Then,

$$\frac{1}{\varphi(t,p)} = \frac{1}{1 + \frac{1}{2}t} \quad \text{and,} \quad \int_0^s \frac{1}{1 + \frac{1}{2}t}\, dt = 2 \ln\left(\frac{2+s}{2}\right).$$

We now consider a nonlinear function of the form

$$g(s, z, \eta) = \begin{bmatrix} \frac{s+2}{10^2}\sin(z_1) + \left(\frac{s+1}{9}\right)^2 \eta_1 \\ \\ \frac{s+2}{10^2}\sin(z_2) + \left(\frac{s+1}{9}\right)^2 \eta_2 \end{bmatrix}, \quad \eta = \int_0^s h(s, r, z)dr, \ \ h(s, r, z) = e^{-(s-r)}z.$$

Substituting the definition of h into the integral term, we obtain

$$\int_0^s h(s, r, z(r))dr = \begin{bmatrix} \int_0^s e^{-(s-r)} z_1(r)dr \\ \\ \int_0^s e^{-(s-r)} z_2(r)dr \end{bmatrix}.$$

Therefore, the non-linear function becomes

$$g\left(s, z(s), \int_0^s h(s, r, z(r))dr\right) = \begin{bmatrix} \frac{s+2}{10^2}\sin(z_1(s)) + \left(\frac{s+1}{9}\right)^3 \int_0^s e^{-(s-r)} z_1(r)dr \\ \\ \frac{s+2}{10^2}\sin(z_1(s)) + \left(\frac{s+1}{9}\right)^3 \int_0^s e^{-(s-r)} z_2(r)dr \end{bmatrix}.$$

1. **Computation of the product** ${}^{GC}E^p_\varphi(A, b, 0)\ {}^{GC}E^p_\varphi(-A, s, 0)$**.**

We have ${}^{GC}E^p_\varphi(A,b,0) = \begin{bmatrix} \left(\frac{3}{2}\right)^2 & 0 \\ 0 & \left(\frac{2}{3}\right)^4 \end{bmatrix}$, ${}^{GC}E^p_\varphi(-A,s,0) = \begin{bmatrix} \left(\frac{2}{2+s}\right)^2 & 0 \\ 0 & \left(\frac{2+s}{2}\right)^4 \end{bmatrix}$.

Hence

$$ {}^{GC}E^p_\varphi(A,b,t) = \begin{bmatrix} \left(\frac{3}{2+s}\right)^2 & 0 \\ 0 & \left(\frac{2+s}{3}\right)^4 \end{bmatrix}. $$

2. **Derivation of the conformable Gramian matrix W.**

$$ W = \int_0^1 \frac{1}{\varphi(t,p)} \left[{}^{GC}E^p_\varphi(A,b,t)B\right]\left[{}^{GC}E^p_\varphi(A,b,t)B\right]^* dt. $$

Using

$$ \left[{}^{GC}E^p_\varphi(A,b,t)B\right]\left[{}^{GC}E^p_\varphi(A,b,t)B\right]^* = 10^{10} \times \begin{bmatrix} 5^2 \times \left(\frac{3}{2+t}\right)^4 & 0 \\ 0 & 2^2 \times \left(\frac{2+t}{3}\right)^8 \end{bmatrix}, $$

We obtain

$$ \begin{aligned} W &= \int_0^1 \frac{2\times 10^{10}}{2+t} \times \begin{bmatrix} 5^2 \times \left(\frac{3}{2+t}\right)^4 & 0 \\ 0 & 2^2 \times \left(\frac{2+t}{3}\right)^8 \end{bmatrix} dt \\ &= 10^{10} \times \int_0^1 \begin{bmatrix} \frac{2\times 5^2\times 3^4}{(2+t)^5} & 0 \\ 0 & \frac{2^3\times(2+t)^7}{3^8} \end{bmatrix} dt \\ &= 10^{10} \times \begin{bmatrix} \frac{1625}{32} & 0 \\ 0 & \frac{2^3\times 6305}{6561} \end{bmatrix}. \end{aligned} $$

Which gives that

$$ W^{-1} = 10^{-10} \begin{bmatrix} \frac{32}{1625} & 0 \\ 0 & \frac{6561}{2^3\times 6305} \end{bmatrix} \approx 10^{-10} \begin{bmatrix} 0.0196 & 0 \\ 0 & 0.13007 \end{bmatrix}. $$

3. **Calculation of Q:**

$$ \begin{aligned} Q(b) &= \int_0^s \frac{1}{\varphi(t,p)} {}^{GC}E^p_\varphi(A,s,t)Bdt \\ &= \int_0^b \frac{2}{2+t} \times 10^5 \times \begin{bmatrix} 5 \times \left(\frac{2+b}{2+t}\right)^2 & 0 \\ 0 & 2 \times \left(\frac{2+t}{2+b}\right)^4 \end{bmatrix} dt \\ &= 10^{10} \times \begin{bmatrix} \left(\frac{5}{3}\right)^2 & 0 \\ 0 & \frac{65}{81} \end{bmatrix} \end{aligned} $$

Hence the inverse of $\mathcal{Q}$ is given b

$$ \mathcal{Q}^{-1} = 10^{-10} \times \begin{bmatrix} \left(\frac{3}{5}\right)^2 & 0 \\ 0 & \frac{81}{65} \end{bmatrix} $$

4. **Verification of Lipschitz condition:**
For $z_1, z_2 \in \mathbb{R}$ and $z_1', z_2' \in \mathbb{R}$,

$$ \begin{aligned} \|g_1(s,z_1,\eta_1) - g_1(s,z_1',\eta_1')\| &= \left|\frac{s+2}{10^2}\right| \cdot \|\sin(z_1) - \sin(z_1')\| + \left|\frac{s+1}{9}\right|^3 \cdot \|\eta_1 - \eta_1'\| \\ &\leq L_{g,z}(s)\|z_1 - z_1'\| + L_{g,\eta}(s,r)\|\eta_1 - \eta_1'\| \end{aligned} $$

$$ \begin{aligned} \|g_2(s,z_2,\eta_2) - g_2(s,z_2',\eta_2')\| &= \left|\frac{s+2}{10^2}\right| \cdot \|\sin(z_2) - \sin(z_2')\| + \left|\frac{s+1}{9}\right|^3 \cdot \|\eta_2 - \eta_2'\| \\ &\leq L_{g,z}(s)\|z_2 - z_2'\| + L_{g,\eta}(s,r)\|\eta_2 - \eta_2'\| \end{aligned} $$

$$\text{where}\quad L_{g,z} = \sup_{s\in[0,1]} \frac{s+2}{10^2} = \frac{3}{100} = 0.03 \quad \text{and} \quad L_{g,\eta} = \sup_{s\in[0,1]} \left(\frac{s+1}{9}\right)^3 \approx 0.01097.$$

Thus

$$\|g(s,z,\eta) - g(s,z',\eta')\| \le 0.03\,\|z-z'\| + 0.01097\,\|\eta-\eta'\|.$$

Thus g is Lipschitz in (z,η) with Lipschitz constant

$$L_g \le 0.03 + 0.01097 \approx 0.04097.$$

Lipschitz property of the integral operator:
We recall that $\eta = \int_0^s h(s,r,z(r))\,dr = \int_0^s e^{-(s-r)}\,dr$.
Compute the integral

$$\int_0^s e^{-(s-r)}\,dr = \int_0^s e^{-u}\,du = 1 - e^{-s}.$$

Hence the norm of L_h on $[0,1]$ is

$$L_h = \sup_{s\in[0,1]} (1-e^{-s}) = 1 - e^{-1} \approx 0.6321$$

$$\|\eta_1 - \eta_1'\| \le \int_0^s e^{-(s-r)}\|z_1 - z_1'\|\,dr \le L_h(s)\ \|z_1 - z_1'\|$$

$$\|\eta_2 - \eta_2'\| \le \int_0^s e^{-(s-r)}\|z_2 - z_2'\|\,dr \le L_h(s)\ \|z_2 - z_2'\|$$

Combining these estimates yields:

$$\|g(s,z,\eta) - g(s,z',\eta')\| \le L_{g,z}\|z-z'\| + L_{g,\eta}\ L_h\|z-z'\| \le (L_{g,z} + L_{g,\eta}\ L_h)\ \|z-z'\|$$

Combining $L_{g,z}$, $L_{g,\eta}$ and L_h we obtain the following:

$$q = L_{g,z} + L_{g,\eta}L_h \approx 0.03 + (0.01097 \times 0.6321) \approx 0.0369$$

5. **Computation of H_4:**
We recall that

$$H_4 = H\ {}^{GC}E^p_\varphi(\mathcal{N}_\mathcal{A}, b, 0)\Big(\int_0^b \frac{1}{\varphi^2(t,p)}{}^{GC}E^p_\varphi(2\mathcal{N}_\mathcal{A}, t, 0)dt\Big)^{\frac{1}{2}}$$

First compute

$$\mathcal{N}_\mathcal{A} = \|A\| = 2 \quad , \quad \text{and} \quad {}^{GC}E^p_\varphi(\mathcal{N}_\mathcal{A}, b, 0) = e^{2\int_0^1 \frac{2}{2+t}dt} = e^{4\ \ln\frac{3}{2}} = (\frac{3}{2})^4$$

Similarly

$${}^{GC}E^p_\varphi(2\mathcal{N}_\mathcal{A}, t, 0) = e^{4\int_0^t \frac{2}{2+t}dt} = e^{8\ \ln\frac{2+t}{2}} = (\frac{2+t}{2})^8$$

Hence

$$\int_0^b \frac{1}{\varphi^2(t,p)}{}^{GC}E^p_\varphi(2\mathcal{N}_\mathcal{A}, t, 0)dt = \int_0^1 (\frac{2+t}{2})^6 dt = \frac{2}{7}\Big((\frac{3}{2})^7 - 1\Big) = \frac{2059}{448}$$

Since

$$H = \|\mathcal{Q}^{-1}\| = \|\mathcal{W}^{-1}\|^{\frac{1}{2}} \approx (0.13007 \times 10^{-10})^{\frac{1}{2}} \approx 0.3606 \times 10^{-5}$$

We obtain

$$\begin{aligned} H_4 &= H\ .{}^{GC}E^p_\varphi(\mathcal{N}_\mathcal{A}, b, 0)\Big(\int_0^b \frac{1}{\varphi^2(t,p)}{}^{GC}E^p_\varphi(2\mathcal{N}_\mathcal{A}, t, 0)dt\Big)^{\frac{1}{2}} \\ &= 0.3606 \times 10^{-5} \times \frac{9^4}{2^4} \times \left(\frac{2059}{448}\right)^{\frac{1}{2}} \\ &\approx 0.00317 \end{aligned}$$

6. **Computation of H_2** Using

$${}^{GC}E^p_\varphi(\mathcal{N}_\mathcal{A}, b, 0) = (\frac{3}{2})^4 \qquad \text{and} \qquad q \approx 0.0369.$$

We obtain

$$\begin{aligned} H_2 &= \frac{1}{\mathcal{N}_{\mathcal{A}}} {}^{GC}E_{\varphi}^{p}(\mathcal{N}_{\mathcal{A}}, b, 0)\left({}^{GC}E_{\varphi}^{p}(\mathcal{N}_{\mathcal{A}}, b, 0) - 1\right) \times q \\ &= \frac{1}{2} \times \frac{81}{16} \times \frac{65}{16} \times 0.0369 \\ &\approx 0.0369 \end{aligned}$$

Using

$$H_2 \left[1 + H_4 \|B\|\right] \approx 0.0369 \left(1 + 0.00317\right) \approx 0.037 < 1.$$

Consequently, all the conditions of theorem (3.3) are satisfied and the system (18) is completely controllable on $[0, 1]$.

5. Conclusion

This paper established a novel set of verifiable sufficient conditions for the complete controllability of semilinear integrodifferential systems whose dynamics are controlled by the generalized conformable fractional derivative. These results hold under the assumption that the corresponding linear component is completely controllable.

6. Data Availability Statement

Data sharing is not applicable to this paper as no new data were created or analyzed in this work.

7. Conflict of Interest

The authors declare that they have no conflict of interest.

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On a two-layer conjugate problem for the heat conduction equation in a rectangle

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Abstract. The analytical theory of nonstationary heat conduction is a rapidly developing field of mathematical physics and is widely applied to solving important technical problems in heat and mass transfer processes that express the energy balance and the balance of the corresponding substances. This paper investigates the initial–boundary value problem for the heat conduction equation in a two-layer rectangular medium with a given initial temperature distribution, boundary conditions, and coupling conditions at the contact surface. The boundary coupling condition at the line of contact between the media corresponds to heat exchange between the body's surface and the surrounding environment according to Fourier's law of heat conduction, under the assumption that the temperatures of the contacting surfaces are ideal thermal contact. In addition, the flow of a fluid (or gas) around a solid body determines the heat transfer in the region close to the body's surface, where it occurs according to the law of thermal conductivity (molecular heat transfer). This process corresponds to the heat exchange between contacting solid bodies that satisfy the boundary coupling conditions. An analytical solution to the problem of unsteady heat conduction in a two-layer rectangular medium is constructed using the classical method of separation of variables in combination with integral transforms—specifically, the Laplace transform with respect to the time variable and the finite Fourier transform with respect to the spatial variables—in the form of convolution heat potentials whose kernels satisfy the initial and boundary conditions.

2020 Mathematics Subject Classifications: 35A09, 35K05, 35K20

Keywords: two-layer conduction, conjugate problem, unsteady temperature, discontinuous coefficients, heat exchange

Introduction

Thermodynamic processes are accompanied by heat transfer between structural elements within the working space of thermal installations and the surrounding environment. In such processes, heat serves as a quantitative measure of energy exchange between bodies, and the regularities of heat transfer are governed by the theory of heat exchange and heat transfer. The motivation for research in this field arises from the practical significance of the heat conduction equation with discontinuous coefficients in engineering applications, as well as from its deep connection to various branches of mathematics, including function theory, integral equation theory, series and integral theory, functional analysis, and approximation theory. According to the second law of thermodynamics, spontaneous heat transfer occurs in a medium whenever a temperature difference exists, proceeding in the direction of decreasing temperature. In addition to classical heat conduction and diffusion problems, similar mathematical models appear in the theory of heat and mass transfer—such as in the description of drying and cooling processes—as well as in the theory of nuclear chain reactions for studying neutron moderation, in signal theory for the macroscopic description of random processes at the output of radio engineering devices, and in the study of many processes in biological and chemical kinetics, among other technical applications. The study of nonstationary initial–boundary value problems for the heat conduction equation with discontinuous coefficients is one of the key problems in the theory of partial differential equations and is of great interest not only to mathematicians but also to researchers in engineering and applied sciences. Boundary value problems for the heat conduction equation with discontinuous coefficients constitute one of the classical objects of study.Boundary value problems for equations with discontinuous coefficients that are most closely related to our research are presented in [1]–[4].

In the work of Samara [3], the well-posedness of the first initial–boundary value problem for the heat conduction equation with a discontinuous coefficient is established using the methods of Green's functions and thermal potentials. Meanwhile, Kazakh mathematicians Kim E.I. and Baimukhanov B.B. [4] proved the correctness of the first initial–boundary value problem for a two-dimensional heat conduction equation with a discontinuous heat conduction coefficient by applying the potential method and reducing the problem to an integral equation in a half-space. In works [3]–[5], the existence of classical solutions to various boundary value problems for parabolic equations is also demonstrated using thermal potential methods.The novelty of the present study lies, first, in the construction of the Green's function for the Cauchy problem in the corresponding domain, which makes it possible to avoid the use of integral equations that usually arise in similar conjugation problems. Second, knowing the Green's function in the given domain allows one to obtain an analytical solution by multiplying the given equation by the constructed Green's function—as the solution of the conjugate heat conduction equation—then integrating by parts and taking the limit as $t \to 0$.

1. Mathematical formulation of the problem

We denote by $u(x, y, t)$ a temperature at time t and in position x, y.
Let us consider

$$\Omega = \Omega_1 \cup \Omega_2, \quad \Omega_T = \Omega \times (0, T)$$
$$\Omega_1 = \{(x, y) : 0 < x < l_0,\ 0 < y < h\}, \quad \Omega_2 = \{(x, y) : l_0 < x < l_1,\ 0 < y < h\}$$
$$\omega_{1T} = \{x : 0 < x < l_0\},\ 0 < t < T,\ y = 0, \quad \omega_{2T} = \{x : l_0 < x < l_1\},\ 0 < t < T,\ y = 0,$$
$$\sigma_{1T} = \{y : 0 < y < h,\ 0 < t < T,\ \text{if } x = 0\}, \quad \sigma_{2T} = \{y : 0 < y < h,\ 0 < t < T,\ \text{if } x = l\}$$

We consider the following problem: find a solution of the heat equation

$$\frac{\partial u(x, y, t)}{\partial t} = a^2 \Delta u(x, y, t) + f(x, y, t) \quad \text{on the region } \Omega_T \tag{1.1}$$

that satisfies:
The initial condition

$$u|_{t=0} = u_0(x, y) \quad \text{on the region } \Omega \tag{1.2}$$

Boundary conditions

$$u|_{x=0} = \Phi_0(y, t), \quad u|_{x=l} = \Phi_1(y, t) \tag{1.3}$$

$$u|_{y=0} = \Psi_0(x, t), \quad u|_{y=h} = \Psi_1(x, t) \tag{1.4}$$

Conjugation conditions

$$u|_{x=l_0-0} = u|_{x=l_0+0}, \quad \kappa_1 \frac{\partial u}{\partial x}\bigg|_{x=l_0-0} = \kappa_2 \frac{\partial u}{\partial r}\bigg|_{x=l_0+0} \tag{1.5}$$

2. Main results

Considering the Cauchy problem and applying the finite Fourier transform with respect to the variables x , y and and Laplace transform

$$X_{in}(x) = \begin{cases} X_{1n}(x) = \dfrac{\sin \frac{\lambda_n}{a_1} x}{\sin \frac{\lambda_n}{a_1} l_0}, & 0 \le x \le l_0 \\ X_{2n}(x) = \dfrac{\sin \frac{\lambda_n}{a_2}(l-x)}{\sin \frac{\lambda_n}{a_2}(l-l_0)}, & l_0 \le x \le l \end{cases}$$

$$Y_m(x) = \sin \frac{m\pi y}{h}, \quad \text{if } 0 \le y \le h \tag{2.2}$$

then, after performing some transformations and applying the inverse Fourier and Laplace transforms, we obtain the fundamental solution of the Cauchy problem

$$u_i(x, y, t) = u_{0i} * G_{i1} + u_{02} * G_{i2} \tag{2.3}$$

for i=1,2
where

$$G(x,\xi,y,\zeta,t)=\begin{cases} G_{11}(x,\xi,y,\zeta,t), & \text{if } 0<x<l_0,\ 0<\xi<l_0\\ G_{12}(x,\xi,y,\zeta,t), & \text{if } 0<x<l_0,\ l_0<\xi<l\\ G_{21}(x,\xi,y,\zeta,t), & \text{if } l_0<x<l_1,\ 0<\xi<l_0\\ G_{22}(x,\xi,y,\zeta,t), & \text{if } l_0<x<l_1,\ l_0<\xi<l \end{cases} \tag{2.4}$$

$$G_{ij}(x,\xi,y,\zeta,t)=\begin{cases} g_{ij}(x,\xi,y,\zeta,t)+g_{ij}*(\frac{\partial}{\partial t}-a_i^2)g_\alpha, & \text{if } i=j\\ g_{ij}*(\frac{\partial}{\partial t}-a_i^2)g_\alpha, & \text{if } i\neq j \end{cases} \tag{2.5}$$

Here the symbol * denotes the convolution between two functionals.

$$g_{ij}(x,\xi,y,\zeta,t)=\sum_{m,n=1}^{\infty}\frac{-e^{\beta_{mn}^2 t}X_{in}(x)X_{jn}(\xi)Y_m(y)Y_m(\zeta)}{\|X_n\|^2\,\|Y_m\|^2}. \tag{2.6}$$

$$\beta_{mn}^2=\begin{cases}\lambda_n^2+a_1^2\mu_m^2, & \text{if } 0<x<l_0\\ \lambda_n^2+a_2^2\mu_m^2, & \text{if } l_0<x<l\end{cases}$$

$$g_\alpha(x,\xi,y,\zeta,t)=\sum_{m,n=1}^{\infty}\frac{e^{-\alpha_{nm}^2 t}X_{in}(x)X_{in}(\xi)Y_m(y)Y_m(\zeta)}{\|X_n\|^2\|Y_m\|^2} \tag{2.7}$$

$$\alpha_{nm}^2=\kappa_1\lambda_n^2+\kappa_2\lambda_n^2+(a_1^2+a_2^2)\mu_m^2 \tag{2.7}$$

λ_n - *root of transcendental equation*

$$\kappa_1 a_1\cot\frac{\lambda\lambda_0}{a_1}+\kappa_2 a_2\cot\frac{\lambda(l-l_0)}{a_2}=0 \tag{2.8}$$

$$\|X_n\|^2=\frac{\kappa_1 l_0^2}{\sin^2\frac{\lambda_m l_0}{a_1}}+\frac{\kappa_2(l-l_0)^2}{\sin^2\frac{\lambda_m(l-l_0)}{a_2}} \tag{2.9}$$

$$\|Y_m\|^2=\frac{h}{2} \tag{2.10}$$

Theorem. For any functions

$$f(x,y,t)\in C(\overline{\Omega}_T)\cap C^{2,1}(\Omega_T),\quad u_0(x,y)\in C(\overline{\Omega})\cap C^2(\Omega),$$
$$\Phi_i(y,t)\in C(\overline{\sigma}_{iT})\cap C^{2,1}(\sigma_{iT}),\quad \Psi_i(x,t)\in C(\overline{\omega}_{iT})\cap C^{2,1}(\omega_{iT}),\quad i=0,1$$

then there is a unique classical solution.

$$u(x,y,t)=\begin{cases}u_1(x,y,t)\in C(\overline{\Omega}_{1T})\cap C^{2,1}(\Omega_{1T})\\ u_2(x,y,t)\in C(\overline{\Omega}_{2T})\cap C^{2,1}(\Omega_{2T})\end{cases}$$

satisfying the initial condition (1.2), boundary conditions (1.3)-(1.4) and conjugation conditions (1.5)-(1.6) — which are presented as a sum of thermal potentials:

$$\begin{aligned}u_i(x,y,t)=\ & f_1*G_{i1}+f_2*G_{i2}f_1*G_{i1}+u_{0i}*G_{i1}+u_{02}*G_{i2}\\ &+\Phi_1*\left.\frac{\partial G_{i2}}{\partial\xi}\right|_{\xi=l}-\Phi_0*\left.\frac{\partial G_{i1}}{\partial\xi}\right|_{\xi=0}+\Psi_1*\left.\frac{\partial G_{i2}}{\partial\zeta}\right|_{\zeta=h}-\Psi_0*\left.\frac{\partial G_{i1}}{\partial\zeta}\right|_{\zeta=0}\end{aligned}$$

3. Conclusion

The practical significance of the obtained results lies in the constructive analytical solution of thermophysical energy transfer problems, which enables a better understanding of internal processes and the calculation of nonstationary temperature fields and heat fluxes in two-layer sheet products, structures, and installations. This also applies to rectangular samples of devices and systems in which the thermophysical parameters functionally depend on the flow boundaries, the initial temperature, and the interfaces between media. Such an approach makes it possible to describe two-layer heat exchange processes based on temperature fields and the corresponding initial and boundary conditions.

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On the three-layer conjugation problem for the heat conduction equation in a rectangle

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ABSTRACT. The study of initial–boundary value problems for the heat conduction equation and the development of analytical methods for their solution represent an important and rapidly developing field of mathematical physics. Using the method of integral transforms—Laplace for the time variable and the finite Fourier transform for the spatial variables—an analytical solution is constructed for the nonstationary heat conduction problem in a three-layer medium. Solutions to such problems can be used to calculate transient temperature fields and heat fluxes in three-layer sheet products, structures, and components, as well as in flat samples of devices whose thermophysical parameters depend on the initial temperature and on the boundaries of the three-layer medium. This problem is formulated and solved for the first time in this form. The interface boundary condition of the fourth kind describes heat exchange between a body's surface and the surrounding environment in accordance with Fourier's law of heat conduction, or heat exchange between contacting solid bodies when the temperatures of the contacting surfaces are equal (ideal, or perfect thermal contact).

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KEYWORDS: three lower heat conduction equation, initial boundary value problems, heat conduction equation, three lower temperature field, three lower heat exchange process, temperature field with discontinuous coefficients

1. INTRODUCTION

The study of non-stationary initial–boundary value problems for the heat conduction equation with discontinuous coefficients represents one of the fundamental issues in the theory of partial differential equations and attracts considerable attention from both mathematicians and engineers. Boundary value problems with discontinuous coefficients for the heat conduction equation are classical research topics in this field. Such problems have been investigated in works [1]–[4]. Kazakh mathematicians E.I. Kim and B.B. Baimukhanov [5] considered the first initial–boundary value problem for the two-dimensional heat conduction equation with a discontinuous thermal conductivity coefficient and demonstrated the existence of solutions using potential methods by reducing the problem to an integral equation in a half-space. In study [6], the well-posedness of the first initial–boundary value problem for the heat conduction equation with discontinuous coefficients was established using Green's functions and heat potential methods. In work [8], two-dimensional unsteady heat conduction was examined in a plate consisting of k layers, where the temperature of each medium was measured from the so-called "false zero." A separate temperature scale and a corresponding variable substitution were introduced for each layer, but without a consistent connection to the original coordinate system. In works [9]–[11], the existence of classical solutions to various boundary value problems for parabolic equations was established using heat potential methods. However, this approach is inadequate because the differential equations are derived separately for each region in its own coordinate system, which leads to the loss of variables from the original coordinate system. Similar problems arise in the theory of heat and mass transfer when describing drying and cooling processes, in the theory of nuclear chain reactions when studying neutron moderation processes, in signal theory when providing a macroscopic description of random processes at the output of radio engineering devices, as well as in the analysis of many processes in biological and chemical kinetics and other technical applications. The motivation for research in this direction stems both from the practical importance of heat conduction equations with discontinuous coefficients in engineering applications and from their deep connections with several areas of mathematics, including function theory, integral equations, series and integrals, functional analysis, and approximation theory.

2. Mathematical Formulation of the problem

Let $u(x,y,t)$ denote temperature at time t in the Cartesian coordinate system x, y.

$\Omega_1 = \{(x,y) : 0 < x < l_0,\ 0 < y < h\}$, $\Omega_2 = \{(x,y) : l_0 < x < l_1,\ 0 < y < h\}$,

$\Omega_3 = \{(x,y) : l_1 < x < l,\ 0 < y < h\}$, $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$,

$\Omega_T = \Omega \times (0,T)$, $\omega_{1T} = \{x : 0 < x < l,\ 0 < t < T,\ \text{if } y = 0\}$,

$\omega_{2T} = \{x : 0 < x < l,\ 0 < t < T,\ \text{if } y = h\}$ $\sigma_{1T} = \{y : 0 < y < h,\ 0 < t < T,\ \text{if } x = 0\}$,

$\sigma_{2T} = \{y : 0 < y < h,\ 0 < t < T,\ \text{if } x = l\}$

We consider the following problem: Find the solution of the heat conduction equation

$$\frac{\partial u(x,y,t)}{\partial t} = a^2 \Delta u(x,y,t) + f(x,y,t), \quad \text{in domain } \Omega_T \tag{2.1}$$

$$a^2 = \begin{cases} a_1^2 \in \Omega_{1T}, \\ a_2^2 \in \Omega_{2T}, \\ a_3^2 \in \Omega_{3T}. \end{cases}$$

satisfying the following conditions:
the initial condition

$$u|_{t=0} = u_0(x,y) \quad \text{on the region } \Omega \tag{2.2}$$

the boundary conditions:

$$u|_{x=0} = \Phi_0(y,t), \quad u|_{x=l} = \Phi_1(y,t) \tag{2.3}$$

$$u|_{y=0} = \Psi_0(x,t), \quad u|_{y=h} = \Psi_1(x,t) \tag{2.4}$$

and the conjugation conditions

$$u|_{x=l_0-0} = u|_{x=l_0+0}, \quad \kappa_1 \frac{\partial u}{\partial x}\bigg|_{x=l_0-0} = \kappa_2 \frac{\partial u}{\partial r}\bigg|_{x=l_0+0} \tag{2.5}$$

$$u|_{x=l_1-0} = u|_{x=l_1+0}, \quad \kappa_2 \frac{\partial u}{\partial x}\bigg|_{x=l_1-0} = \kappa_3 \frac{\partial u}{\partial r}\bigg|_{x=l_1+0} \tag{2.6}$$

κ_i — heat transfer coefficient for $i = 1, 2, 3$.

3. The Main results

3.1. Green's function and it's properties. Let us consider Cauchy's problem

$$\frac{\partial u(x,y,t)}{\partial t} = a^2 \Delta u(x,y,t) \quad \text{in domain } \Omega_T \tag{3.1}$$

satisfying the following conditions:
the initial condition

$$u|_{t=0} = u_0(x,y) \quad \text{on the region } \Omega \tag{3.2}$$

the boundary conditions:

$$u|_{x=0} = 0 \quad u|_{x=l} = 0 \tag{3.3}$$

$$u|_{y=0} = 0, \quad u|_{y=h} = 0 \tag{3.4}$$

and the conjugation conditions

$$u|_{x=l_0-0} = u|_{x=l_0+0}, \quad \kappa_1 \frac{\partial u}{\partial x}\bigg|_{x=l_0-0} = \kappa_2 \frac{\partial u}{\partial r}\bigg|_{x=l_0+0} \tag{3.5}$$

$$u|_{x=l_1-0} = u|_{x=l_1+0}, \quad \kappa_2 \frac{\partial u}{\partial x}\bigg|_{x=l_1-0} = \kappa_3 \frac{\partial u}{\partial r}\bigg|_{x=l_1+0} \tag{3.6}$$

Using the method of integral transforms : Laplace for the time variable and finite Fourier transformfor spa where $G(x,\xi,y,\zeta,t)$ Green's function for (2.1)-(2.6) problem which was defined in the form:

$$G(x,\xi,y,\zeta,t)=\begin{cases}G_{11}(x,\xi,y,\zeta,t), & \text{if } 0<x<l_0,\ 0<\xi<l_0\\ G_{12}(x,\xi,y,\zeta,t), & \text{if } 0<x<l_0,\ l_0<\xi<l_1\\ G_{13}(x,\xi,y,\zeta,t), & \text{if } 0<x<l_0,\ l_1<\xi<l\\ G_{21}(x,\xi,y,\zeta,t), & \text{if } l_0<x<l_1,\ 0<\xi<l_0\\ G_{22}(x,\xi,y,\zeta,t), & \text{if } l_0<x<l_1,\ l_0<\xi<l_1\\ G_{23}(x,\xi,y,\zeta,t), & \text{if } l_0<x<l_1,\ l_1<\xi<l\\ G_{31}(x,\xi,y,\zeta,t), & \text{if } l_1<x<l,\ 0<\xi<l_0\\ G_{32}(x,\xi,y,\zeta,t), & \text{if } l_1<x<l,\ l_0<\xi<l_1\\ G_{33}(x,\xi,y,\zeta,t), & \text{if } l_1<x<l,\ l_1<\xi<l\end{cases} \tag{3.7}$$

where

$$G_{ij}(x,\xi,y,\zeta,t)=\begin{cases}g_i(x,\xi,y,\zeta,t)h+(\alpha^2-a_i^2)\Delta g_i*g_\alpha, & \text{if } i=j\\ \left(\frac{\partial}{\partial t}-a_i^2\Delta\right)g_i*g_\alpha, & \text{if } i\neq j\end{cases} \tag{3.8}$$

$$g_i(x,\xi,y,\zeta,t)=\sum_{m,n=1}^{\infty}\frac{e^{-a_i^2[\lambda_n^2+\left(\frac{m\pi}{h}\right)^2]t}[\lambda_n^2+\left(\frac{m\pi}{h}\right)^2]X_{in}(x)X_{in}(\xi)Y_m(y)Y_m(\zeta)}{\|X_{in}\|^2\|Y_{im}\|^2}$$

$$g_\alpha(x,\xi,y,\zeta,t)=\sum_{m,n=1}^{\infty}\frac{e^{-\alpha^2[\lambda_{mn}^2+\left(\frac{n\pi}{h}\right)^2]t}[\lambda_{mn}^2+\left(\frac{n\pi}{h}\right)^2]X_{in}(x)X_{in}(\xi)Y_m(y)Y_m(\zeta)}{\|X_{in}\|^2\|Y_{im}\|^2} \tag{3.9}$$

$$\alpha^2=\frac{(\kappa_1+\kappa_2)a_1^2a_2^2+(\kappa_2+\kappa_3)a_2^2a_3^2}{\kappa_1a_2^2+\kappa_2(a_1^2+a_3^2)+\kappa_3a_2^2}$$

$$X_{in}(x)=\begin{cases}X_{1n}(x)=\frac{\sin\lambda_nx}{\sin\lambda_nl_0}, & 0\le x\le l_0\\ X_{2n}(x)=\frac{\sin\lambda_n(l_1-x)}{\sin\lambda_n(l_1-l_0)}+\frac{\sin\mu_n(x-l_0)}{\sin\mu_n(l_1-l_0)}, & l_0\le x\le l_1\\ X_{3n}(x)=\frac{\sin\mu_n(l-x)}{\sin\mu_n(l-l_1)}, & l_1\le x\le l\end{cases}$$

λ_n,μ_n – *roots of the system of transcendental equations*

$$\begin{cases}\kappa_1\cot\lambda l_0+\kappa_2\cot\lambda(l_1-l_0)=\frac{\kappa_2}{\sin\mu(l_1-l_0)}\\ \kappa_2\cot\mu(l_1-l_0)+\kappa_3\cot\mu(l_2-l_1)=\frac{\kappa_2}{\sin\lambda(l_1-l_0)}\end{cases}$$

$$\|X_{in}\|^2=\frac{1}{\kappa_1\alpha_2^2+\kappa_2(a_1^2+a_3^2)+\kappa_3a_2^2}\left[\frac{\kappa_1a_2^2l_0^2}{\sin^2\lambda_nl_0}+\frac{\kappa_2(a_1^2+a_3^2)(l_1-l_0)^2}{\sin\lambda_n(l_1-l_0)\sin\mu_n(l_1-l_0)}+\frac{\kappa_3a_2^2(l-l_1)^2}{\sin^2\mu_n(l-l_1)}\right]$$

$$Y_m(x)=\sin\frac{m\pi y}{h},\quad \text{if } 0\le y\le h,\quad \|Y_m\|^2=\frac{h}{2}$$

The main properties of the Green's function $G(x,\xi,y-\varsigma,t)$ for the first boundary-value problem (2.1)-(2.6) of the heat conduction equation:

(1) Green's function $G(x,\xi,y-\varsigma,t)$ is symmetric with respect to the variables x and y.
(2) Green's function $G(x,\xi,y-\varsigma,t)$ is infinitely differentiable with respect to x,y, and t for $0<x<l,\ 0<y<h,\ 0<t\le T$.
(3) For each ξ,ς,x,y,t, the function $G(x,\xi,y-\varsigma,t)$ satisfies the heat conduction equation with respect to x,y,t:

$$\frac{\partial G(x,\xi,y-\varsigma,t)}{\partial t}-a^2\left(\frac{\partial^2G(x,\xi,y-\varsigma,t)}{\partial x^2}+\frac{\partial^2G(x,\xi,y-\varsigma,t)}{\partial y^2}\right)=0.$$

(4) Green's function satisfies the normalization condition:

$$\lim_{t\to0}\int_\Omega G(x,\xi,y-\varsigma,t)\,d\Omega=1.$$

(5) Green's function satisfies the boundary conditions:

$$\begin{cases} G(x,\xi,y-\varsigma,t)\big|_{x=0}=0, \\ G(x,\xi,y-\varsigma,t)\big|_{x=l}=0, \\ G(x,\xi,y-\varsigma,t)\big|_{y=0}=0, \\ G(x,\xi,y-\varsigma,t)\big|_{y=h}=0, \end{cases}$$

(6) Green's function satisfies the conjugation conditions:

$$G_{ji}\big|_{x=l_{j-1}-0}=G_{j+1,i}\big|_{x=l_{j-1}+0},$$

$$k_j\frac{\partial G_{ji}}{\partial x}\bigg|_{x=l_{j-1}-0}=k_{j+1}\frac{\partial G_{j+1,i}}{\partial x}\bigg|_{x=l_{j-1}+0}$$

, for i=1,2,3, j=1,2.

3.2. **Existence and Uniqueness of the Solution.** Using the formula

$$\frac{\partial}{\partial\tau}(Gu)=G\frac{\partial u}{\partial\tau}+u\frac{\partial G}{\partial\tau}=a^2\big(G\Delta u-u\Delta G\big)+Gf$$

and integrating by parts and then passing to the limit as $\tau\to 0$, we obtain the solution of problem (2.1)–(2.6).

Theorem. For any functions

$$f(x,y,t)\in C(\overline{\Omega}_T)\cap C^{2,1}(\Omega_T),\quad u_0(x,y)\in C(\overline{\Omega})\cap C^2(\Omega)$$

,

$$\Phi_i(y,t)\in C(\overline{\sigma}_{iT})\cap C^{2,1}(\sigma_{iT}),\quad \Psi_i(x,t)\in C(\overline{\omega}_{iT})\cap C^{2,1}(\omega_{iT}),\ i=0,1$$

,

a unique classical solution:

$$u(x,y,t)=\begin{cases} u_1(x,y,t)\in C(\overline{\Omega}_{1T})\cap C^{2,1}(\Omega_{1T}) \\ u_2(x,y,t)\in C(\overline{\Omega}_{2T})\cap C^{2,1}(\Omega_{2T}) \\ u_3(x,y,t)\in C(\overline{\Omega}_{3T})\cap C^{2,1}(\Omega_{3T}) \end{cases}$$

satisfying the initial condition (2.2), the boundary conditions (2.3)-(2.4)the conjugation conditions (2.5)-(2.6)which are presented as a sum of the thermal potentials:

$$\begin{aligned} u_i(x,y,t)=&\ f_1*G_{i1}+f_2*G_{i2}f_1*G_{i1}+f_3*G_{i3}f_1*G_{i1} \\ &+u_{0i}*G_{i1}+u_{02}*G_{i2}+u_{03}*G_{i3} \\ &+\Phi_1*\frac{\partial G_{i3}}{\partial\xi}\bigg|_{\xi=l}-\Phi_0*\frac{\partial G_{i1}}{\partial\xi}\bigg|_{\xi=0} \\ &+\Psi_1*\frac{\partial G_{i3}}{\partial\zeta}\bigg|_{\zeta=h}-\Psi_0*\frac{\partial G_{i1}}{\partial\zeta}\bigg|_{\zeta=0} \end{aligned}$$

where $G(x,\xi,y,\zeta,t)$ is Green's function for (2.1)-(2.6) problem which was defined by the formulas (3.7)-(3.9).

4. Conclusion

The practical significance of the obtained results lies in solving specific initial–boundary value problems for the heat conduction equation in three-layer media. This enables the analysis of internal processes and the calculation of nonstationary temperature fields and heat fluxes in three-layer sheet products, structural elements, and constructions, as well as in rectangular samples of devices and apparatuses where thermophysical parameters depend on flow boundaries, initial temperature, and medium interface boundaries.

These results can be applied to describe three-layer heat exchange processes under prescribed temperature fields and corresponding initial and boundary conditions. The potential method provides powerful tools for analyzing and solving heat conduction problems, simplifying the solution process and extending its applicability beyond mathematics and physics.

The analytical approach to constructing the Green's function, followed by its transition to the thermal potential method, is presented here for the first time. This approach offers valuable insights into the behavior of heat propagation in multilayer media and represents an effective and preferred method for studying such problems.

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On the two-layer conjugation problem for the heat conduction equation in a cylinder

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Abstract. The analytical theory of non-stationary heat conduction in cylindrical surfaces finds wide application in solving important technical and technological problems related to heat and mass transfer processes, which express the balance of energy or matter. This article considers an initial-boundary value problem for the heat conduction equation on a cylindrical surface with a given initial temperature distribution, boundary conditions, and conjugation conditions on the contact cylindrical surface. The boundary conjugation condition (of the fourth kind) corresponds to heat exchange between the surface of a body and the surrounding medium according to Fourier's law of heat conduction, or to heat exchange between contacting solid bodies when the temperatures of the contacting surfaces are equal (ideal or perfect thermal contact). In addition, the flow around a solid body is determined by the flow of a fluid (or gas), where heat transfer from the fluid (or gas) to the body surface in the immediate vicinity occurs in accordance with the law of heat conduction (molecular heat transfer). As a result, the heat exchange process corresponds to the boundary condition of the fourth kind. Using the method of separation of variables combined with integral transformations—specifically, the Laplace transform with respect to time and the zero-order Bessel transform with respect to the cylindrical radius—an analytical solution is obtained for the problem of non-stationary heat conduction in a two-layer medium. The study of initial-boundary value problems for the heat conduction equation and the development of analytical methods for their solution represent an important and rapidly developing area of mathematical physics. Solutions to such problems can be used to calculate non-stationary temperature fields and heat fluxes in two-layer sheet materials, structural elements, and buildings, as well as in cylindrical specimens of materials and devices whose thermophysical parameters depend functionally on temperature and medium interfaces. Furthermore, these solutions can be applied to determine the conditions of two-layer heat exchange processes based on temperature fields and corresponding initial and boundary conditions.

2020 Mathematics Subject Classifications: 35A09, 35K05, 35K20

Keywords: two-layer medium, temperature field, non-stationary conduction, Bessel function, conjugation conditions, discontinuous coefficients

1. Introduction

The study of non-stationary initial–boundary value problems for the heat conduction equation with discontinuous coefficients is one of the key topics in the theory of partial differential equations and attracts great interest not only among mathematicians but also among researchers in technical and applied sciences. Boundary value problems for the heat conduction equation with discontinuous coefficients are classical objects of research. Such problems have been studied extensively in works [1]–[4]. In [3], using the method of Green's functions and thermal potentials, the well-posedness of the first initial–boundary value problem for the heat conduction equation with a discontinuous coefficient was established. Furthermore, in the work of Kazakhstani mathematicians E. I. Kim and B. B. Baimukhanov [4], the well-posedness of the first initial–boundary value problem for the two-dimensional heat conduction equation with a discontinuous thermal conductivity coefficient was proved using the potential method and reduction to an integral equation in a half-space. In [3]–[5], by employing thermal potentials, the existence of classical solutions to various boundary value problems for parabolic equations was demonstrated.

The motivation for studying this class of problems arises from the practical importance of the heat conduction equation with discontinuous coefficients in engineering and applied sciences. It is also connected with the development of various branches of mathematics, including function theory, the theory of integral equations, the theory of series and integrals, functional analysis, and approximation theory. In addition to classical problems of heat conduction and diffusion, similar mathematical models occur in heat and mass transfer theory (for example, in the description of drying and cooling processes), in nuclear chain reaction theory for studying neutron moderation, in signal theory for the macroscopic description of random processes at the output of radio-engineering devices, as well as in various processes of biological and chemical kinetics and other technical applications.

2. Mathematical formulation of the problem

Let us denote by

$$\Omega = \Omega_1(0 < r < r_0) \cup \Omega_2(r_0 < r < R), \quad \Omega_T = [\Omega_1(0 < r < r_0) \cup \Omega_2(r_0 < r < R)] \times (0, T)$$

We consider the following problem: find a solution of the heat equation

$$\frac{\partial u}{\partial t} = a^2(r)\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + f(r,t) \tag{2.1}$$

where

$$a^2(r) = \begin{cases} a_1^2, & 0 < r < r_0 \\ a_2^2, & r_0 < r < R \end{cases}, \quad f(r,t) = \begin{cases} f_1(r,t), & 0 < r < r_0 \\ f_2(r,t), & r_0 < r < R \end{cases}$$

that satisfies:
the initial condition

$$u(r,0) = \begin{cases} u_{01}(r), & 0 < r < r_0, \\ u_{02}(r), & r_0 < r < R, \end{cases} \quad \text{on the region } \Omega \tag{2.1}$$

boundary conditions:

$$|u(0,t)| < \infty, \quad u(R,t) = \psi(t) \tag{2.3}$$

conjugation conditions:

$$u_1|_{r=r_0-0} = u_2|_{r=r_0+0}, \quad k_1 \left.\frac{\partial u_1}{\partial r}\right|_{r=r_0-0} = k_2 \left.\frac{\partial u_2}{\partial r}\right|_{r=r_0+0} \tag{2.4}$$

3. Main results

Considering Cauchy's problem, there is constructed the Green's function using the finite Hankel transform with respect to the variable r and the Laplace transform with respect to time variable, which is represented in the following

$$G_(r,r',t) = \begin{cases} G_{11} = \sum_{n=1}^{\infty} \frac{e^{-\lambda_n^2 t}}{\|R_n\|^2} R_{1n}(r)R_{1n}(r'), & \text{if } 0 < r < r_0,\ 0 < r' < r_0 \\ G_{12} = \sum_{n=1}^{\infty} \frac{e^{-\lambda_n^2 t}}{\|R_n\|^2} R_{1n}(r)R_{2n}(r'), & \text{if } 0 < r < r_0,\ r_0 < r' < R \\ G_{21} = \sum_{n=1}^{\infty} \frac{e^{-\lambda_n^2 t}}{\|R_n\|^2} R_{2n}(r)R_{1n}(r'), & \text{if } r_0 < r < R,\ 0 < r' < r_0 \\ G_{22} = \sum_{n=1}^{\infty} \frac{e^{-\lambda_n^2 t}}{\|R_n\|^2} R_{2n}(r)R_{2n}(r'), & \text{if } r_0 < r < R,\ r_0 < r' < R \end{cases} \tag{3.1}$$

where
(3.2)

$$R_n(r) = \begin{cases} R_{1n}(r) = \alpha_{1n} J_0\left(\frac{\lambda_n}{a_1} r\right), & \text{if } 0 < r < r_0, \\ R_{2n}(r) = \alpha_{2n}\left\{J_0\left(\frac{\lambda_n}{a_2} r\right) N_0\left(\frac{\lambda_n}{a_2} R\right) - N_0\left(\frac{\lambda_n}{a_2} r\right) J_0\left(\frac{\lambda_n}{a_2} R\right)\right\}, & \text{if } r_0 < r < R, \end{cases}$$

$$\alpha_n = \begin{cases} \alpha_{1n} = J_0\left(\frac{\lambda_n}{a_2} r_0\right) N_0\left(\frac{\lambda_n}{a_2} R\right) - N_0\left(\frac{\lambda_n}{a_2} r_0\right) J_0\left(\frac{\lambda_n}{a_2} R\right), \text{if} & 0 < r < r_0 \\ \alpha_{2n} = J_0\left(\frac{\lambda_n}{a_1} R\right), \text{if} & r_0 < r < R, \end{cases} \tag{3.3}$$

with squared norm:

$$\|R_n\|^2 = \alpha_{1n}^2 \frac{r_0^2}{2} J_0^2\left(\frac{\lambda_n}{a_1} r_0\right) + \alpha_{2n}^2 \frac{\pi^2}{2J_0^2\left(\frac{\lambda_n}{a_2} r_0\right)}\left\{J_0^2\left(\frac{\lambda_n}{a_2} r_0\right) - J_0^2\left(\frac{\lambda_n}{a_2} R\right)\right\} \tag{3.4}$$

here λ_n are roots of the transcendental equation:

$$\det \begin{pmatrix} J_0(\lambda r_0) & -N_0(\lambda r_0) & -J_0(\lambda r_0) \\ k_1 J_0'(\lambda r_0) & k_2 N_0'(\lambda r_0) & k_2 J_0'(\lambda r_0) \\ 0 & N_0(\lambda R) & J_0(\lambda R) \end{pmatrix} = 0 \qquad (3.5)$$

Theorem 3.1. *For any functions* $f(r,t) \in C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T) \cap L_2(\Omega_T)$, $u_0(r) \in C(\overline{\Omega}) \cap C^2(\Omega) \cap L_2(\Omega)$, *and* $\psi(t) \in C[0,T] \cap C^1(0,T)$, *there exists a unique classical solution*

$$u(r,t) = \begin{cases} u_1(r,t) \in C(\Omega_{1T}) \cap C^{2,1}(\Omega_{1T}) \\ u_2(r,t) \in C(\Omega_{2T}) \cap C^{2,1}(\Omega_{2T}) \end{cases} \qquad (3.6)$$

satisfying (2.2)–(2.4), *presented as:*

$$u_i(r,t) = f_1 * G_{i1} + f_2 * G_{i2} + u_{01} * G_{i1} + u_{02} * G_{i2} + (-1)^i \psi * \left.\frac{\partial G_{i2}}{\partial r'}\right|_{r'=R} \qquad (3.7)$$

Conclusion

The practical significance of the obtained results lies in solving specific initial–boundary value problems for the heat conduction equation in two-layer media, aimed at understanding internal processes and calculating non-stationary temperature fields and heat fluxes in two-layer and two-layer sheet products, structural elements, and constructions, as well as in cylindrical samples of devices and apparatuses where thermophysical parameters functionally depend on flow boundaries, initial temperature, and medium interface boundaries. This makes it possible to apply the conditions of two-layer heat exchange processes in accordance with given temperature fields and the corresponding initial and boundary conditions.

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Nonlinear Schwarz problem in triangular domains

Bahriye Karaca

Abstract. In this paper, we study the Schwarz boundary value problem for nonlinear higher-order elliptic equations in a triangular domain. By employing Pompeiu-type operators, we reduce the nonlinear differential problem to an equivalent integro-differential system and derive explicit integral representations for the solution. Under suitable continuity and Lipschitz conditions on the nonlinear term, we establish the existence and uniqueness of solutions via the Banach fixed point theorem.

2020 Mathematics Subject Classifications: 35J60, 35A09, 30E25

Keywords: Schwarz boundary value problem, Nonlinear differential equation, Triangular domain

1. Introduction and Preliminaries

Boundary value problems for complex partial differential equations, especially of the Schwarz type, have been widely investigated because of their central role in complex analysis and their applications in physics and engineering [1, 2, 3, 4]. Early studies primarily considered classical domains such as the unit disc, the half-plane, and annular regions, focusing on the Cauchy–Riemann and polyanalytic equations through the use of integral representations, operator hierarchies, and explicit solution formulas [5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 19].

In this paper, we address the Schwarz problem for nonlinear higher-order equations in a triangular domain. To set the groundwork, we first recall results on the inhomogeneous Cauchy–Riemann and polyanalytic equations in this setting.

Let T denote a triangular domain in the complex plane $\mathbb{C}$, defined as

$$T = \left\{ z \in \mathbb{C} : |z - \epsilon_k| < \sqrt{2}, \quad k = 1, 2, 3 \right\},$$

where

$$\epsilon_1 = 1, \quad \epsilon_2 = -1, \quad \epsilon_3 = -i\sqrt{3}.$$

The circles

$$C_1 = \left\{ z \in \mathbb{C} : |z - 1| = \sqrt{2} \right\}, \quad C_2 = \left\{ z \in \mathbb{C} : |z + 1| = \sqrt{2} \right\}, \quad C_3 = \left\{ z \in \mathbb{C} : |z + i\sqrt{3}| = \sqrt{2} \right\}$$

are centered at ϵ_1, ϵ_2, and ϵ_3 each with radius $\sqrt{2}$. The domain T is the intersection of the interiors of these three circles.

For $z \in T$, define the following transformations:

$$z_2 = -\frac{1}{z}, \quad z_3 = \frac{-(1+i\sqrt{3})z + 1 - i\sqrt{3}}{(1-i\sqrt{3})z + 1 + i\sqrt{3}}, \quad z_4 = \frac{(i\sqrt{3}-1)z - 1 - i\sqrt{3}}{(-1+i\sqrt{3})z + 1 - i\sqrt{3}},$$

$$z_5 = \frac{\overline{z}+1}{\overline{z}-1}, \quad z_6 = \frac{-\overline{z}+1}{\overline{z}+1}, \quad z_7 = \frac{-i\sqrt{3}\,\overline{z} - 1}{\overline{z} - i\sqrt{3}}, \quad z_8 = \frac{\overline{z} - i\sqrt{3}}{i\sqrt{3}\,\overline{z} + 1}.$$

It is evident that

$$z_2, z_3, z_4, z_5, z_6, z_7, z_8 \notin T.$$

The boundary of T, denoted by ∂T, consists of three circular arcs. Its vertices are given by

$$a = -\tfrac{1}{2}(\sqrt{3}-1)(1+i), \quad b = \tfrac{1}{2}(\sqrt{3}-1)(1-i), \quad c = -i,$$

which correspond to the pairwise intersections of C_1, C_2, and C_3.

The next theorem provides the solution of the classical Schwarz problem for the inhomogeneous Cauchy–Riemann equation in T, forming the basis for higher-order polyanalytic cases.

Theorem 1.1 ([14]). *The Schwarz boundary value problem for the inhomogeneous polyanalytic equation in the triangular domain T,*

$$\partial_{\bar z}^n w(z) = f(z) \quad in\ T, \qquad \text{Re}\ \partial_{\bar z}^s w(z) = \vartheta_s \quad on\ \partial T,$$

together with the additional conditions

$$\sum_{j=1}^{3} \frac{2}{\pi i} \int_{\partial T \cap C_j} \text{Im}\left(\partial_{\bar z}^s w(\varsigma)\right) \frac{d\varsigma}{\varsigma - \epsilon_j} = c_s, \quad s = 0, \dots, n-1,$$

is uniquely solvable. Its solution is given by

$$w(z) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \sum_{j=1}^{3} \frac{1}{\pi i} \int_{\partial T \cap C_j} (\varsigma - z + \overline{\varsigma - z})^k \vartheta_k(\varsigma) \left(K_1(z,\varsigma) - \frac{2}{\varsigma - \epsilon_j} \right) d\varsigma$$

$$(1) \quad + \frac{(-1)^n}{(n-1)!\pi} \iint_T (\varsigma - z + \overline{\varsigma - z})^{n-1} \left[f(\varsigma) K_1(z,\varsigma) - \overline{f(\varsigma) K_2(z,\varsigma)} \right] d\kappa d\delta + \sum_{s=0}^{n-1} \frac{(z+\bar z)^s}{s!} i c_s$$

where

$$(2) \qquad K_1(z,\varsigma) = \sum_{k=1}^{4} \frac{1}{\varsigma - z_k}, \quad K_2(z,\varsigma) = \sum_{k=5}^{8} \frac{1}{\varsigma - z_k}, \quad \varsigma = \kappa + i\delta.$$

2. Integral Operators in the Study of Higher-Order Schwarz Problems

Integral representation formulas are fundamental in studying boundary value problems, as they provide explicit constructions of solutions and allow for a detailed analysis of both analytic and functional-analytic properties of associated operators. Within the framework of the Schwarz problem for higher-order equations in a triangular domain, such formulas naturally arise and serve as the main tool for constructing solutions.

In particular, the domain integral appearing in (1) is associated with the so-called T-*operator*, which plays a central role in solving the Schwarz problem in triangular domains. By iterating this operator, one can define higher-order Pompeiu-type operators, which allow systematic handling of derivatives of arbitrary order. More precisely, we introduce the following definition. Here, $L_{p,2}(T)$ denotes the space of measurable functions $f : T \to \mathbb{C}$ such that $\|f\|_{L_{p,2}(T)} := \left(\iint_T |f(z)|^p \, d\kappa \, d\delta\right)^{1/p} < \infty$. Moreover, $\mathbb{D}$ denotes the unit disk in the complex plane: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Definition 2.1. *Let $n \in \mathbb{N}$ and $f \in L_{p,2}(T)$, $p > 2$. For $z \in T$, define*

$$(3) \qquad \tilde T_{n,T} f(z) := \frac{(-1)^n}{(n-1)!\,\pi} \iint_T (\varsigma - z + \overline{\varsigma - z})^{n-1} \left[f(\varsigma) K_1(z,\varsigma) - \overline{f(\varsigma) K_2(z,\varsigma)} \right] d\kappa d\delta,$$

with the convention

$$\tilde T_{0,T} f := f.$$

The operators $\tilde T_{n,T}$ possess a recursive structure that allows them to be expressed as repeated applications of the first-order operator $\tilde T_{1,T}$. This property is crucial for establishing explicit solution formulas and for analyzing the mapping properties of higher-order operators.

Theorem 2.2. *For $n \in \mathbb{N}$, $f \in L_{p,2}(T)$, $p > 2$, and $z \in T$, the operator $\tilde T_{n,T}$ can be represented as an n-fold composition of $\tilde T_{1,T}$:*

$$(4) \qquad \tilde T_{n,T} f(z) = \left(\tilde T_{1,T}\right)^n f(z),$$

where $\left(\tilde T_{1,T}\right)^n$ denotes the n-fold iteration of $\tilde T_{1,T}$.

The following result highlights key properties of the operators $\tilde T_{n,T}$ regarding derivatives and boundary behavior, which are essential for solving the Schwarz problem.

Theorem 2.3. *Let $f \in L_{p,2}(T)$ with $p > 2$ and $n \in \mathbb{N}$. Then, for each $0 \leq l \leq n-1$, the function $\tilde{T}_{n,T} f$ satisfies:*

$$\partial_{\bar{z}}^{l} \tilde{T}_{n,T} f(z) = \tilde{T}_{n-1,T} f(z), \tag{5}$$

$$\operatorname{Re} \partial_{\bar{z}}^{l} \tilde{T}_{n,T} f(z) = 0 \quad \text{on } \partial T, \tag{6}$$

$$\sum_{j=1}^{3} \frac{2}{\pi i} \int_{\partial T \cap C_j} \operatorname{Im} \left(\partial_{\bar{z}}^{l} \tilde{T}_{n,T} f(\varsigma) \right) \frac{d\varsigma}{\varsigma - \epsilon_j} = 0 \quad \text{on } \partial T. \tag{7}$$

Moreover, these operators are linear and bounded on $L_{p,2}(T)$, and they provide a systematic framework for constructing solutions of the Schwarz problem for polyanalytic equations of arbitrary order in triangular domains.

3. Main results

In this section, we study the Schwarz boundary value problem for nonlinear higher-order elliptic equations in a triangular domain. The main goal is to establish existence and uniqueness of solutions by reducing the differential problem to an equivalent integro-differential system using Pompeiu-type operators, see [13]. We also formulate precise conditions on the nonlinear term to ensure well-posedness of the problem.

We consider the elliptic equation

$$\partial_{\bar{z}}^{n} w = F(z, w, D^{\alpha_1} w, D^{\alpha_2} w, \ldots, D^{\alpha_n} w), \tag{8}$$

of order n in $\mathbb{D}$. Following the notation of [15], we define

$$D = (\partial_z, \partial_{\bar{z}}), \quad \alpha_j = (k, l), \quad |\alpha_j| = k + l = j, \quad j = 1, 2, \ldots, n, \tag{9}$$

with the restriction $(k, l) \neq (0, n)$.

For simplicity, we denote

$$D^{\alpha_1} w, D^{\alpha_2} w, \ldots, D^{\alpha_n} w \quad \text{by} \quad w_{k,l}.$$

With this notation, Equation (8) can be equivalently written as

$$\partial_{\bar{z}}^{n} w = F(z, w_{k,l}). \tag{10}$$

Assumptions on F. We impose the following assumptions on the nonlinear function F to ensure well-posedness of the problem:

1. $F(z, w_{k,l})$ is continuous in all its arguments, ensuring smooth dependence on z and the derivatives of w.
2. If $w_{k,l} \in L^p(\mathbb{D})$ with $p > 2$, then $F(z, w_{k,l})$ also belongs to $L^p(\mathbb{D})$, guaranteeing integrability.
3. $F(z, w_{k,l})$ satisfies a Lipschitz-type condition:

$$|F(z, w_{k,l}) - F(z, \tilde{w}_{k,l})| \leq \sum_{k+l \leq n} L_{k,l} |w_{k,l} - \tilde{w}_{k,l}| \leq L \sum_{k+l \leq n} |w_{k,l} - \tilde{w}_{k,l}|, \tag{11}$$

   where $L = \max_{k+l \leq n} L_{k,l}$, controlling the growth of F with respect to the derivatives of w.

Before formulating the Schwarz boundary value problem, we note that using the Pompeiu operators, the nonlinear differential equation can be reduced to an equivalent integro-differential equation. This reduction is essential for analyzing existence and uniqueness of solutions. In the following problem, $W^{n,p}(T)$, with $p > 2$, denotes the Sobolev space of complex-valued functions on T whose derivatives up to order n are p-integrable:

$$W^{n,p}(T) = \left\{ f \in L^p(T) : D^{\alpha} f \in L^p(T), \ |\alpha| \leq n \right\},$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index and $D^{\alpha} f = \frac{\partial^{|\alpha|} f}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$. This space measures both the integrability and smoothness of functions on T.

Schwarz Problem. Find a solution $w \in W^{n,p}(\mathbb{D})$, $p > 2$, of Equation (8) satisfying the Schwarz boundary conditions

(12)
$$\operatorname{Re} \partial_{\bar z}^s w(z) = 0 \quad \text{on } \partial T, \quad \text{and} \quad \sum_{j=1}^{3} \frac{2}{\pi i} \int_{\partial T \cap C_j} \operatorname{Im}\left(\partial_{\bar z}^s w(\zeta)\right) \frac{d\zeta}{\zeta - \epsilon_j} = 0, \quad s = 0, \dots, n-1,$$

Using the Pompeiu operators introduced in the previous section, the problem can be rewritten as the system

$$w(z) = \psi(z) + \tilde{T}_{0,n} F(z, w_{k,l}), \tag{13}$$

$$w_{k,l}(z) = \frac{\partial^{k+l}\psi(z)}{\partial z^k \partial \bar z^l} + \partial_z^k \tilde{T}_{0,n-l} F(z, w_{k,l}), \tag{14}$$

for $0 \le k + l \le n$, $(k,l) \neq (0,n)$, where

$$\psi(z) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \sum_{j=1}^{3} \frac{1}{\pi i} \int_{\partial T \cap C_j} (\zeta - z + \overline{\zeta - z})^k \gamma_k(\zeta) \left(K_1(\zeta) - \frac{2}{\zeta - \epsilon_j} \right) d\zeta. \tag{15}$$

3.1. **Solvability.** Define the Banach space

$$H^p(\mathbb{D}) = \{w_{k,l} \in L^p(\mathbb{D}) : 0 \le k+l \le n,\ (k,l) \neq (0,n)\}, \quad \|w_{k,l}\|_{H^p} = \max_{0\le k+l\le n} \|w_{k,l}\|_{L^p}. \tag{16}$$

We rewrite the Schwarz problem in operator form using the Pompeiu operators:

$$W(z) = \psi(z) + \tilde{T}_{0,n} F(z, w_{k,l}), \quad W_{k,l}(z) = \frac{\partial^{k+l}\psi(z)}{\partial z^k \partial \bar z^l} + \partial_z^k \tilde{T}_{0,n-l} F(z, w_{k,l}). \tag{17}$$

Define the operator $Q : H^p(\mathbb{D}) \to H^p(\mathbb{D})$ by

$$Q(w_{k,l}) = (W_{k,l}). \tag{18}$$

For two elements $w_{k,l}, \tilde w_{k,l} \in H^p(\mathbb{D})$, we estimate

$$\begin{aligned} \|Q(w_{k,l}) - Q(\tilde w_{k,l})\|_{H^p} &= \max_{0\le k+l\le n} \|W_{k,l} - \tilde W_{k,l}\|_{L^p} \\ &\le \max_{0\le k+l\le n} \Big(\|\tilde T_{0,n}(F(z, w_{k,l}) - F(z, \tilde w_{k,l}))\|_{L^p} \\ &\quad + \|\partial_z^k \tilde T_{0,n-l}(F(z, w_{k,l}) - F(z, \tilde w_{k,l}))\|_{L^p} \Big). \end{aligned} \tag{19}$$

Using the Lipschitz property of F, we have

$$\|F(z, w_{k,l}) - F(z, \tilde w_{k,l})\|_{L^p} \le \sum_{k+l\le n} L_{k,l} \|w_{k,l} - \tilde w_{k,l}\|_{L^p} \le L \|w_{k,l} - \tilde w_{k,l}\|_{H^p}, \tag{20}$$

where

$$L = \max_{k+l\le n} L_{k,l}. \tag{21}$$

Combining the boundedness of the Pompeiu operators with the Lipschitz estimate, we obtain

$$\|Q(w_{k,l}) - Q(\tilde w_{k,l})\|_{H^p} \le L \max\left\{ \|\tilde T_{0,n}\|, \|\partial_z^k \tilde T_{0,n-l}\|, \|\Pi_{n,\mathbb{D}}\| \right\} \|w_{k,l} - \tilde w_{k,l}\|_{H^p}. \tag{22}$$

If

$$L \max\left\{ \|\tilde T_{0,n}\|, \|\partial_z^k \tilde T_{0,n-l}\|, \|\Pi_{n,\mathbb{D}}\| \right\} < 1, \tag{23}$$

then Q is a contraction. By the Banach fixed point theorem, Q admits a unique fixed point in $H^p(\mathbb{D})$, which corresponds to the unique solution of the Schwarz boundary value problem (8)–(12).

4. Conclusion

We have established existence and uniqueness results for the Schwarz boundary value problem of nonlinear higher-order elliptic equations in a triangular domain. The reduction to an integro-differential system via Pompeiu operators proved effective in deriving explicit solution formulas, and the Banach fixed point theorem provided a rigorous framework for solvability. These results extend classical boundary value theory to more general nonlinear settings.

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Some basic results of eccentricity sombor index

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ABSTRACT. This paper is related to investigating the upper and lower bounds of eccentricity Sombor index, stated as $SO_{ec}(\mathcal{G})$. In mathematical terms, this index is defined as $SO_{ec}(\mathcal{G}) = \sum\limits_{uv \in E(\mathcal{G})} \sqrt{ec^2(u) + ec^2(v)}$ where $E(\mathcal{G})$ is the set of edges in the graph $\mathcal{G}$, and $ec(.)$ is the eccentricity of a vertex of the edge in $E(\mathcal{G})$. The eccentricity of a vertex is defined as the maximum distance between the vertex and any other vertex in the graph. By analyzing graph structural aspects, the work provides new perspectives on the features and extremal behaviors of this graph invariant. The calculated constraints contribute to our knowledge of the eccentricity Sombor index and its possible applications in graph theory and other domains.

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1. INTRODUCTION

In this study, we exclusively consider connected simple graphs, ensuring that every graph discussed has a path connecting any pair of vertices and contains no loops or multiple edges. Let $\mathcal{G}$ denote a graph with $n = |V_{\mathcal{G}}|$ vertices and $m = |E_{\mathcal{G}}|$ edges where $V_{\mathcal{G}}$ represents the vertex set and $E_{\mathcal{G}}$ the edge set of the graph. If two vertices $u, v \in V_{\mathcal{G}}$ are adjacent, the edge connecting u and v is denoted as uv. The degree of a vertex $v \in V_{\mathcal{G}}$, represented as $d(v)$ is defined as the number of edges incident to v. Additionally, the minimum vertex degree of the graph $\mathcal{G}$ is denoted by $\delta = \delta_{\mathcal{G}}$, while the maximum vertex degree is expressed as $\Delta = \Delta_{\mathcal{G}}$ [4, 11, 21].

The eccentricity of a vertex v, denoted $ec(v)$ of $v \in V_{\mathcal{G}}$, is a measure of the maximum distance from v to any other vertex in the graph. Formally, it is defined as:

$$ec(v) = \max_{u \in V_{\mathcal{G}}} d(u, v)$$

where $d(u, v)$ is the length of a shortest path between u and v in $\mathcal{G}$. Two important graph invariants derived from eccentricity are the radius and the diameter of the graph. The radius of a graph, $r_{\mathcal{G}}$ is defined as the minimum eccentricity across all vertices, representing the centrality of the graph. On the other hand, the diameter, $D_{\mathcal{G}}$, is the maximum eccentricity among all vertices, signifying the greatest distance between any two vertices in the graph [4, 11, 21].

Graph invariants, particularly those based on distance and eccentricity, play a pivotal role in the field of chemical graph theory. These invariants, often referred to as topological indices, are widely applied in chemistry to study and establish relationships between molecular structures and their properties. Examples of such indices include those based on distance and eccentricity, as highlighted in foundational works such as [3, 5, 7, 8, 10, 12, 15, 16, 17, 19, 20]. These indices provide valuable tools for understanding molecular characteristics and facilitating the prediction of chemical behaviors [1, 2].

In [18], Sharma et al. gave the definition of eccentric connectivity index which is a distance–based molecular structure descriptor and which is defined as

$$\xi^C(\mathcal{G}) = \sum_{uv \in E_{\mathcal{G}}} ec(u) + ec(v).$$

Firstly, Gutman [9] gave the definition of the Sombor index:

$$SO(\mathcal{G}) = \sum_{uv \in E_{\mathcal{G}}} \sqrt{d^2(u) + d^2(v)}.$$

The eccentricity Sombor index defined by Kulli [13] as following:

$$SO_{ec}(G) = \sum_{uv \in E_{\mathcal{G}}} \sqrt{ec^2(u) + ec^2(v)}.$$

In addition to the reduced Sombor index [6] definition, the reduced eccentricity Sombor index has also been defined. Liu et al. [14], gave the definiton of the reduced eccentricity Sombor index:

$$SO_{red-ec}(\mathcal{G}) = \sum_{uv \in E_{\mathcal{G}}} \sqrt{(ec(u) - 1)^2 + (ec(v) - 1)^2}.$$

In this work, we obtain some basic results for eccentricity Sombor index.

2. Preliminaries

In this part of the paper, we give some lemmas which are used to obtain the main results.

Lemma 2.1. [5] *Let $\rho_1, \rho_2, ..., \rho_n$ and $\tau_1, \tau_2, ..., \tau_n$ be real numbers. Then*

$$\sum_{l=1}^{n} \tau_l^2 + pt \sum_{l=1}^{n} \rho_l^2 \leq (p+t) \sum_{l=1}^{n} \rho_l \tau_l$$

where p and t are real constants such that $p\rho_l \leq \tau_l \leq t\rho_l$ for any l $(l = 1, 2, ..., n)$.

Lemma 2.2. [22] *Consider $\mathcal{G}$ to be a connected graph with $n \geq 3$ vertices. Then for all $u \in V_{\mathcal{G}}$ we have*

$$ec(u) \leq n - d(u)$$

with equality if and only if $\kappa_n - \omega e$, for $\omega = 0, 1, 2, ..., \left\lfloor \frac{n}{2} \right\rfloor$, or $\mathcal{G} = P_4$.

Lemma 2.3. *Consider $\mathcal{G}$ to be a connected simple graph with n vertices. It holds that*

$$r_{\mathcal{G}} \leq ec(u) \leq D_{\mathcal{G}}$$

for each vertex $u \in V_{\mathcal{G}}$.

3. Main results

In our main results, we give some bounds for eccentricity Sombor index.

Example 3.1. *Let S_n $(n \geq 1)$ and K_n $(n \geq 2)$ be a star and complete graph with n vertices; C_n be the cycle with $n \geq 3$ vertices and $K_{m,n}$ be the complete bipartite graph with m and n bipartiton. Eccentricity Sombor index of these graphs given as*

$$\begin{aligned}
SO_{ec}(S_n) &= (n-1)\sqrt{5},\\
SO_{ec}(K_n) &= \frac{n(n-1)\sqrt{2}}{2},\\
SO_{ec}(C_n) &= n\left\lfloor \frac{n}{2} \right\rfloor \sqrt{2},\\
SO_{ec}(K_{m,n}) &= 2\sqrt{2}mn \text{ where } m, n \geq 2,\\
SO_{ec}(P_n) &= \begin{cases} 2\sum\limits_{i=1}^{\frac{n}{2}-1} \sqrt{(n-i)^2 + (n-i-1)^2} + \frac{\sqrt{2}n}{2} & \text{if } n \text{ is even},\\ 2\sum\limits_{i=1}^{\left\lceil \frac{n}{2} \right\rceil - 1} \sqrt{(n-i)^2 + (n-i-1)^2} & \text{if } n \text{ is odd}. \end{cases}
\end{aligned}$$

Theorem 3.2. *Let $\mathcal{G}$ be a graph with n vertices. Assume that the radius of $\mathcal{G}$ is denoted by $r_{\mathcal{G}}$ and the diameter of $\mathcal{G}$ is denoted by $D_{\mathcal{G}}$. Then*

$$m\sqrt{2}r_{\mathcal{G}} \leq SO_{ec}(\mathcal{G}) \leq m\sqrt{2}D_{\mathcal{G}}.$$

Proof. By using Lemma 2.3, we have

$$\begin{aligned}
\sum_{uv \in E_{\mathcal{G}}} \sqrt{ec^2(u) + ec^2(v)} &\leq \sum_{uv \in E_{\mathcal{G}}} \sqrt{D_{\mathcal{G}}^2 + D_{\mathcal{G}}^2}\\
&= m\sqrt{2}D_{\mathcal{G}}
\end{aligned}$$

and

$$\begin{aligned}\sum_{uv\in E_{\mathcal{G}}}\sqrt{ec^2(u)+ec^2(v)} &\geq \sum_{uv\in E_{\mathcal{G}}}\sqrt{r_{\mathcal{G}}^2+r_{\mathcal{G}}^2}\\ &= m\sqrt{2}r_{\mathcal{G}}.\end{aligned}$$

Clearly, $m\sqrt{2}r_{\mathcal{G}} \leq SO_{ec}(\mathcal{G}) = \sum_{uv\in E(\mathcal{G})}\sqrt{ec^2(u)+ec^2(v)} \leq m\sqrt{2}D_{\mathcal{G}}$.

□

Theorem 3.3. *Let $\mathcal{G}$ be a graph with n vertices and m edges. If $\delta=\delta_{\mathcal{G}}$ is the minimum vertex degree of $\mathcal{G}$, we get*

$$SO_{ec}(\mathcal{G}) \leq \sqrt{2}m(n-\delta).$$

Proof. By using Lemma 2.2, we obtain

$$\begin{aligned}\sum_{uv\in E_{\mathcal{G}}}\sqrt{ec^2(u)+ec^2(v)} &\leq \sum_{uv\in E_{\mathcal{G}}}\sqrt{(n-d(u))^2+(n-d(v))^2}\\ &= \sum_{uv\in E_{\mathcal{G}}}\sqrt{2n^2+d(u)^2+d(v)^2-2nd(u)-2nd(v)}.\end{aligned}$$

Then, we have

$$\begin{aligned}\sum_{uv\in E_{\mathcal{G}}}\sqrt{2n^2+d(u)^2+d(v)^2-2nd(u)-2nd(v)} &\leq \sum_{uv\in E_{\mathcal{G}}}\sqrt{2n^2+2\delta^2-4n\delta}\\ &\leq \sum_{uv\in E_{\mathcal{G}}}\sqrt{2(n^2+\delta^2-2n\delta)}\\ &= \sum_{uv\in E_{\mathcal{G}}}\sqrt{2(n-\delta)^2}\\ &= \sqrt{2}m(n-\delta).\end{aligned}$$

Finally,

$$\sum_{uv\in E_{\mathcal{G}}}\sqrt{ec^2(u)+ec^2(v)} \leq \sqrt{2}m(n-\delta).$$

□

Theorem 3.4. *Let $\mathcal{G}$ be a graph of size m and $\xi^C(\mathcal{G})$ be the eccentric connectivity index of $\mathcal{G}$. Then*

$$\xi^C(\mathcal{G}) > SO_{ec}(\mathcal{G}) \geq \frac{1}{\sqrt{2}}\xi^C(\mathcal{G}).$$

Proof. Let $ec(u) \geq ec(v) \geq 1$ be the eccentricity of any u, v such that $uv \in E_{\mathcal{G}}$. Then we have

$$\begin{aligned}(ec(u)+ec(v))^2 &> ec^2(u)+ec^2(v)\\ ec(u)+ec(v) &> \sqrt{ec^2(u)+ec^2(v)}.\end{aligned}$$

Clearly,

$$\begin{aligned}\sum_{uv\in E_{\mathcal{G}}} ec(u)+ec(v) &> \sum_{uv\in E_{\mathcal{G}}}\sqrt{ec^2(u)+ec^2(v)}\\ \xi^C(\mathcal{G}) &> SO_{ec}(\mathcal{G}).\end{aligned}$$

For the second part of the proof we have

$$\begin{aligned}(ec(u)-ec(v))^2 &\geq 0\\ ec^2(u)+ec^2(v) &\geq 2ec(u)\,ec(v)\\ 2ec^2(u)+2ec^2(v) &\geq (ec(u)+ec(v))^2.\end{aligned}$$

Then we can obtain

$$\begin{aligned}\sqrt{2}\sum_{uv\in E_{\mathcal{G}}}\sqrt{ec^2(u)+ec^2(v)} &\geq \sum_{uv\in E_{\mathcal{G}}} ec(u)+ec(v)\\ SO_{ec}(\mathcal{G}) &\geq \frac{1}{\sqrt{2}}\xi^C(\mathcal{G}).\end{aligned}$$

□

Theorem 3.5. *Let $\mathcal{G}$ be a graph of order n and size m. Assume that the radius of $\mathcal{G}$ is denoted by $r_{\mathcal{G}}$ and the diameter of $\mathcal{G}$ is denoted by $D_{\mathcal{G}}$. Then,*

$$2mr_{\mathcal{G}} + \sqrt{2}\chi\xi^{C}(\mathcal{G}) \leq \left(2\chi + \sqrt{2}\right) SO_{ec}(\mathcal{G})$$

where $\chi = \frac{\sqrt{D_{\mathcal{G}}^2 + r_{\mathcal{G}}^2}}{D_{\mathcal{G}} + r_{\mathcal{G}}}$ and $\xi^{C}(\mathcal{G})$ is the eccentric connectivity index of the graph $\mathcal{G}$.

Proof. From Theorem 3.4 we have

$$\sqrt{2}\sqrt{ec^2(u) + ec^2(v)} \geq ec(u) + ec(v)$$

that is,

$$\frac{\sqrt{ec^2(u) + ec^2(v)}}{ec(u) + ec(v)} \geq \frac{1}{\sqrt{2}}.$$

Since $r_{\mathcal{G}} \leq ec(u) \leq D_{\mathcal{G}}$ for any $u \in V(\mathcal{G})$, we get

$$\frac{r_{\mathcal{G}}}{D_{\mathcal{G}}} \leq \frac{ec(u)}{ec(v)} \leq \frac{D_{\mathcal{G}}}{r_{\mathcal{G}}}.$$

So,

$$\frac{\sqrt{ec^2(u) + ec^2(v)}}{ec(u) + ec(v)} \leq \frac{\sqrt{D_{\mathcal{G}}^2 + r_{\mathcal{G}}^2}}{D_{\mathcal{G}} + r_{\mathcal{G}}}.$$

In Lemma 2.1, if we take $\rho_l = \sqrt{ec(u) + ec(v)}$, $\tau_l = \sqrt{\frac{ec^2(u)+ec^2(v)}{ec(u)+ec(v)}}$, $p = \frac{1}{\sqrt{2}}$, $t = \frac{\sqrt{D_{\mathcal{G}}^2 + r_{\mathcal{G}}^2}}{D_{\mathcal{G}} + r_{\mathcal{G}}}$, we have

$$\sum_{uv \in E_{\mathcal{G}}} \frac{ec^2(u) + ec^2(v)}{ec(u) + ec(v)} + \frac{1}{\sqrt{2}} \frac{\sqrt{D_{\mathcal{G}}^2 + r_{\mathcal{G}}^2}}{D_{\mathcal{G}} + r_{\mathcal{G}}} \sum_{uv \in E_{\mathcal{G}}} ec(u) + ec(v)$$
$$\leq \left(\frac{1}{\sqrt{2}} + \frac{\sqrt{D_{\mathcal{G}}^2 + r_{\mathcal{G}}^2}}{D_{\mathcal{G}} + r_{\mathcal{G}}}\right) \sum_{uv \in E_{\mathcal{G}}} \sqrt{ec(u) + ec(v)} \sqrt{\frac{ec^2(u) + ec^2(v)}{ec(u) + ec(v)}}.$$

One can easily see that

$$mr_{\mathcal{G}} \leq \sum_{uv \in E_{\mathcal{G}}} \frac{ec^2(u) + ec^2(v)}{ec(u) + ec(v)}$$

and let $\chi = \frac{\sqrt{D_{\mathcal{G}}^2 + r_{\mathcal{G}}^2}}{D_{\mathcal{G}} + r_{\mathcal{G}}}$. So we obtain

$$mr_{\mathcal{G}} + \frac{\chi}{\sqrt{2}}\xi^{C}(\mathcal{G}) \leq \left(\frac{1}{\sqrt{2}} + \chi\right) SO_{ec}(\mathcal{G}),$$
$$2mr_{\mathcal{G}} + \sqrt{2}\chi\xi^{C}(\mathcal{G}) \leq \left(2\chi + \sqrt{2}\right) SO_{ec}(\mathcal{G}).$$

□

Theorem 3.6. *Let $\mathcal{G}$ be a order n and size m. We have,*

$$\frac{r_{\mathcal{G}} - 1}{r_{\mathcal{G}}} SO_{ec}(\mathcal{G}) \leq SO_{red-ec}(\mathcal{G}) \leq \frac{D_{\mathcal{G}} - 1}{D_{\mathcal{G}}} SO_{ec}(\mathcal{G})$$

where $D_{\mathcal{G}}$ and $r_{\mathcal{G}}$ are the diameter and the radius of $\mathcal{G}$, respectively.

Proof. When $D_{\mathcal{G}} = r_{\mathcal{G}} = 1$, $SO_{red-ec}(\mathcal{G}) = 0$. Now, assume that $D_{\mathcal{G}} \geq 2$ and $f : [r_{\mathcal{G}}, D_{\mathcal{G}}] \times [r_{\mathcal{G}}, D_{\mathcal{G}}] \to (0, \infty)$ defined as

$$f(a, b) = \frac{(a-1)^2 + (b-1)^2}{a^2 + b^2}$$

where $a = ec(u)$ and $b = ec(v)$. If $X \leq f(a, b) \leq Y$, then we can obtain

$$\sqrt{X}\sqrt{ec^2(u)+ec^2(v)} \leq \sqrt{(ec(u)-1)^2+(ec(v)-1)^2} \leq \sqrt{Y}\sqrt{ec^2(u)+ec^2(v)}$$

$$\begin{aligned}\sqrt{X}\sum_{uv\in E_G}\sqrt{ec^2(u)+ec^2(v)} &\leq \sum_{uv\in E_G}\sqrt{(ec(u)-1)^2+(ec(v)-1)^2}\\ &\leq \sqrt{Y}\sum_{uv\in E_G}\sqrt{ec^2(u)+ec^2(v)}.\end{aligned}$$

So we have

$$\sqrt{X}SO_{ec}(\mathcal{G}) \leq SO_{red-ec}(\mathcal{G}) \leq \sqrt{Y}SO_{ec}(\mathcal{G}).$$

Now we should find the maximum and minimum values of the $f(a,b)$.

One can easily see that

$$f(a,a) = \frac{(a-1)^2}{a^2} \geq \frac{(r_{\mathcal{G}}-1)^2}{r_{\mathcal{G}}^2} = f(r_{\mathcal{G}}, r_{\mathcal{G}})$$

when $r_{\mathcal{G}} \leq a = b \leq D_{\mathcal{G}}$.

Since $D_{\mathcal{G}} \geq r_{\mathcal{G}}$, we have

$$\begin{aligned}\frac{(D_{\mathcal{G}}-1)^2}{D_{\mathcal{G}}^2} &\geq \frac{(r_{\mathcal{G}}-1)^2}{r_G^2}\\ (D_{\mathcal{G}}-1)^2\left(D_{\mathcal{G}}^2+r_{\mathcal{G}}^2\right) &\geq D_{\mathcal{G}}^2\left((D_{\mathcal{G}}-1)^2+(r_{\mathcal{G}}-1)^2\right)\\ \frac{(D_{\mathcal{G}}-1)^2}{D_{\mathcal{G}}^2} &\geq \frac{(D_{\mathcal{G}}-1)^2+(r_{\mathcal{G}}-1)^2}{D_{\mathcal{G}}^2+r_{\mathcal{G}}^2}.\end{aligned}$$

Thus,

$$f(a,b) \leq \frac{(D_{\mathcal{G}}-1)^2}{D_{\mathcal{G}}^2}$$

for all $a, b \in [r_{\mathcal{G}}, D_{\mathcal{G}}]$. Hereby,

$$\frac{(r_{\mathcal{G}}-1)^2}{r_{\mathcal{G}}^2} \leq \frac{(a-1)^2+(b-1)^2}{a^2+b^2} \leq \frac{(D_{\mathcal{G}}-1)^2}{D_{\mathcal{G}}^2}.$$

Consequently for all $uv \in E_{\mathcal{G}}$ we have

$$\begin{aligned}\frac{r_{\mathcal{G}}-1}{r_{\mathcal{G}}}\sum_{uv\in E_{\mathcal{G}}}\sqrt{ec^2(u)+ec^2(v)} &\leq \sum_{uv\in E_{\mathcal{G}}}\sqrt{(ec(u)-1)^2+(ec(v)-1)^2} \leq \sum_{uv\in E_{\mathcal{G}}}\frac{D_{\mathcal{G}}-1}{D_{\mathcal{G}}}\\ \frac{r_{\mathcal{G}}-1}{r_{\mathcal{G}}}SO_{ec}(\mathcal{G}) &\leq SO_{red-ec}(\mathcal{G}) \leq \frac{D_{\mathcal{G}}-1}{D_{\mathcal{G}}}SO_{ec}(\mathcal{G}).\end{aligned}$$

□

4. Conclusion

In this paper, we have explored various bounds for the structural invariant $SO_{ec}(\mathcal{G})$ of a graph $\mathcal{G}$, leveraging different graph indices such as vertex degrees, radius, diameter, and average degree. These bounds provide valuable insights into the interplay between the structural properties of a graph and its topological indices.

Relationships between the eccentricity Sombor index, reduced eccentricity Sombor index, and eccentric connectivity index were found.

Future work may focus on extending the results to specific classes of graphs, such as planar graphs, trees, or bipartite graphs and investigating the applications of $SO_{ec}(\mathcal{G})$ in real-world networks, such as social, biological, and communication networks.

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Estimation of a finite population mean under random non-response using improved Nadaraya-Watson kernel weights

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ABSTRACT. Non-response is a potential source of errors in sample surveys. It introduces bias and large variance in the estimation of finite population parameters. Regression models have been recognized as one of the techniques of reducing bias and variance due to random non-response using auxiliary data. In this study, it is assumed that random non-response occurs in the survey variable in the second stage of cluster sampling assuming full auxiliary information is available throughout. Auxiliary information is used at the estimation stage via a regression model to address the problem of random non-response. In particular, auxiliary information is used via an improved Nadaraya-Watson kernel regression technique to compensate for random non-response. The asymptotic bias and mean squared error of the estimator proposed are derived. Besides, a simulation study conducted indicates that the proposed estimator has smaller values of the bias and smaller mean squared error values compared to existing estimators of finite population mean. The proposed estimator is also shown to have tighter confidence interval lengths at 95% coverage rate. The results obtained in this study are useful for instance in choosing efficient estimators of finite population mean in demographic sample surveys.

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1. INTRODUCTION

Many authors such as [10], [11] and [12] among others have looked at estimation of a finite population mean in the presence of non-response using various assumptions. However, the estimators developed in these studies need improvements on the efficiency and the bias. In the sequence of improving estimation of finite population mean in the presence of random non-response, an improved Nadaraya- Watson kernel regression estimator is proposed in this study. The improved Nadaraya-Watson kernel regression technique was first fronted by [4]. To compensate for random non-response, auxiliary information is used in this study via an improved Nadaraya-Watson kernel regression technique due to [4].

2. MAIN RESULTS

2.1. **The Proposed Estimator of Finite Population Mean.** An improvement of the Nadaraya-Watson estimator [14, 8] has been proposed by [4] using local bandwidth factor λ_{ij} determined using [6] algorithm. The improved Nadaraya-Watson estimator is given by

$$m_{IMW}(\hat{x}) = \frac{\sum_{i\in s}\sum_{j\in s}\frac{Y_{ij}}{\lambda_{ij}}K\left(\frac{x-X_{ij}}{\lambda_{ij}b}\right)}{\sum_{i\in s}\sum_{j\in s}\frac{1}{\lambda_{ij}}K\left(\frac{x-X_{ij}}{\lambda_{ij}b}\right)}. \quad (1)$$

where b is a smoothing parameter while the local bandwidth, λ_{ij}, is given by

$$\lambda_{ij} = \left\{m(\hat{X}_{ij})/a\right\}^{-\alpha}. \quad (2)$$

where a is an arithmetic mean given by $a = \sum_{i=1}^{n}\sum_{j=1}^{m} m(\hat{X}_{ij})/mn$ while α is a sensitivity parameter which satisfies $0 \le \alpha \le 1$. It has been suggested by [6] that taking $\alpha = \frac{1}{2}$ produce good results. Consider a finite population of size N consisting of M clusters with N_j elements in the j^{th} cluster. A sample of m clusters is selected so that n_{1i} units respond and n_{2i} units fail to respond. Let y_{ij} denote the value of the survey variable y for unit j in cluster i, for $i = 1, 2, \cdots, N, j = 1, 2, \cdots, N_i$ and let population mean be given by

$$\overline{\overline{Y}} = \frac{1}{N}\sum_{i=1}^{M}\sum_{j=1}^{N_i} Y_{ij}\,. \tag{3}$$

The proposed estimator is given by

$$\hat{\overline{\overline{Y}}}_{INW} = \frac{1}{M}\Big\{\frac{1}{n_{1i}}\sum_{i\in s}\sum_{j\in s} y_{ij} + \frac{1}{n_{2i}}\sum_{i\in s}\sum_{j\notin s}\hat{y}_{ij}\Big\}. \tag{4}$$

where $\hat{y}_{ij}$ is an estimator of the non-response component of the sample. Assuming auxiliary information, X_{ij}, is known throughout, $\hat{y}_{ij}$ can be obtained using the improved Nadaraya-Watson regression technique by

$$\hat{y}_{ij} = m_{INW}(\hat{x}_{ij}) = \frac{\sum_{i\in s}\sum_{j\in s}\frac{1}{\lambda_{ij}}K\Big(\frac{x-X_{ij}}{\lambda_{ij}b}\Big)Y_{ij}}{\sum_{i\in s}\sum_{j\in s}\frac{1}{\lambda_{ij}}K\Big(\frac{x-X_{ij}}{\lambda_{ij}b}\Big)} \tag{5}$$

so that the estimator of finite population mean can be re-written as

$$\hat{\overline{\overline{Y}}}_{INW} = \frac{1}{M}\Big\{\frac{1}{n_{1i}}\sum_{i\in s}\sum_{j\in s} y_{ij} + \frac{1}{n_{2i}}\sum_{i\in s}\sum_{j\notin s} m_{INW}(\hat{x}_{ij})\Big\}. \tag{6}$$

A special case where $n_{1i} = n_{2i} = n$ is assumed in this study. This simplifies mathematical computations so that equation (7) can be re-written as

$$\hat{\overline{\overline{Y}}}_{INW} = \frac{1}{Mn}\Big\{\sum_{i\in s}\sum_{j\in s} y_{ij} + \sum_{i\in s}\sum_{j\notin s} m_{INW}(\hat{x}_{ij})\Big\}. \tag{7}$$

where $m_{INW}(\hat{x}_{ij})$ is the improved Nadaraya-Watson kernel regression estimator given in (1), which is a weighted sum of the values of the survey variable Y_{ij}'s. Data is generated using a regression model given by

$$\hat{Y}_{ij} = m(\hat{x}_{ij}) + \hat{e}_{ij} \tag{8}$$

where $m(.)$ is an unknown smooth function of auxiliary random variables, X_{ij}. It is assumed that the error term, $\hat{e}_{ij}$, satisfies the following conditions:

$$E(\hat{e}_{ij}) = 0, Var(\hat{e}_{ij}) = \sigma^2_{ij}, Cov(\hat{e}_i, \hat{e}_j) = 0, for \quad i \neq j \tag{9}$$

Hence the unspecified function of the auxiliary random variables, $m(\hat{x}_{ij})$, is replaced by the improved Nadaraya-Watson kernel estimator, $m_{INW}(\hat{x}_{ij})$. The estimator can be re-written as

$$m_{INW}(\hat{x}_{ij}) = \sum_{i\in s}\sum_{j\in s} w(x_{ij})Y_{ij}. \tag{10}$$

where $w(x_{ij}) = \dfrac{\frac{1}{\lambda_{ij}}K\Big(\frac{x-X_{ij}}{\lambda_{ij}b}\Big)}{\sum_{i\in s}\sum_{j\in s}\frac{1}{\lambda_{ij}}K\Big(\frac{x-X_{ij}}{\lambda_{ij}b}\Big)}$ are the improved Nadaraya-Watson kernel weights where $K(.)$ is a given kernel function assumed to be symmetrical. Since the choice of the kernel function is not critical for the performance of the kernel regression estimator, a simplified Gaussian kernel with mean 0 and variance 1 is used in this study. This is given by

$$K(w) = \frac{1}{\sqrt{2\pi}}e^{-\left(\frac{w^2}{2}\right)} = \frac{1}{\sqrt{2\pi}}e^{-\left(\frac{\left(\frac{x-X_{ij}}{\lambda_{ij}b}\right)^2}{2}\right)} \tag{11}$$

In this case, the improved Nadaraya-Watson kernel estimation at any point x_{ij} is given by

$$\hat{y}_{ij} = m_{INW}(\hat{x}_{ij}) = \frac{\sum_i\sum_j \frac{1}{\sqrt{2\pi}}e^{-\left(\frac{\left(\frac{x-X_{ij}}{\lambda_{ij}b}\right)^2}{2}\right)}Y_{ij}}{\sum_i\sum_j \frac{1}{\sqrt{2\pi}}e^{-\left(\frac{\left(\frac{x-X_{ij}}{\lambda_{ij}b}\right)^2}{2}\right)}} \tag{12}$$

where b is the bandwidth while λ_{ij} is given in equation (2) due to [4].
This provides a way of estimating the non-response values of the survey variable Y_{ij}, in the i^{th} cluster given the auxiliary values x_{ij}, for a specified kernel function.

2.2. **Asymptotic Bias of the of the Proposed Estimator.** The expected value of the estimator may be given as

$$E(\hat{\bar{\bar{Y}}}_{INW}) = \frac{1}{Mn}\Big\{\sum_{i=1}^{n}\sum_{j=1}^{m} Y_{ij} + \sum_{i=n+1}^{N}\sum_{j=m+1}^{M} m_{INW}\left(\hat{x}_{ij}\right)\Big\} \tag{13}$$

Re-writing equation (5) using the property of symmetry associated with Nadaraya-Watson estimator,

$$m_{INW}\left(\hat{x}_{ij}\right) = \frac{\sum_{i\in s}\sum_{j\in s} K\left(\frac{X_{ij}-x_{ij}}{\lambda_{ij}b}\right) Y_{ij}}{\sum_{i\in s}\sum_{j\in s} K\left(\frac{X_{ij}-x_{ij}}{\lambda_{ij}b}\right)}, i = 1, 2, \cdots, n; j = 1, 2, \cdots, m \tag{14}$$

Following the procedure by [13], equation (14) can be re-written as

$$m_{INW}\left(\hat{x}_{ij}\right) = \frac{1}{g(\hat{x}_{ij})}\left[\frac{1}{mn(\lambda_{ij}b)}\sum_{i}\sum_{j} K\left(\frac{X_{ij}-x_{ij}}{\lambda_{ij}b}\right) Y_{ij}\right] \tag{15}$$

where $g(\hat{x}_{ij})$ is the estimated marginal density of auxiliary variables X_{ij}. The bias of the estimator can be written as

$$Bias\,(\hat{\bar{\bar{Y}}}_{INW}) = E(\hat{\bar{\bar{Y}}}_{INW} - \bar{\bar{Y}}) \tag{16}$$

$$\begin{aligned} Bias\,(\hat{\bar{\bar{Y}}}_{INW}) = E\Big\{&\frac{1}{Mn}\Big[\sum_{i=1}^{n}\sum_{j=1}^{m} Y_{ij} + \sum_{i=n+1}^{N}\sum_{j=m+1}^{M} m_{INW}(\hat{x}_{ij})\Big] \\ &- \frac{1}{Mn}\Big[\sum_{i=1}^{n}\sum_{j=1}^{m} Y_{ij} + \sum_{i=n+1}^{N}\sum_{j=m+1}^{M} Y_{ij}\Big]\Big\} \end{aligned} \tag{17}$$

which reduces to

$$Bias\,(\hat{\bar{\bar{Y}}}_{INW}) = \frac{1}{Mn}E\left\{\sum_{i=n+1}^{N}\sum_{j=m+1}^{M} m_{INW}(\hat{x}_{ij}) - \sum_{i=n+1}^{N}\sum_{j=m+1}^{M} Y_{ij}\right\} \tag{18}$$

Re-writing the regression model given by $Y_{ij} = m(X_{ij}) + e_{ij}$ as

$$Y_{ij} = m(x_{ij}) + \big[m(X_{ij}) - m(x_{ij})\big] + e_{ij} \tag{19}$$

and substituting it in equation (15) gives

$$\begin{aligned} m_{INW}\left(\hat{x}_{ij}\right) = \frac{1}{g(\hat{x}_{ij})}\Big[\frac{1}{mn(\lambda_{ij}b)}\sum_{i}\sum_{j} K\Big(\frac{X_{ij}-x_{ij}}{\lambda_{ij}b}\Big)\Big(m(x_{ij}) + \big[m(X_{ij}) - m(x_{ij})\big] \\ + e_{ij}\Big)\Big] \end{aligned} \tag{20}$$

Hence the first term in equation (18) before taking expectation is given as:

$$\begin{aligned} &\frac{1}{Mn}\left\{\frac{\frac{1}{mnb}\sum_{i=n+1}^{N}\sum_{j=m+1}^{M} K\left(\frac{X_{ij}-x_{ij}}{\lambda_{ij}b}\right) Y_{ij}}{g(\hat{x}_{ij})}\right\} \\ &\quad= \frac{1}{Mng(\hat{x}_{ij})}\left\{\frac{1}{mn(\lambda_{ij}b)}\sum_{i=n+1}^{N}\sum_{j=m+1}^{M} K\left(\frac{X_{ij}-x_{ij}}{\lambda_{ij}b}\right) m(x_{ij})\right. \\ &\quad+ \frac{1}{mn(\lambda_{ij}b)}\sum_{i=n+1}^{N}\sum_{j=m+1}^{M} K\left(\frac{X_{ij}-x_{ij}}{\lambda_{ij}b}\right)\big[m(X_{ij}) - m(x_{ij})\big] \\ &\quad\left.+ \frac{1}{mn(\lambda_{ij}b)}\sum_{i=n+1}^{N}\sum_{j=m+1}^{M} K\left(\frac{X_{ij}-x_{ij}}{\lambda_{ij}b}\right) e_{ij}\right\} \end{aligned} \tag{21}$$

Simplifying equation (21), the following is obtained:

$$\frac{1}{Mn}\left\{\frac{1}{mn(\lambda_{ij}b)\,g(\hat{x}_{ij})}\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}K\left(\frac{X_{ij}-x_{ij}}{\lambda_{ij}b}\right)Y_{ij}\right\}$$
$$(22)\qquad =\frac{1}{Mn}\left(\frac{1}{mn(\lambda_{ij}b)\,g(\hat{x}_{ij})}\right)\left\{\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}g(\hat{x}_{ij})m(x_{ij})\right.$$
$$\left.+m_1(\hat{x}_{ij})+m_2(\hat{x}_{ij})\right\}$$

where

$$(23)\qquad m_1(\hat{x}_{ij})=\frac{1}{mn(\lambda_{ij}b)}\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}K\left(\frac{X_{ij}-x_{ij}}{\lambda_{ij}b}\right)\Big[m(X_{ij})-m(x_{ij})\Big]$$

$$(24)\qquad m_2(\hat{x}_{ij})=\frac{1}{mn(\lambda_{ij}b)}\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}K\left(\frac{X_{ij}-x_{ij}}{\lambda_{ij}b}\right)e_{ij}$$

Taking conditional expectation of equation (22) leads to

$$(25)\qquad E\Big[\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}m_{INW}(\hat{x}_{ij})/x_{ij}\Big]=\frac{1}{Mn}E\left[\frac{1}{mn(\lambda_{ij}b)}\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}\left[m(x_{ij})+\frac{m_1(\hat{x}_{ij})}{g(\hat{x}_{ij})}+\frac{m_2(\hat{x}_{ij})}{g(\hat{x}_{ij})}\right]\right]$$

The following theorem due to [5] and applied by [9] was used in obtaining asymptotic bias and variance of the estimator using conditional expectations.

Theorem 2.1. *Let $K(w)$ be a symmetric density function with $\int wk(w)dw=0$ and $\int w^2k(w)dw=k_2$. Assume n and N increase together such that $\frac{n}{N}\to\pi$ with $0<\pi<1$. Besides, assume the sampled and non-sampled values of x are in the interval $[c,d]$ and are obtained by densities d_s and d_{p-s} respectively where both are bounded away from zero on $[c,d]$ with continuous second derivatives. If for any variable $Z, E(Z/U=u)=A(u)+O(B)$ and $Var(Z/U=u)=O(C)$, then $Z=A(u)+O_p(B+C^{\frac{1}{2}})$.*

Using this theorem the asymptotic bias and variance are derived. The conditional asymptotic bias is therefore given by

$$(26)\qquad Bias\,(\hat{\bar{\bar{Y}}}_{INW}/x_{ij})=\frac{1}{Mn}\left\{(\frac{Mn-mn}{mn})(\lambda_{ij}b)^2d_kC(x)+o((\lambda_{ij}b)^2)\right\}$$

where $C(x)=[g(\hat{x}_{ij})]^{-1}\left[\frac{1}{2}m''(x_{ij})g(x_{ij})+g'(x_{ij})m'(x_{ij})\right]$ and $d_k=\int w^2k(w)dw$.
The conditional asymptotic variance is also given by

$$(27)\qquad Var(\hat{\bar{\bar{Y}}}_{INW}/x_{ij})=\frac{1}{(Mn)^2}\left\{\frac{(Mn-mn)^2H(w)\sigma^2_{x_{ij}}}{mn(\lambda_{ij}b)g(\hat{x}_{ij})}+o\left[\frac{(Mn-mn)^2}{mn(\lambda_{ij}b)}+\frac{1}{mn(\lambda_{ij}b)}\right]\right\}$$

where $H(w)=\int K(w)^2dw$.
The proofs of the asymptotic bias and variance are provided in the appendix.

2.3. **Mean Squared Error of the Estimator.** The conditional MSE of the estimator of finite population mean combines the conditional squared bias and the conditional variance of the estimator, that is,

$$(28)\qquad MSE(\hat{\bar{\bar{Y}}}_{INW})=Var(\hat{\bar{\bar{Y}}}_{INW})+Bias^2(\hat{\bar{\bar{Y}}}_{INW})$$

Which on simplification leads to

$$MSE(\hat{\bar{\bar{Y}}}/X_{ij}=x_{ij}) = \frac{1}{(Mn)^2}\Bigg\{\frac{(Mn-mn)^2 H(w)\sigma^2_{x_{ij}}}{mn(\lambda_{ij}b)g(\hat{x}_{ij})} + \Bigg[\frac{(Mn-mn)^2}{4(mn)^2(Mn^2)}(\lambda_{ij}b)^2 d_k^2\Big[m''(x_{ij})g(x_{ij}) + \frac{2g'(x_{ij})m'(x_{ij})}{g(\hat{x}_{ij})}\Big]^2 + o\Big(\frac{1}{Mn}\Big\{\frac{(Mn-mn)^2}{mn(\lambda_{ij}b)} + \frac{1}{mn(\lambda_{ij}b)}\Big\}\Big)\Bigg]\Bigg\} \tag{29}$$

where $H(w)=\int K(w)^2dw$, $d_k=\int w^2K(w)dw$.
From equation (62), it is noted that if the sample size is large, that is as $n\to N$ and $m\to M$, the MSE of $\hat{\bar{\bar{Y}}}_{INW}$ due to the kernel tends to zero for a sufficiently small bandwidth. The estimator $\hat{\bar{\bar{Y}}}$ is therefore asymptotically consistent since its MSE converges to zero in probability.

3. Simulation Study

A simulation experiment was conducted using R code in order to compare the performance of the proposed estimator in two-stage cluster sampling with the transformed estimator due to [2] and the non-parametric regression estimator due to [3]. An asymptotic framework is used where both the population number of clusters and the sample number of clusters are large. The number of clusters within each cluster, N_i, is held constant so that no cluster dominates the population.
Both linear and non-linear mean functions of auxiliary random variables due to [3] were considered in generating data, where $x\in(0,1)$. The mean functions are given in table 1 below.

Table 1. Equations of Mean Functions Simulated

Mean function	Equation
Linear	$m_1(\hat{x}) = 1+2(x-0.5)$
Quadratic	$m_2(\hat{x}) = 1+2(x-0.5)^2$
Sine	$m_3(\hat{x}) = 2+\sin(2\pi x)$
Exponential	$m_4(\hat{x}) = exp(-8x)$
Bump	$m_5(\hat{x}) = 1+2(x-0.5)+exp\big\{-200(x-0.5)^2\big\}$
Jump	$m_6(\hat{x}) = 1+2(x-0.5)I_{x\leq 0.65}+0.65I_{x\geq 0.65}$

The mean functions in table 1 are useful in statistical data simulations as noted by [3].
The population auxiliary values, x_{ij}, of size $M=2000$ are generated as identical and independently distributed uniform $(0,1)$ random variables. The survey values are only known for the respondents in the selected sample. Using the auxiliary values, the non-response values are generated, that is, for every generated value $x_{ij}, i=1,2,\cdots,M; j=1,2,\cdots,N_i$, the mean survey non-response values are generated as

$$\hat{y}_{ij}=\frac{1}{M}\Big\{\frac{m(\hat{x}_{ij})}{N_i}+\frac{\hat{e}_{ij}}{N_i}\Big\} \tag{30}$$

where $\hat{e}_{ij}$ are identically and independently distributed normal random variables with mean zero and variance one. Besides, a Gaussian kernel with mean zero and variance one was used. A Gaussian kernel was used since it has smooth and continuous derivatives at every data point. Besides, an optimal bandwidth generated using cross-validation technique due to [7] was used. It has been noted by [7] that this bandwidth would lead to more informative estimates compared to other choices. The local bandwidth, λ_{ij}, given in equation (2) were generated using the algorithm due to [6].

At stage one, a sample of clusters is generated first by simple random sampling using a sample of size $m=200$. At stage two, sub-samples of elements within every selected cluster are generated by simple random sampling with replacement using a random sample of size n_i. The non-response mean survey values were then generated using equation (30). The estimates of finite population mean were then computed using the estimator in equation (7). The values of bias and mean squared error values were also computed. The 95% confidence intervals were then constructed for the estimators of the finite population means for comparative purposes.

3.1. **Simulation Results.** The values of the bias, mean squared error and confidence interval lengths are given below in tables 1, 2 and 3 respectively. Note that $\hat{\hat{Y}}_{INW}$ is the estimator of finite population mean proposed in this study, $\hat{\hat{Y}}_{TDM}$ is the transformation of data method estimator of finite population mean due to [2] whereas $\hat{\hat{Y}}_{REG}$ is the non-parametric regression estimator due to [3]. Both $\hat{\hat{Y}}_{TDM}$ and $\hat{\hat{Y}}_{REG}$ were used for comparative purposes with the proposed estimator.

TABLE 2. Summary Results of Bias

Estimators	$\hat{\hat{Y}}_{INW}$	$\hat{\hat{Y}}_{TDM}$	$\hat{\hat{Y}}_{REG}$
Linear	-0.00213	-0.1667	-0.3312
Quadratic	-0.0132	0.04171	-0.0966
Sine	-0.0521	-0.6416	-1.2311
Exponential	-0.0041	0.3592	0.7225
Bump	-0.0032	-0.2358	-0.4685
Jump	-0.0188	-0.2466	-0.4743

Negative values of the bias imply underestimation while positive values of the bias indicate overestimation of the finite population mean by the different estimators. As noted in table 2, the proposed estimator has relatively smaller values of the bias followed by transformation of data method estimator due to [2]. The non-parametric-based estimator due to [3] has larger values compared to the other two estimators. It is also observed that the three estimators have relatively closer values of the bias in the quadratic mean function though the transformation of data method has positive bias at this mean function. Generally, among the three estimators of finite population mean, the proposed estimator using improved Nadaraya-Watson kernel regression technique performs better than the other two estimators in terms of bias.

TABLE 3. Summary Results of MSE Values

Estimators	$\hat{\hat{Y}}_{INW}$	$\hat{\hat{Y}}_{TDM}$	$\hat{\hat{Y}}_{REG}$
Linear	0.0334	0.1097	0.1321
Quadratic	0.0093	0.1455	0.5835
Sine	0.4215	1.5157	1.555
Exponential	0.3430	0.5220	1.3780
Bump	0.0634	0.2195	0.2508
Jump	0.2250	0.2951	1.1611

Mean squared error combines both the variance and the squared bias terms of an estimator. The mean squared error values presented in table 3 were simulated using the different mean functions indicated. The quadratic mean function gives the smallest value of the mean squared error of the proposed estimator followed by the linear function. The estimator due to [3] has the largest value of the mean squared error in the jump function. Generally, it is noted from table 3 that the mean squared error values for the proposed estimator are relatively smaller than the rest of the estimators considered. The transformation of data method estimator due to [2] follows closely in the second place with smaller mean squared error values compared to non-parametric regression-based estimator due to [3]. From this comparison of the mean squared error values, it can be concluded that the proposed estimator is more efficient than the other two estimators considered. It has got smaller MSE values in all the mean functions and thus outperforms the others in terms of efficiency.

Table 4. Summary Results of 95% CI Lengths

Estimators	$\hat{\hat{Y}}_{INW}$	$\hat{\hat{Y}}_{TDM}$	$\hat{\hat{Y}}_{REG}$
Linear	0.7164	1.2984	1.4249
Quadratic	0.8270	1.4951	2.994
Sine	1.5269	2.5451	4.888
Exponential	2.2958	2.8322	4.6016
Bump	0.9872	1.8365	1.9630
Jump	1.8594	2.1297	4.2239

The 95% upper and lower confidence intervals were constructed for the estimators of finite population mean. Confidence interval lengths were then obtained. The results are given in table 4. From the values obtained, it is noted that the confidence interval lengths for the proposed estimator are much tighter than those of the estimators due to [3] and [2]. Hence, at 95% level of confidence, the estimator proposed in this study performs better than its rival estimators.

4. Conclusion

This study has developed an estimator of finite population mean in two-stage cluster sampling assuming random non-response occurs in the survey variable in the second stage of cluster sampling. Complete auxiliary information is assumed to be available in both stage one and stage two of cluster sampling. Kernel weights developed using improved Nadaraya-Watson regression technique were used in the estimation process. The theoretical properties of the proposed estimator such as asymptotic bias, variance and mean squared error were derived. Simulation results show that the proposed estimator has smaller values of the bias, smaller mean squared error values and tighter confidence interval lengths compared to the other estimators. Therefore, the estimator of finite population mean proposed in this study dominates the estimators due to [3] and [2] respectively.

Data Availability

Simulation data generated using R statistical package were used to support the theoretical findings.

Conflict of Interest

The authors declare that they have no conflicts of interest.

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Appendix: Proofs of Asymptotic Bias and Variance

Proofs of Asymptotic Bias of the Estimator

From the conditions of the error term stated in (9), it follows that $E(e_{ij}/X_{ij}) = 0$. Therefore, $E\big[m_2(\hat{x}_{ij}) = 0\big]$. Thus, $E\big[m_1(\hat{x}_{ij})\big]$ can be obtained as follows:

$$E\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}\Big[m_1(\hat{x}_{ij})\Big] = \frac{1}{Mn}\big(\frac{1}{mn(\lambda_{ij}b)}\big)E\Big\{\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}K\Big(\frac{X_{ij}-x_{ij}}{\lambda_{ij}b}\Big) \times\Big[m(X_{ij})-m(x_{ij})\Big]\Big\} \tag{31}$$

Using substitution and change of variable technique given by

$$\left.\begin{array}{c} w=\frac{V-x_{ij}}{\lambda_{ij}b} \\ V = x_{ij}+(\lambda_{ij}b)w \\ dV=(\lambda_{ij}b)dw \end{array}\right\} \tag{32}$$

Equation (31) can be simplified to:

$$E\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}\Big[m_1(\hat{x}_{ij})\Big] = \frac{1}{Mn}\Big\{\frac{Mn-mn}{mn}\int k(w)\Big[m(x_{ij}+(\lambda_{ij}b)w) - m(x_{ij})\Big]\int g\Big(x_{ij}+(\lambda_{ij}b)w\Big)dw\Big\} \tag{33}$$

Using Taylor's series expansion about the point x_{ij}, the k^{th} order kernel can be derived as follows:

$$g\big(x_{ij}+(\lambda_{ij}b)w\big) = g(x_{ij}) + g\prime(x_{ij})(\lambda_{ij}b)w + \frac{1}{2}g\prime\prime(x_{ij})(\lambda_{ij}b)^2w^2+\cdots + \frac{1}{k!}g^k(x_{ij})(\lambda_{ij}b)^kw^k + o((\lambda_{ij}b)^2) \tag{34}$$

Similarly,

$$m\big(x_{ij} + (\lambda_{ij}b)w\big) = m(x_{ij}) + m\prime(x_{ij})(\lambda_{ij}b)w + \frac{1}{2}m\prime\prime(x_{ij})(\lambda_{ij}b)^2w^2 + \cdots + \frac{1}{k!}m^k(x_{ij})(\lambda_{ij}b)^kw^k + o((\lambda_{ij}b)^2) \tag{35}$$

Therefore, expanding equation (33) up to order $o((\lambda_{ij}b)^2)$ and simplifying gives

$$\begin{aligned} E\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}\big[m_1(\hat{x}_{ij})\big] = \frac{1}{Mn}\Big\{ &(\frac{Mn-mn}{mn})g(x_{ij})m\prime(x_{ij})(\lambda_{ij}b)\int wk(w)dw \\ &+ (\frac{Mn-mn}{mn})g\prime(x_{ij})m\prime(x_{ij})(\lambda_{ij}b)^2\int w^2k(w)dw \\ &+ (\frac{Mn-mn}{mn})\frac{1}{2}g(x_{ij})m\prime\prime(x_{ij})(\lambda_{ij}b)^2 \\ &\times\int w^2k(w)dw + o((\lambda_{ij}b)^2)\Big\} \end{aligned} \tag{36}$$

Using the conditions due to [5] given by $\int_{-\infty}^{\infty}k(w)dw = 1$, $\int_{-\infty}^{\infty}wk(w)dw = 0$ and $\int_{-\infty}^{\infty}w^2k(w)dw = d_k$, the derivation in equation (36) can further be simplified to obtain:

$$\begin{aligned} E\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}\Big[m_1(\hat{x}_{ij})\Big] = \frac{1}{Mn}(\frac{Mn-mn}{mn})\Big[&g\prime(x_{ij})m\prime(x_{ij}) \\ &+ \frac{1}{2}g(x_{ij})m\prime\prime(x_{ij})\Big](\lambda_{ij}b)^2d_k + o((\lambda_{ij}b)^2) \end{aligned} \tag{37}$$

Hence the expected value of the second term in equation (25) then becomes:

$$\begin{aligned} E\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}\big[m_1(\hat{x}_{ij})\big] = \frac{1}{Mn}\Big\{(\frac{Mn-mn}{mn})\Big[&\frac{1}{2g(\hat{x}_{ij})}m\prime\prime(x_{ij})g(x_{ij}) \\ &+ \frac{g\prime(x_{ij})m\prime(x_{ij})}{g(\hat{x}_{ij})}\Big](\lambda_{ij}b)^2d_k + o((\lambda_{ij}b)^2)\Big\} \end{aligned} \tag{38}$$

Simplifying equation (38) gives:

$$E\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}\big[m_1(\hat{x}_{ij})\big] = \frac{1}{Mn}\Big\{(\frac{Mn-mn}{mn})(\lambda_{ij}b)^2d_kC(x) + o((\lambda_{ij}b)^2)\Big\} \tag{39}$$

where $C(x) = [g(\hat{x}_{ij})]^{-1}\Big[\frac{1}{2}m\prime\prime(x_{ij})g(x_{ij}) + g\prime(x_{ij})m\prime(x_{ij})\Big]$ and $d_k = \int w^2k(w)dw$.
Using equation of the bias given in (16) and the conditional expectation in equation (25), the following equation for the conditional bias of the estimator was obtained:

$$Bias\,(\hat{\bar{\bar{Y}}}_{INW}/x_{ij}) = \frac{1}{Mn}\Big\{(\frac{Mn-mn}{mn})(\lambda_{ij}b)^2d_kC(x) + o((\lambda_{ij}b)^2)\Big\} \tag{40}$$

This completes the proof of the bias. Next, the asymptotic variance is also proved.

Proofs of Asymptotic Variance of the Estimator

Using equation (7), the conditional variance of the estimator is given as

$$Var\big(\hat{\bar{\bar{Y}}}_{INW}/x_{ij}\big) = Var\Big\{\frac{1}{Mn}\Big\{\sum_{i=1}^{n}\sum_{j=1}^{m}Y_{ij} + \sum_{i=n+1}^{N}\sum_{j=m+1}^{M}m_{INW}\,(\hat{x}_{ij})\Big\}\Big\} \tag{41}$$

$$= \Big(\frac{1}{Mn}\Big)^2Var\Big\{\sum_{i=1}^{n}\sum_{j=1}^{m}Y_{ij} + \sum_{i=n+1}^{N}\sum_{j=m+1}^{M}m_{INW}\,(\hat{x}_{ij})\Big\} \tag{42}$$

where $m_{INW}\,(\hat{x}_{ij})$ is given by

$$m_{INW}\Big(\hat{x}_{ij}\Big) = \frac{1}{g(\hat{x}_{ij})}\Big[\frac{1}{mn(\lambda_{ij}b)}\sum_{i}\sum_{j}K\Big(\frac{X_{ij}-x_{ij}}{\lambda_{ij}b}\Big)Y_{ij}\Big] \tag{43}$$

where $g(\hat{x}_{ij}) = \frac{1}{mn(\lambda_{ij}b)}\sum_i\sum_j K\left(\frac{x-X_{ij}}{\lambda_{ij}b}\right)$ is the estimated marginal density of auxiliary variables X_{ij}, for details see [8, 14]. Re-writing the regression model $Y_{ij} = m(X_{ij}) + e_{ij}$ as $Y_{ij} = m(x_{ij}) + \Big[m(X_{ij}) - m(x_{ij})\Big] + e_{ij}$ and substituting in equation (43) leads to

(44)

$$Var\big(\hat{\bar{\bar{Y}}}_{INW}/x_{ij}\big) = \Big(\frac{1}{Mn}\Big)^2 Var\Bigg\{\sum_{i=1}^{n}\sum_{j=1}^{m}Y_{ij} + \Bigg(\frac{1}{mn(\lambda_{ij}b)\,g(\hat{x}_{ij})}\Bigg)\Bigg\{\sum_{i=n+1}^{N}\sum_{j=m+1}^{M} g(\hat{x}_{ij})m(x_{ij}) + m_1(\hat{x}_{ij}) + m_2(\hat{x}_{ij})\Bigg\}\Bigg\}$$

From equation (24),

$$m_2(\hat{x}_{ij}) = \frac{1}{mnb}\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}K\big(\frac{X_{ij}-x_{ij}}{\lambda_{ij}b}\big)e_{ij} \tag{45}$$

Hence

$$Var\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}\big[m_2(\hat{x}_{ij})\big] = \frac{1}{(Mn)^2}\bigg(\frac{Mn-mn}{mn(\lambda_{ij}b)}\bigg)^2\sum_{i=1}^{n}\sum_{j=1}^{m}Var(D_x) \tag{46}$$

where $D_x = K\big(\frac{X_{ij}-x_{ij}}{\lambda_{ij}b}\big)e_{ij}$. Expressing equation (46) in terms of expectation the following equation is obtained

$$Var\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}\big[m_2(\hat{x}_{ij})\big] = \frac{1}{(Mn)^2}\bigg[\frac{(Mn-mn)^2}{mn(\lambda_{ij}b)^2}\bigg]\Big\{E[D_x]^2 - [E(D_x)]^2\Big\} \tag{47}$$

Using the fact that the conditional expectation $E(e_{ij}/X_{ij}) = 0$, the second term in equation (47) reduces to zero. Therefore,

$$Var\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}\big[m_2(\hat{x}_{ij})\big] = \frac{1}{(Mn)^2}\bigg[\frac{(Mn-mn)^2}{mn(\lambda_{ij}b)^2}\bigg]\sigma_{ij}^2 \tag{48}$$

where $E\big(e_{ij}/X_{ij}\big)^2 = \sigma_{ij}^2$.
Let $X = X_{ij}$, and $x = x_{ij}$ and make the following substitutions

$$\left.\begin{array}{c} w = \frac{X-x}{\lambda_{ij}b} \\ X - x = (\lambda_{ij}b)w \\ dX = (\lambda_{ij}b)dw. \end{array}\right\} \tag{49}$$

so that

$$Var\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}\big[m_2(\hat{x}_{ij})\big] = \frac{(Mn-mn)^2}{mn(\lambda_{ij}b)^2(Mn)^2}\int K\bigg(\frac{X-x}{\lambda_{ij}b}\bigg)^2\sigma_x^2\,g(X)dX \tag{50}$$

Using the change of variables technique and simplifying, equation (50) reduces to

$$Var\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}\big[m_2(\hat{x}_{ij})\big] = \frac{(Mn-mn)^2}{mn(\lambda_{ij}b)(Mn)^2}\int K(w)^2\sigma_x^2\,g(x)dw + o\big(\frac{1}{mn(\lambda_{ij}b)}\big) \tag{51}$$

Following the same procedure for getting the variance of $m_2(\hat{x}_{ij})$,
$Var\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}\big[m_1(\hat{x}_{ij})\big]$ can similarly be obtained as follows:

(52)

$$Var\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}\big[m_1(\hat{x}_{ij})\big] = \frac{1}{(Mn)^2}Var\sum_{i=n+1}^{N}\sum_{j=m+1}^{M}\bigg[\frac{1}{mn(\lambda_{ij}b)}\sum_{i=1}^{n}\sum_{j=1}^{m}K\bigg(\frac{X_{ij}-x_{ij}}{\lambda_{ij}b}\bigg)\bigg] \times\Big[m(X_{ij}) - m(x_{ij})\Big]$$

Equation (52) can be re-written as

$$Var \sum_{i=n+1}^{N} \sum_{j=m+1}^{M} \Big[m_1(\hat{x}_{ij})\Big] = \frac{(Mn-mn)^2}{mnb^2(Mn)^2} Var K\Big(\frac{X_{ij}-x_{ij}}{\lambda_{ij}b}\Big)^2 \Big[m(X_{ij}) - m(x_{ij})\Big]^2 g(X)dX \tag{53}$$

where $X = (\lambda_{ij}b)w + x$ so that $dX = (\lambda_{ij}b)dw$. Changing variables and applying Taylor's series expansion about the point x_{ij} leads to

$$Var \sum_{i=n+1}^{N} \sum_{j=m+1}^{M} \Big[m_1(\hat{x}_{ij})\Big] = \frac{(Mn-mn)^2}{mn(\lambda_{ij}b)^2(Mn)^2} \int K(w^2)\Big[m(x+(\lambda_{ij}b)w) - m(x)\Big]^2 \times g(x+(\lambda_{ij}b)w)dw \tag{54}$$

which gives

$$Var \sum_{i=n+1}^{N} \sum_{j=m+1}^{M} \big[m_1(\hat{x}_{ij})\big] = \frac{(Mn-mn)^2}{mn(\lambda_{ij}b)^2(Mn)^2} \int K(w^2)\Big[m(x) + m\prime(x)(\lambda_{ij}b)w + \cdots - m(x)\Big]^2 \Big(g(x) + g\prime(x)(\lambda_{ij}b)w\Big)dw \tag{55}$$

Following the procedure by [1] and simplifying, equation (55) reduces to

$$Var \sum_{i=n+1}^{N} \sum_{j=m+1}^{M} \big[m_1(\hat{x}_{ij})\big] = o\Big[\frac{(Mn-mn)^2 b^2}{mn(\lambda_{ij}b)}\Big] \tag{56}$$

For large samples, as $n \to N, m \to M$ and $b \to 0$, then $mn(\lambda_{ij}b) \to \infty$. Hence the variance in equation (55) asymptotically tends to zero, i.e, $Var \sum_{i=1}^{N} \sum_{j=1}^{M} \big[m_1(\hat{x}_{ij})\big] \to 0$ so that the variance of the estimator of the population mean reduces to

$$Var\,\big(\hat{\bar{\bar{Y}}}_{INW}/x_{ij}\big) = \frac{(Mn-mn)^2}{mn(\lambda_{ij}b)(Mn)^2} \sum_{i=n+1}^{N} \sum_{j=m+1}^{M} Var\Big[m(x_{ij}) + \frac{m_1(\hat{x}_{ij}) + m_2(\hat{x}_{ij})}{g(\hat{x}_{ij})}\Big] \tag{57}$$

Simplifying equation (57) leads to

$$Var\,\big(\hat{\bar{\bar{Y}}}_{INW}/x_{ij}\big) = \frac{(Mn-mn)^2}{mn(\lambda_{ij}b)(Mn)^2\big[g(\hat{x}_{ij})\big]^2} Var\Big\{\sum_{i=n+1}^{N} \sum_{j=m+1}^{M} \big[m_2(\hat{x}_{ij})\big]\Big\} \tag{58}$$

Substituting equation (51) into (58) yields the following:

$$Var\,\big(\hat{\bar{\bar{Y}}}_{INW}/x_{ij}\big) = \frac{1}{(Mn)^2}\Big\{\frac{(Mn-mn)^2 \int K(w)^2 \sigma^2_{x_{ij}} dw}{mn(\lambda_{ij}b)g(\hat{x}_{ij})} + o\Big[\frac{(Mn-mn)^2}{mn(\lambda_{ij}b)} + \frac{1}{mn(\lambda_{ij}b)}\Big]\Big\} \tag{59}$$

This completes the proof of the asymptotic variance.
The mean squared error is obtained by summing the variance and the squared bias.

MSE of the Estimator of Finite Population Mean. The conditional MSE combines the conditional bias and the variance terms of the estimator, $(\hat{\bar{\bar{Y}}}_{INW})$, that is,

$$MSE(\hat{\bar{\bar{Y}}}_{INW}/x_{ij}) = E\big(\hat{\bar{\bar{Y}}}_{INW} - \bar{\bar{Y}}\big)^2 \tag{60}$$

Expanding and simplifying, equation (60) gives

$$MSE(\hat{\bar{\bar{Y}}}_{INW}/x_{ij}) = Var\big(\hat{\bar{\bar{Y}}}_{INW}/x_{ij}\big) + \Big(Bias\big(\hat{\bar{\bar{Y}}}_{INW}/x_{ij}\big)\Big)^2 \tag{61}$$

Combining the conditional bias in equation (40) and the conditional variance in equation (27) leads to

$$MSE\big(\hat{\bar{\bar{Y}}}_{INW}/X_{ij}=x_{ij}\big)=\frac{1}{(Mn)^2}\Bigg\{\frac{(Mn-mn)^2H(w)\sigma^2_{x_{ij}}}{mn(\lambda_{ij}b)g(\hat{x}_{ij})}$$

$$+\Bigg[\frac{(Mn-mn)^2}{4(mn)^2(Mn^2)}(\lambda_{ij}b)^2d_k^2\bigg[m\prime\prime(x_{ij})g(x_{ij})+\frac{2g\prime(x_{ij})m\prime(x_{ij})}{g(\hat{x}_{ij})}\bigg]^2 \tag{62}$$

$$+o\Bigg(\frac{1}{Mn}\Big\{\frac{(Mn-mn)^2}{mn(\lambda_{ij}b)}+\frac{1}{mn(\lambda_{ij}b)}\Big\}\Bigg)\Bigg]\Bigg\}$$

Where

$$H(w)=\int K(w)^2dw,\ d_k=\int w^2K(w)dw \tag{63}$$

and

$$C(x)=[g(\hat{x}_{ij})]^{-1}\Big[\frac{1}{2}m\prime\prime(x_{ij})g(x_{ij})+g\prime(x_{ij})m\prime(x_{ij})\Big]$$

On the existence and convergence theorems for weakly rational contraction mappings in M-metric spaces

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Abstract. The aim of this paper is to introduce and investigate the concept of weakly rational contraction mappings within the framework of M-metric spaces. By relaxing classical contractive conditions and incorporating suitable auxiliary functions, we establish a new fixed point theorem that extends and generalizes several well-known results in the literature. To demonstrate the applicability of the main results, an illustrative example is also provided. Furthermore, the paper concludes with a discussion on potential extensions and open problems, including the possibility of weakening or omitting the Picard-sequentially monotone condition in the established theorem, thereby paving the way for further investigations in generalized metric spaces.

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1. Introduction

The Banach contraction principle is one of the most celebrated and fundamental results in fixed point theory. It provides a simple yet powerful tool in nonlinear analysis and has become a cornerstone for the development of modern fixed point theory. This classical theorem not only guarantees the existence and uniqueness of fixed points for Banach contraction mappings but also serves as a foundation for diverse applications in functional analysis, optimization, and dynamic systems. Over the years, numerous authors have extended, refined, and generalized this principle within different mathematical structures, thereby broadening its theoretical scope and practical relevance. Among these extensions, a significant milestone was introduced by Matthews [8], who proposed the concept of a partial metric space, a generalization of the conventional metric space in which the self-distance of a point is not necessarily zero. This modification has allowed researchers to model various phenomena where symmetry or the zero self-distance property fails to hold, such as in computer science, information theory, and topology. Building upon this concept, several fixed point theorems were developed by many authors (see, for example, [3], [4], [5], [6]), demonstrating that partial metric spaces provide a more flexible framework for convergence analysis.

In 2014, Asadi *et al.* [2] introduced the notion of an M-metric space, which further extends the idea of partial metric spaces by relaxing certain structural conditions while preserving the essential properties needed for fixed point results. They established fixed point theorems for Banach and Kannan contraction mappings in this setting, laying the groundwork for several subsequent studies. This framework has since inspired many researchers to explore additional generalizations and to obtain new existence and uniqueness results in the context of M-metric spaces (see [1], [9], [10]).

Motivated by the above progress, the present paper aims to continue this line of investigation by introducing and studying a new class of weakly rational contraction mappings within the framework of M-metric spaces. We establish a new fixed point theorem that both extends and generalizes known results in the literature, while also providing deeper insight into the structure of contractive mappings in these spaces. Furthermore, we include a constructive example to illustrate the validity and applicability of our main results. The findings presented herein contribute to the further advancement of fixed point theory in generalized metric spaces and offer potential foundations for future research in nonlinear analysis and applied mathematics.

2. Preliminaries

In this section, we begin by recalling some fundamental definitions and preliminary results from the existing literature.

Definition 2.1 ([8]). *Let X be a nonempty set and let $p : X \times X \to [0, \infty)$ be a function satisfying the following properties for all $x, y, z \in X$:*

(p_1) $p(x, x) = p(y, y) = p(x, y)$ *if and only if* $x = y$ *(equality);*
(p_2) $p(x, x) \leq p(x, y)$ *(small self-distance);*
(p_3) $p(x, y) = p(y, x)$ *(symmetry);*
(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$ *(triangle inequality).*

Then the function p is called a partial metric (or more specifically, a p-metric) on X, and the pair (X, p) is called a partial metric space.

It is clear that every metric space is a partial metric space. However, the converse does not necessarily hold. The following examples illustrate functions that define a partial metric, but it is not a metric.

Example 2.2 ([8]). *Let $X = [0, \infty)$ and define $p : X \times X \to [0, \infty)$ by*

$$p(x, y) = \max\{x, y\} \quad \text{for all } x, y \in X.$$

Then p is a partial metric on X, but it is not a metric on X.

Example 2.3 ([8]). *Let $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$ and define $p : X \times X \to [0, \infty)$ by*

$$p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\} \quad \text{for all } [a, b], [c, d] \in X.$$

Then p is a partial metric on X, but it is not a metric on X.

Each partial metric p on X generates a T_0-topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where

$$B_p(x, \epsilon) := \{y \in X : p(x, y) < p(x, x) + \epsilon\}$$

for all $x \in X$ and $\epsilon > 0$.

Definition 2.4 ([8]). *Let (X, p) be a partial metric space.*

(1) *A sequence $\{x_n\}$ in X is said to converge to a point $x \in X$ if and only if*

$$p(x, x) = \lim_{n\to\infty} p(x, x_n).$$

In this case, we denote the convergence by $x_n \to x$ as $n \to \infty$.

(2) *A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if*

$$\lim_{n,m\to\infty} p(x_n, x_m)$$

exists and is finite.

(3) *The partial metric space (X, p) is said to be complete if and only if every Cauchy sequence $\{x_n\}$ in X converges with respect to its topology τ_p to some $x \in X$ such that*

$$\lim_{n,m\to\infty} p(x_n, x_m) = p(x, x).$$

Next, we recall some notations, fundamental definitions, and preliminary results in the setting of M-metric spaces that will be used throughout the paper. Let X be a nonempty set and let $m : X \times X \to [0, \infty)$ be a given function. The following notations will be used in the sequel:

(1) $m_{x,y} := \min\{\, m(x, x), m(y, y)\,\}$ for all $x, y \in X$;
(2) $M_{x,y} := \max\{\, m(x, x), m(y, y)\,\}$ for all $x, y \in X$.

Definition 2.5 ([2]). *Let X be a nonempty set and let $m : X \times X \to [0, \infty)$ be a function satisfying the following properties for all $x, y, z \in X$:*

(m_1) $m(x, x) = m(y, y) = m(x, y) \Leftrightarrow x = y$;
(m_2) $m_{x,y} \leq m(x, y)$;
(m_3) $m(x, y) = m(y, x)$;
(m_4) $(m(x, y) - m_{x,y}) \leq (m(x, z) - m_{x,z}) + (m(z, y) - m_{z,y})$.

Then the function m is said to be an m-metric and the pair (X, m) is called an M-metric space.

Lemma 2.6 ([2]). *Every partial metric space is an M-metric space, but the converse does not hold in general.*

Example 2.7 ([2]). *Let $X = [0,\infty)$ and a function $m : X \times X \to [0,\infty)$ be defined by $m(x,y) = \frac{x+y}{2}$ for all $x, y \in X$. Then m is an m-metric on X, but it is not a partial metric.*

Example 2.8 ([2]). *Let $X = \{1,2,3\}$ and a function $m : X \times X \to [0,\infty)$ defined by*

$$m(x,y) = \begin{cases} 1, & x = y = 1, \\ 9, & x = y = 2, \\ 5, & x = y = 3, \\ 10, & x, y \in \{1,2\} \text{ and } x \neq y, \\ 7, & x, y \in \{1,3\} \text{ and } x \neq y, \\ 8, & x, y \in \{2,3\} \text{ and } x \neq y. \end{cases}$$

Then m is an m-metric, but it is not a partial metric.

From Lemma 2.6, we refer the reader to the relation between metric spaces, partial metric spaces, and M-metric spaces in Figure 1.

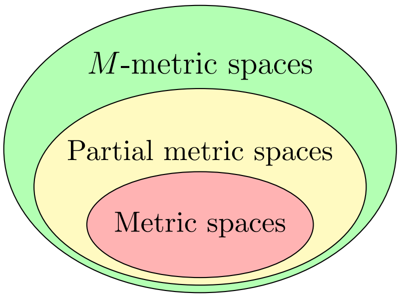

FIGURE 1. Relationship between metric spaces, partial metric spaces and M-metric spaces on a nonempty set X

Definition 2.9 ([2]). *Let (X,m) be an M-metric space.*

(1) *A sequence $\{x_n\}$ in X is said to converge to a point $x \in X$ if and only if*

$$\lim_{n\to\infty} \big(m(x_n, x) - m_{x_n,x}\big) = 0.$$

In this case, we denote the convergence by $x_n \to x$ as $n \to \infty$.

(2) *A sequence $\{x_n\}$ in X is said to be an m-Cauchy sequence if and only if*

$$\lim_{n,m\to\infty} \big(m(x_n, x_m) - m_{x_n,x_m}\big) \quad \text{and} \quad \lim_{n,m\to\infty} \big(M_{x_n,x_m} - m_{x_n,x_m}\big)$$

both exist and are finite.

(3) *The M-metric space (X,m) is said to be complete if and only if every Cauchy sequence $\{x_n\}$ in X converges to some $x \in X$ such that*

$$\lim_{n\to\infty} (m(x_n, x) - m_{x_n,x}) = 0 \quad \text{and} \quad \lim_{n\to\infty} (M_{x_n,x} - m_{x_n,x}) = 0.$$

It is easy to obtain that if m is an m-metric on a nonempty set X, then the function $m^w, m^s : X \times X \to [0,\infty)$ defined for each $x, y \in X$ by

$$m^w(x,y) := m(x,y) - 2m_{x,y} + M_{x,y}$$

and

$$m^s(x,y) := \begin{cases} m(x,y) - m_{x,y}, & x \neq y, \\ 0, & x = y \end{cases}$$

are metrics on X.

Lemma 2.10 ([2]). *Let (X,m) be an M-metric space, and let $\{x_n\}$ be a sequence in X with $x \in X$. Then the following statements hold:*

(1) *The sequence $\{x_n\}$ is an m-Cauchy sequence in (X,m) if and only if it is a Cauchy sequence in the metric space (X, m^w).*

(2) *The space* (X, m) *is complete if and only if the metric space* (X, m^w) *is complete. Moreover,*

$$\lim_{n\to\infty} m^w(x_n, x) = 0 \quad \iff \quad \begin{cases} \lim_{n\to\infty} \left(m(x_n, x) - m_{x_n,x}\right) = 0, \\ \lim_{n\to\infty} \left(M_{x_n,x} - m_{x_n,x}\right) = 0. \end{cases}$$

It is worth noting that the above lemma remains valid when the metric m^w is replaced by m^s. In other words, the equivalence between the notions of m-Cauchy and Cauchy sequences, as well as the completeness of (X, m) and (X, m^s), continues to hold under the same framework. This observation emphasizes the robustness of the characterization of convergence and completeness within the context of M-metric spaces.

Lemma 2.11 ([2])**.** *Let* (X, m) *be an* M*-metric space and* $\{x_n\}$ *be a sequence in* X*. Assume that* $x_n \to x \in X$ *as* $n \to \infty$*. Then*

$$\lim_{n\to\infty} (m(x_n, y) - m_{x_n,y}) = m(x, y) - m_{x,y}$$

for all $y \in X$*.*

Now, we recall some fundamental definitions from [7] and [11], which will play a crucial role in formulating the contraction mappings employed in the present study.

Definition 2.12 ([7])**.** *A function* $\psi : [0, \infty) \to [0, \infty)$ *is called an altering distance function if it satisfies the following two conditions:*

(1) ψ *is continuous and nondecreasing;*
(2) $\psi(t) = 0$ *if and only if* $t = 0$*.*

We denote Ψ *the set of all altering distance functions.*

Definition 2.13 ([11])**.** *A function* $\varphi : [0, \infty) \to [0, \infty)$ *is called a strong-altering distance function if it satisfies the following two conditions:*

(1) φ *is continuous;*
(2) $\varphi(t) \neq 0$ *when* $t \neq 0$*.*

We denote Φ *the set of all strong-altering distance functions.*

Throughout this paper, in an M-metric space (X, m), the notation m^2 denotes the squared value of the function m evaluated at the specified point. We conclude this preliminary section with this notation, which will be consistently used in the subsequent discussions and main results.

3. Main results

In this section, we first present some essential definitions that will be employed in the formulation of our main theorems. An illustrative example and the subsequent results are then provided to demonstrate the validity and applicability of the proposed concepts. We begin with the following new contraction mapping in an M-metric space, which serves as the central concept for establishing the fixed point theorems developed in this paper.

Definition 3.1. *Let* (X, m) *be an* M*-metric space. A mapping* $T : X \to X$ *is called a weakly rational contraction mapping if there exist* $\psi \in \Psi$ *and* $\phi \in \Phi$ *such that*

$$\psi(m^2(Tx, Ty)) \le \psi(M(x, y)) - \varphi(M(x, y)) \tag{1}$$

for all $x, y \in X$*, where*

$$M(x, y) = \max\left\{m^2(x, y), \frac{m(x, Tx)\cdot m(y, Ty)}{1 + m^2(x, y)}, \frac{m(x, Ty)\cdot m(y, Tx)}{2}, \frac{m^2(y, Ty)}{1 + m^2(y, y)}\right\}.$$

Before proceeding to the main theorems of this paper, we present another key definition that plays a crucial role in the formulation of our main results.

Definition 3.2. *Let* (X, m) *be an* M*-metric space. A mapping* $T : X \to X$ *is said to be Picard-sequentially monotone if* $\{m(x_n, x_n)\}$ *is increasing or decreasing for each Picard sequence* $\{x_n\}$ *in* X*.*

Remark 3.3. *In the case of metric spaces, which are particular instances of* M*-metric spaces, we observe that any self-mapping on a metric space is always Picard-sequentially monotone.*

In the following, we present the fixed point results that constitute the core contributions of this paper.

Theorem 3.4. *Let (X, m) be a complete M-metric space and let $T : X \to X$ be a weakly rational contraction that satisfies the Picard-sequential monotonicity condition. Then T is a Picard operator, which means that it admits a unique fixed point $z \in X$, and for any initial point $x_0 \in X$, the sequence $\{T^n(x_0)\}$ converges to z.*

Proof. Assume that $\psi \in \Psi$ and $\varphi \in \Phi$ are functions satisfying (1). Let x_0 be an arbitrary point in X and define a Picard sequence $\{x_n\} \subseteq X$ by

$$x_n = Tx_{n-1}$$

for all $n \in \mathbb{N}$. Now, if there exists an index $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$. So x_{n_0} is a fixed point of T and the proof is finished. Therefore, we assume that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Using (1) for the elements $x = x_{n-1}$ and $y = x_n$, arbitrary $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \psi(m^2(x_n, x_{n+1})) &= \psi(m^2(Tx_{n-1}, Tx_n)) \\ &\leq \psi(M(x_{n-1}, x_n)) - \varphi(M(x_{n-1}, x_n)), \end{aligned} \tag{2}$$

where

$$M(x_{n-1}, x_n) = \max \left\{ \begin{array}{c} m^2(x_{n-1}, x_n), \dfrac{m(x_{n-1}, x_n) \cdot m(x_n, x_{n+1})}{1 + m^2(x_{n-1}, x_n)}, \\ \dfrac{1}{2}[m(x_{n-1}, x_{n+1}) \cdot m(x_n, x_n)], \dfrac{m^2(x_n, x_{n+1})}{1 + m^2(x_n, x_n)} \end{array} \right\}. \tag{3}$$

First, we claim that $M(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$. We suppose, by contradiction, that $M(x_{n^*-1}, x_{n^*}) = 0$ for some $n^* \in \mathbb{N}$. Then we have

$$m(x_{n^*-1}, x_{n^*}) = 0, \tag{4}$$

$$m(x_{n^*}, x_{n^*+1}) = 0 \tag{5}$$

and

$$m(x_{n^*-1}, x_{n^*+1}) \cdot m(x_n^*, x_n^*) = 0.$$

Once again, assume that $m(x_n^*, x_n^*) = 0$. From the Picard-sequential monotonicity of T, we proceed by considering two possible cases.

Case 1: If $\{m(x_n^*, x_n^*)\}$ is increasing, then

$$m(x_{n^*-1}, x_{n^*-1}) \leq m(x_{n^*}, x_{n^*}) = 0.$$

Using the above fact and (4), we get

$$m(x_{n^*-1}, x_{n^*-1}) = m(x_{n^*-1}, x_{n^*}) = m(x_{n^*}, x_{n^*}) = 0.$$

Hence, by condition (m_1), it follows that $x_{n^*-1} = x_{n^*}$, which leads to a contradiction.

Case 2: If $\{m(x_n^*, x_n^*)\}$ is decreasing, then

$$m(x_{n^*+1}, x_{n^*+1}) \leq m(x_{n^*}, x_{n^*}) = 0.$$

Using the above fact and (5) we get

$$m(x_{n^*+1}, x_{n^*+1}) = m(x_{n^*}, x_{n^*+1}) = m(x_{n^*}, x_{n^*}) = 0.$$

Hence, by condition (m_1), it follows that $x_{n^*} = x_{n^*+1}$, which leads to a contradiction.

By the above two cases, we have $m(x_n^*, x_n^*) > 0$. Since $\{m(x_{n^*}, x_{n^*})\}$ is monotone, by (m_2), (4) and (5) we have

$$0 < m(x_{n^*}, x_{n^*}) = m_{x_{n^*}, x_{n^*+1}} \leq m(x_{n^*}, x_{n^*+1}) = 0$$

or

$$0 < m(x_{n^*}, x_{n^*}) = m_{x_{n^*-1}, x_{n^*}} \leq m(x_{n^*-1}, x_{n^*}) = 0,$$

which is a contradiction. Hence, $M(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$, which implies by the property of φ that

$$\varphi(M(x_{n-1}, x_n)) > 0 \tag{6}$$

for all $n \in \mathbb{N}$. Next, we will claim that

$$M(x_{n-1}, x_n) \leq \max\{m^2(x_n, x_{n-1}), m(x_{n-1}, x_n) \cdot m(x_n, x_{n+1}), m^2(x_n, x_{n+1})\}$$

for all $n \in \mathbb{N}$. To establish this result, we consider two distinct cases as follows:

Case 1: If $\{m(x_n, x_n)\}$ is an increasing sequence, then by applying conditions (m_2) and (m_4), for each $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
&\frac{1}{2}[m(x_{n-1}, x_{n+1}) \cdot m(x_n, x_n)] \\
\leq\ & \frac{1}{2}[m(x_{n-1}, x_n) - m_{x_{n-1},x_n} + m(x_n, x_{n+1}) - m_{x_n,x_{n+1}} + m_{x_{n-1},x_{n+1}}]m(x_n, x_n) \\
\leq\ & \frac{1}{2}[m(x_{n-1}, x_n) - m_{x_{n-1},x_n} + m(x_n, x_{n+1}) + m_{x_{n-1},x_{n+1}}]m(x_n, x_n) \\
=\ & \frac{1}{2}[m(x_{n-1}, x_n) - m(x_{n-1}, x_{n-1}) + m(x_n, x_{n+1}) + m(x_{n-1}, x_{n-1})]m_{x_n,x_{n+1}} \\
=\ & \frac{1}{2}[m(x_{n-1}, x_n) + m(x_n, x_{n+1})]m_{x_n,x_{n+1}} \\
\leq\ & \frac{1}{2}[m(x_{n-1}, x_n) + m(x_n, x_{n+1})]m(x_n, x_{n+1}) \\
\leq\ & \max\{m^2(x_n, x_{n+1}), m(x_{n-1}, x_n) \cdot m(x_n, x_{n+1})\}. \qquad (7)
\end{aligned}$$

Case 2: If $\{m(x_n, x_n)\}$ is a decreasing sequence, then by applying conditions (m_2) and (m_4), for each $n \in \mathbb{N}$, we similarly obtain

$$\begin{aligned}
&\frac{1}{2}[m(x_{n-1}, x_{n+1}) \cdot m(x_n, x_n)] \\
\leq\ & \frac{1}{2}[m(x_{n-1}, x_n) - m_{x_{n-1},x_n} + m(x_n, x_{n+1}) - m_{x_n,x_{n+1}} + m_{x_{n-1},x_{n+1}}]m(x_n, x_n) \\
\leq\ & \frac{1}{2}[m(x_{n-1}, x_n) + m(x_n, x_{n+1}) - m_{x_n,x_{n+1}} + m_{x_{n-1},x_{n+1}}]m(x_n, x_n) \\
=\ & \frac{1}{2}[m(x_{n-1}, x_n) + m(x_n, x_{n+1}) - m(x_{n+1}, x_{n+1}) + m(x_{n+1}, x_{n+1})]m_{x_{n-1},x_n} \\
\leq\ & \frac{1}{2}[m(x_{n-1}, x_n) + m(x_n, x_{n+1})]m(x_{n-1}, x_n) \\
\leq\ & \max\{m^2(x_n, x_{n-1}), m(x_{n-1}, x_n) \cdot m(x_n, x_{n+1})\}. \qquad (8)
\end{aligned}$$

From above two cases and (3), we obtain

$$M(x_{n-1}, x_n) \leq \max\{m^2(x_n, x_{n-1}), m(x_{n-1}, x_n) \cdot m(x_n, x_{n+1}), m^2(x_n, x_{n+1})\} \qquad (9)$$

for all $n \in \mathbb{N}$. Now, using (2), (6) and (9) we have

$$\begin{aligned}
\psi(m^2(x_n, x_{n+1})) \leq\ & \psi(\max\{m^2(x_{n-1}, x_n), m(x_{n-1}, x_n) \cdot m(x_n, x_{n+1}), m^2(x_n, x_{n+1})\}) \\
& -\varphi(M(x_{n-1}, x_n)) \qquad (10) \\
<\ & \psi(\max\{m^2(x_{n-1}, x_n), m(x_{n-1}, x_n) \cdot m(x_n, x_{n+1}), m^2(x_n, x_{n+1})\})
\end{aligned}$$

for all $n \in \mathbb{N}$. The nondecreasing property of ψ implies that

$$m^2(x_n, x_{n+1}) < \max\{m^2(x_{n-1}, x_n), m(x_{n-1}, x_n) \cdot m(x_n, x_{n+1}), m^2(x_n, x_{n+1})\} \qquad (11)$$

for all $n \in \mathbb{N}$. If

$$\max\{m^2(x_{n-1}, x_n),\ m(x_{n-1}, x_n)\, m(x_n, x_{n+1}),\ m^2(x_n, x_{n+1})\} = m^2(x_n, x_{n+1})$$

for some $n \in \mathbb{N}$, then a contradiction arises from (11). Hence, it is clear that

$$m^2(x_n, x_{n+1}) < \max\{m^2(x_{n-1}, x_n), m(x_{n-1}, x_n) \cdot m(x_n, x_{n+1})\}$$

for all $n \in \mathbb{N}$. This implies that

$$m(x_n, x_{n+1}) < m(x_n, x_{n-1})$$

for all $n \in \mathbb{N}$. Therefore, $\{m(x_n, x_{n+1})\}$ is strictly decreasing and hence converges to $r \geq 0$. Now, we will show that $r = 0$. On the contrary, suppose that $r > 0$. From (3) and (9), we have

$$m^2(x_n, x_{n-1}) \leq M(x_{n-1}, x_n) \leq \max\{m^2(x_n, x_{n-1}), m(x_{n-1}, x_n) \cdot m(x_n, x_{n+1}), m^2(x_n, x_{n+1})\}$$

for all $n \in \mathbb{N}$. By taking the limit as $n \to \infty$ in the above inequality, we have

$$\lim_{n \to \infty} M(x_{n-1}, x_n) = r^2.$$

From this fact, by taking the limit as $n \to \infty$ in (10) and by the continuity of ψ and ϕ, we obtain

$$\begin{aligned} \psi(r^2) &\leq \psi(\max\{r^2, r^2, r^2\}) - \varphi(r^2) \\ &= \psi(r^2) - \varphi(r^2) \\ &< \psi(r^2), \end{aligned}$$

which is a contradiction. Hence, $r = 0$. Thus,

$$\lim_{n\to\infty} m(x_n, x_{n+1}) = 0. \tag{12}$$

Using (m_2) and (12) we obtain

$$\lim_{n\to\infty} m_{x_n, x_{n+1}} = 0, \tag{13}$$

and so we have

$$\lim_{n\to\infty} m(x_n, x_n) = 0. \tag{14}$$

Now, we will claim that $\{x_n\}$ is an m-Cauchy sequence in (X, m) so by Lemma 2.10 we must prove that $\{x_n\}$ is a Cauchy sequence in (X, m^w). Suppose that $\{x_n\}$ is not a Cauchy sequence in (X, m^w), which means that there exists an $\epsilon > 0$ and two subsequences $\{x_{a_k}\}$ and $\{x_{b_k}\}$ of $\{x_n\}$ with $b_k > a_k > k$ such that

$$m^w(x_{a_k}, x_{b_k}) \geq \epsilon. \tag{15}$$

So we can assume that b_k is the smallest integer such that (15) holds. Then we have

$$m^w(x_{a_k}, x_{b_k - 1}) < \epsilon. \tag{16}$$

Using the triangle inequality for the metric m^w, (15), (16) and the property of m^w we have

$$\begin{aligned} \epsilon &\leq m^w(x_{a_k}, x_{b_k}) \\ &\leq m^w(x_{a_k}, x_{b_k-1}) + m^w(x_{b_k-1}, x_{b_k}) \\ &< \epsilon + m^w(x_{b_k-1}, x_{b_k}) \\ &= \epsilon + m(x_{b_k-1}, x_{b_k}) - 2m_{x_{b_k-1}, x_{b_k}} + M_{x_{b_k-1}, x_{b_k}} \end{aligned} \tag{17}$$

for all $k \in \mathbb{N}$. Taking limit as $k \to \infty$ in (17), and from (12) and (14), we have

$$\lim_{k\to\infty} m^w(x_{a_k}, x_{b_k}) = \epsilon. \tag{18}$$

On the other hand,

$$m^w(x_{b_k}, x_{a_k}) \leq m^w(x_{b_k}, x_{b_k-1}) + m^w(x_{b_k-1}, x_{a_k-1}) + m^w(x_{a_k-1}, x_{a_k}),$$

and

$$m^w(x_{b_k-1}, x_{a_k-1}) \leq m^w(x_{b_k-1}, x_{b_k}) + m^w(x_{b_k}, x_{a_k}) + m^w(x_{a_k}, x_{a_{k-1}}).$$

Letting $k \to \infty$ in above two inequalities, by (12) and (18), we deduce that

$$\lim_{k\to\infty} m^w(x_{b_k-1}, x_{a_k-1}) = \epsilon.$$

Similarly, we can prove that

$$\lim_{k\to\infty} m^w(x_{b_k-1}, x_{a_k}) = \epsilon$$

and

$$\lim_{k\to\infty} m^w(x_{b_k}, x_{a_k-1}) = \epsilon.$$

From (13), (14) and the property of m^w, we obtain

$$\lim_{k\to\infty} m(x_{b_k-1}, x_{a_k-1}) = \lim_{k\to\infty} m(x_{b_k-1}, x_{a_k}) = \lim_{k\to\infty} m(x_{b_k}, x_{a_k-1}) = \epsilon. \tag{19}$$

Now, by substituting $x = x_{b_k}$ and $y = x_{a_k}$ into (1), we obtain

$$\begin{aligned} \psi(m^2(x_{b_k}, x_{a_k})) &= \psi(m^2(Tx_{b_k-1}, Tx_{a_k-1})) \\ &\leq \psi(M(x_{b_k-1}, x_{a_k-1})) - \varphi(M(x_{b_k-1}, x_{a_k-1})) \end{aligned} \tag{20}$$

for all $k \in \mathbb{N}$, where

$$M(x_{b_k-1}, x_{a_k-1}) = \max \left\{ \begin{array}{c} m^2(x_{b_k-1}, x_{a_k-1}), \\ \dfrac{m(x_{b_k-1}, x_{b_k}) \cdot m(x_{a_k-1}, x_{a_k})}{1 + m^2(x_{b_k-1}, x_{a_k-1})}, \\ \dfrac{1}{2}[m(x_{b_k-1}, x_{a_k}) \cdot m(x_{a_k-1}, x_{b_k})], \\ \dfrac{m^2(x_{a_k-1}, x_{a_k})}{1 + m^2(x_{a_k-1}, x_{a_k-1})} \end{array} \right\}.$$

Letting $k \to \infty$ in (20) and using (12), (14), (19), we get

$$\begin{aligned} \psi(\epsilon^2) &\le \psi\left(\max\left\{\epsilon^2, 0, \frac{\epsilon^2}{2}, 0\right\}\right) - \varphi\left(\max\left\{\epsilon^2, 0, \frac{\epsilon^2}{2}, 0\right\}\right) \\ &< \psi(\epsilon^2), \end{aligned}$$

which is a contradiction. Hence, we conclude that $\{x_n\}$ is a Cauchy sequence in the metric space (X, m^w). By item (1) of Lemma 2.10, the sequence $\{x_n\}$ is an m-Cauchy sequence in the complete M-metric space (X, m). Hence, there exists a point $z \in X$ such that

$$\lim_{n\to\infty} \left(m(x_n, z) - m_{x_n,z}\right) = 0 \quad \text{and} \quad \lim_{n\to\infty} \left(M_{x_n,z} - m_{x_n,z}\right) = 0.$$

Using (14), the above equation become

$$\lim_{n\to\infty} m(x_n, z) = 0 \text{ and } \lim_{n\to\infty} M_{x_n,z} = 0. \tag{21}$$

This implies that

$$m(z, z) \le M_{x_n,z} \to 0 \text{ as } n \to \infty,$$

that is,

$$m(z, z) = 0. \tag{22}$$

Now, we will show that z is the fixed point of T. From (1), for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \psi(m^2(x_{n+1}, Tz)) &= \psi(m^2(Tx_n, Tz)) \\ &\le \psi(M(x_n, z)) - \varphi(M(x_n, z)), \end{aligned} \tag{23}$$

where

$$M(x_n, z) = \max \left\{ \begin{array}{c} m^2(x_n, z), \dfrac{m(x_n, Tz) \cdot m(z, Tz)}{1 + m^2(x_n, z)}, \\ \dfrac{1}{2}[m(x_n, Tz) \cdot m(z, x_{n+1})], \dfrac{m^2(z, Tz)}{1 + m^2(z, z)} \end{array} \right\}.$$

Using the continuity of ψ and ϕ, (21), (22) and Lemma 2.11 with (14), we obtain from the above inequality that

$$\begin{aligned} \psi(m^2(z, Tz)) &\le \psi(\max\{0, m^2(z, Tz), 0, m^2(z, Tz)\}) - \varphi(\max\{0, m^2(z, Tz), 0, m^2(z, Tz)\}) \\ &= \psi(m^2(z, Tz)) - \varphi(m^2(z, Tz)), \end{aligned}$$

which implies that $\varphi(m^2(z, Tz)) = 0$. Using the property of φ, we deduce that $m^2(z, Tz) = 0$. So we have

$$m(z, Tz) = 0. \tag{24}$$

Again, by (1) we have

$$\psi(m^2(Tz, Tz)) \le \psi(M(z, z)) - \varphi(M(z, z)),$$

where

$$M(z, z) = \max \left\{ \begin{array}{c} m^2(z, z), \dfrac{m(z, Tz) \cdot m(z, Tz)}{1 + m^2(z, z)}, \\ \dfrac{1}{2}[m(z, Tz) \cdot m(z, Tz)], \dfrac{m^2(z, Tz)}{1 + m^2(z, z)} \end{array} \right\}.$$

Now, from (22) and (24), we have

$$\psi(m^2(Tz, Tz)) \le \psi(0) - \varphi(0) = 0,$$

which implies that $m(Tz, Tz) = 0$. Now, we get

$$m(z, z) = m(z, Tz) = m(Tz, Tz).$$

It follows from (m_1) that $z = Tz$, that is, z is a fixed point of T.

To establish the uniqueness of the fixed point, suppose that $z, w \in X$ are two fixed points of T. If $m(z,z) > 0$, then by applying (1) we obtain

$$\psi\big(m^2(z,z)\big) \le \psi\big(M(z,z)\big) - \varphi\big(M(z,z)\big).$$

Clearly, this inequality cannot hold, and hence $m(z,z) = 0$. By a similar argument, we also have $m(w,w) = 0$. On the other hand, if $m(z,w) > 0$, then by (1), we get

$$\psi\big(m^2(z,w)\big) \le \psi\big(M(z,w)\big) - \varphi\big(M(z,w)\big),$$

where

$$M(z,w) = \max\left\{\begin{array}{l} m^2(z,w), \dfrac{m(z,Tz)\cdot m(w,Tw)}{1+m^2(z,w)}, \\ \dfrac{1}{2}\big[m(z,Tw)\cdot m(w,Tz)\big], \dfrac{m^2(w,Tw)}{1+m^2(w,w)} \end{array}\right\}.$$

This leads to the absurd conclusion that $m^2(z,w) < m^2(z,w)$, which is a contradiction. Therefore, we conclude that

$$m(z,z) = m(z,w) = m(w,w) = 0,$$

and by (m_1), it follows that $z = w$. Hence, the fixed point of T is unique. □

Next, we give an example to support the validity of Theorem 3.4.

Example 3.5. *Let $X = [0,\infty)$ and $m : X \times X \to [0,\infty)$ be defined by*

$$m(x,y) = \frac{x+y}{2}$$

for all $x, y \in X$. Then (X, m) is a complete M-metric space. Define a mapping $T : X \to X$ by

$$T(x) = \begin{cases} \dfrac{\sin x}{2}, & 0 \le x < 1, \\ \dfrac{x}{x+1}, & x \ge 1. \end{cases}$$

for all $x \in X$.

First, we consider the value of $M(x,y)$ for all $x, y \in X$. Since $Tx \le x$ for all $x \in X$, we have

$$\frac{m(x,Tx)\cdot m(y,Ty)}{1+m^2(x,y)} = \frac{\left(\frac{x+Tx}{2}\right)\cdot\left(\frac{y+Ty}{2}\right)}{1+\left(\frac{x+y}{2}\right)^2} \le \frac{(x+x)\cdot(y+y)}{4(1+\left(\frac{x+y}{2}\right)^2)} \le xy \le \left(\frac{x+y}{2}\right)^2 = m^2(x,y)$$

and

$$\frac{m(x,Ty)\cdot m(y,Tx)}{2} = \frac{1}{2}\left[\frac{(x+Ty)}{2}\cdot\frac{(y+Tx)}{2}\right] \le \frac{(x+y)\cdot(y+x)}{8} \le \left(\frac{x+y}{2}\right)^2 = m^2(x,y)$$

for all $x, y \in X$. This implies that

$$M(x,y) = \max\left\{m^2(x,y), \frac{m^2(y,Ty)}{1+m^2(y,y)}\right\}$$

for all $x, y \in X$. Observe that T is Picard-sequentially monotone.

Next, we will claim that T is a weakly rational contraction mapping with the functions $\psi \in \Psi$ and $\varphi \in \Phi$ defined by

$$\psi(t) = \sqrt{t}, \quad \varphi(t) = \tfrac{1}{2}\sqrt{t} \quad \text{for all } t \in [0,\infty),$$

that is, T satisfies condition (1). Let $x, y \in X$. We consider the following three cases:

Case 1: *Let $x, y \in [0,1)$. If $M(x,y) = m^2(x,y)$, then we get*

$$\begin{aligned} \psi(m^2(Tx,Ty)) &= m(Tx,Ty) \\ &= \frac{\sin x + \sin y}{4} \\ &\le \frac{x+y}{4} \\ &= \frac{x+y}{2} - \frac{x+y}{4} \\ &= \psi(m^2(x,y)) - \varphi(m^2(x,y)) \\ &= \psi(M(x,y)) - \varphi(M(x,y)). \end{aligned}$$

If $M(x,y)=\dfrac{m^2(y,Ty)}{1+m^2(y,y)}$, *then we get* $\dfrac{m^2(y,Ty)}{1+m^2(y,y)}\geq(\frac{x+y}{2})^2$ *and so*

$$\begin{aligned}
\psi(m^2(Tx,Ty)) &= m(Tx,Ty)\\
&= \frac{\sin x+\sin y}{4}\\
&\leq \frac{1}{2}\left(\frac{x+y}{2}\right)\\
&\leq \frac{1}{2}\left(\frac{m(y,Ty)}{\sqrt{1+m^2(y,y)}}\right)\\
&= \left(\frac{m(y,Ty)}{\sqrt{1+m^2(y,y)}}\right)-\left(\frac{m(y,Ty)}{2\sqrt{1+m^2(y,y)}}\right)\\
&= \psi\left(\frac{m^2(y,Ty)}{1+m^2(y,y)}\right)-\varphi\left(\frac{m^2(y,Ty)}{1+m^2(y,y)}\right)\\
&= \psi(M(x,y))-\varphi(M(x,y)).
\end{aligned}$$

Case 2: *Let* $x,y\in[1,\infty)$. *If* $M(x,y)=m^2(x,y)$, *then we get*

$$\begin{aligned}
\psi(m^2(Tx,Ty)) &= m(Tx,Ty)\\
&= \frac{\left(\frac{x}{x+1}\right)+\left(\frac{y}{y+1}\right)}{2}\\
&\leq \frac{x+y}{4}\\
&= \frac{x+y}{2}-\frac{x+y}{4}\\
&= \psi(m^2(x,y))-\varphi(m^2(x,y))\\
&= \psi(M(x,y))-\varphi(M(x,y)).
\end{aligned}$$

If $M(x,y)=\dfrac{m^2(y,Ty)}{1+m^2(y,y)}$, *then we get*

$$\begin{aligned}
\psi(m^2(Tx,Ty)) &= m(Tx,Ty)\\
&= \frac{\left(\frac{x}{x+1}\right)+\left(\frac{y}{y+1}\right)}{2}\\
&\leq \frac{1}{2}\left(\frac{x+y}{2}\right)\\
&\leq \frac{1}{2}\left(\frac{m(y,Ty)}{\sqrt{1+m^2(y,y)}}\right)\\
&= \left(\frac{m(y,Ty)}{\sqrt{1+m^2(y,y)}}\right)-\left(\frac{m(y,Ty)}{2\sqrt{1+m^2(y,y)}}\right)\\
&= \psi\left(\frac{m^2(y,Ty)}{1+m^2(y,y)}\right)-\varphi\left(\frac{m^2(y,Ty)}{1+m^2(y,y)}\right)\\
&= \psi(M(x,y))-\varphi(M(x,y)).
\end{aligned}$$

Case 3: *Let* $(x,y)\in[0,1)\times[1,\infty)\cup[1,\infty)\times[0,1)$. *Without loss of generality, we may assume that* $x\in[0,1)$ *and* $y\in[1,\infty)$. *If* $M(x,y)=m^2(x,y)$, *then we get*

$$\begin{aligned}
\psi(m^2(Tx,Ty)) &= m(Tx,Ty)\\
&= \frac{\left(\frac{\sin x}{2}\right)+\left(\frac{y}{y+1}\right)}{2}\\
&\leq \frac{x+y}{4}\\
&= \frac{x+y}{2}-\frac{x+y}{4}
\end{aligned}$$

$$
\begin{aligned}
&= \psi(m^2(x,y)) - \varphi(m^2(x,y)) \\
&= \psi(M(x,y)) - \varphi(M(x,y)).
\end{aligned}
$$

If $M(x,y) = \dfrac{m^2(y,Ty)}{1+m^2(y,y)}$, *then we get*

$$
\begin{aligned}
\psi(m^2(Tx,Ty)) &= m(Tx,Ty) \\
&= \frac{\left(\frac{\sin x}{2}\right) + \left(\frac{y}{y+1}\right)}{2} \\
&\leq \frac{1}{2}\left(\frac{x+y}{2}\right) \\
&\leq \frac{1}{2}\left(\frac{m(y,Ty)}{\sqrt{1+m^2(y,y)}}\right) \\
&= \left(\frac{m(y,Ty)}{\sqrt{1+m^2(y,y)}}\right) - \left(\frac{m(y,Ty)}{2\sqrt{1+m^2(y,y)}}\right) \\
&= \psi\left(\frac{m^2(y,Ty)}{1+m^2(y,y)}\right) - \varphi\left(\frac{m^2(y,Ty)}{1+m^2(y,y)}\right) \\
&= \psi(M(x,y)) - \varphi(M(x,y)).
\end{aligned}
$$

Therefore, all the conditions of Theorem 3.4 are satisfied, and hence T *admits a unique fixed point. In this case,* $x = 0$ *is the unique fixed point of* T*.*

Furthermore, some numerical experiments illustrating the approximation of the fixed point of T *are presented in Table 1, while the convergence behavior of the corresponding iterations is depicted in Figure 2.*

TABLE 1. Iterates of Picard iterations for Example 3.5

x_0	0.3	0.6	1	1.4
x_1	0.148	0.282	0.500	0.583
x_2	0.074	0.139	0.240	0.275
x_3	0.037	0.069	0.119	0.136
x_4	0.018	0.035	0.059	0.068
x_5	0.009	0.017	0.030	0.034
x_6	0.005	0.009	0.015	0.017
x_7	0.002	0.004	0.007	0.008
x_8	0.001	0.002	0.004	0.004
x_9	0.001	0.001	0.002	0.002
x_{10}	0.000	0.001	0.001	0.001
⋮	⋮	⋮	⋮	⋮

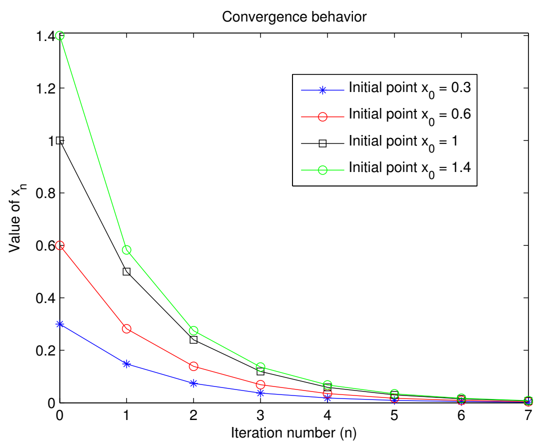

FIGURE 2. Convergence behavior for Example 3.5

In the following, we present several results that can be derived directly from the main theorem of this paper. By taking $\psi(t) = t$ and $\varphi(t) = (1-\alpha)t$ for all $t \in [0,\infty)$, where $\alpha \in [0,1)$, in Theorem 3.4, we immediately obtain the following corollary:

Corollary 3.6. *Let (X,m) be a complete M-metric space and $T : X \to X$ be a mapping satisfying*

$$m^2(Tx,Ty) \le \alpha M(x,y) \tag{25}$$

for all $x,y \in X$, where $\alpha \in [0,1)$ and

$$M(x,y) = \max\left\{m^2(x,y), \frac{m(x,Tx)\cdot m(y,Ty)}{1+m^2(x,y)}, \frac{m(x,Ty)\cdot m(y,Tx)}{2}, \frac{m^2(y,Ty)}{1+m^2(y,y)}\right\}.$$

Assume that T is Picard-sequentially monotone. Then T is a Picard operator.

The following result is the immediate consequence of Corollary 3.6.

Corollary 3.7. *Let (X,m) be a complete M-metric space and $T : X \to X$ be a mapping satisfying*

$$\begin{aligned} m^2(Tx,Ty) \quad &\le \quad a_1 m^2(x,y) + a_2\left[\frac{m(x,Tx)\cdot m(y,Ty)}{1+m^2(x,y)}\right] \\ &\quad + a_3[m(x,Ty)\cdot m(y,Tx)] + a_4\left[\frac{m^2(y,Ty)}{1+m^2(y,y)}\right] \end{aligned} \tag{26}$$

for all $x,y \in X$, where $0 \le a_1 + a_2 + 2a_3 + a_4 < 1$. Assume that T is Picard-sequentially monotone. Then T is a Picard operator.

Remark 3.8. *The Picard-sequentially monotone condition can be omitted if we replace the contractive condition (1) by*

$$\psi(m^2(Tx,Ty)) \le \psi(m^2(x,y)) - \varphi(m^2(x,y)) \tag{27}$$

for all $x,y \in X$, where $\psi \in \Psi$ and $\varphi \in \Phi$.

By taking $\psi(t) = t$ and $\varphi(t) = (1-k^2)t$ with $k \in [0,1)$ for all $t \in [0,\infty)$ in (27), we immediately get the Banach contraction principle in M-metric space as follows:

Corollary 3.9 ([2])**.** *Let (X,m) be a complete M-metric space and $T : X \to X$ satisfies the Banach contractive condition, that is, there exists $k \in [0,1)$ such that*

$$m(Tx,Ty) \le k m(x,y) \tag{28}$$

for all $x,y \in X$. Then T is a Picard operator.

Remark 3.10. *From Figure 1, it is inferred that Theorem 3.4 can be viewed as the generalization and the extension of many comparable results in metric spaces and partial metric spaces.*

4. Conclusions and an open problem

In this paper, we have introduced and studied the weakly rational contraction mappings in the framework of M-metric spaces. By employing suitable auxiliary functions and relaxation of classical contractive conditions, we established a new fixed point theorem that extends and generalizes several existing results in the literature. The proposed approach not only preserves the essential properties of fixed point theory in generalized metric structures but also provides a unified analytical framework for further exploration. The presented example verifies the validity of our main theorem and demonstrates its applicability to nontrivial cases, thereby illustrating the broader scope of weakly rational contractions beyond traditional metric settings. Our findings reveal that M-metric spaces offer a flexible and efficient structure for analyzing nonlinear mappings, especially when the standard assumptions of symmetry or zero self-distance are not satisfied.

Finally, we propose an open problem for future investigation: can the assumption of Picard-sequentially monotone in Theorem 3.4 be weakened or even omitted while still ensuring the existence and uniqueness of a fixed point? This question remains an intriguing challenge for researchers seeking to refine and broaden the applicability of fixed point theory in M-metric spaces.

Acknowledgments

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A new generalization of the Frank Matrix

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Abstract. Matrices play a fundamental role in various areas of mathematics, including combinatorics, numerical analysis, and linear algebra. The literature includes significant papers on matrices whose entries are binomial coefficients, such as Pascal matrices. Max matrices, whose entries are defined as the maximum of their respective row and column indices are another special type of matrix in matrix theory. Among them, the Frank matrix has recently attracted attention from researchers. Several studies have examined various generalizations of the Frank matrix and analysed their basic properties. This paper introduces a new generalization of the Frank matrix, called the binomial Frank matrix, and investigates some of its properties, including the determinant, inverse, adjoint matrix, LU decomposition, and certain norms. This study contributes to the literature by offering a novel approach to extending Frank matrices and providing insights into their algebraic and analytical properties.

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1. Introduction

Matrices whose entries are defined by special number sequences or combinatorial coefficients play an important role in linear algebra and its applications. Among these, the Pascal matrix, constructed from binomial coefficients and representing the classical Pascal triangle in matrix form, is a well-known example widely used since the 1980s. Pascal matrices can be defined in three forms [8, 10, 13]: The lower triangular matrix is most commonly referred to as the Pascal matrix. This matrix constitutes the matrix representation of the classical Pascal triangle and is defined as follows

$$P_n = [p_{ij}]_{i,j=1}^{n} = \begin{cases} \binom{i-1}{j-1}, & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

That is, an $n \times n$ lower triangular Pascal matrix is

$$P_n = \begin{bmatrix} \binom{0}{0} & 0 & 0 & \cdots & 0 \\ \binom{1}{0} & \binom{1}{1} & 0 & \cdots & 0 \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n-1}{0} & \binom{n-1}{1} & \binom{n-1}{2} & \cdots & \binom{n-1}{n-1} \end{bmatrix}.$$

An upper triangular Pascal matrix is expressed by the rule

$$Q_n = [q_{ij}]_{i,j=1}^{n} = \begin{cases} \binom{j-1}{i-1}, & \text{if } j \geq i, \\ 0, & \text{otherwise.} \end{cases}$$

A symmetric Pascal matrix $R_n = [r_{ij}]_{i,j=1}^{n}$ is characterized by

$$r_{ij} = \binom{i+j-2}{i-1}$$

for $i, j = 1, 2, \ldots, n$.

Edelman and Strang [8] studied determinants, inverses, LU decompositions, and some norms of the Pascal matrices. Call and Velleman [3] examined the structure and certain properties of Pascal matrices, establishing their connections with some special matrices. Zhizheng [40] introduced generalizations of the Pascal matrix and conducted a detailed investigation of their algebraic properties, which were further extended by Zhizheng and Liu [39]. Ratliff and Rush [32] focused on the determinants of triangular matrices formed by binomial coefficients, revealing their connection to powers of certain integers. Brawer and Pirovino [2] studied algebraic properties of Pascal matrices, derived explicit formulas for their powers, and analysed their spectral characteristics. Carlitz [4] initiated the study of characteristic polynomials for matrices with binomial coefficient entries. Melham and Cooper [23] conducted a detailed analysis of the eigenvalues of such matrices, revealing their links to fundamental combinatorial identities. Lewis [21] presented a generalization of Pascal matrices through operator-based constructions, extending key algebraic properties and enabling new applications in combinatorics and linear algebra.

In addition to these fundamental studies, there have been significant works on generalized Pascal-type matrices. Tuğlu et al. [37] presented an analogue of the Riordan representation of Pascal matrices via Fibonomial coefficients. Kılıç [17] showed that matrices involving Fibonomial coefficients share eigenvalues with generalized Pascal matrices, highlighting a connection between Fibonacci-based combinatorics and classical Pascal matrices. Kızılaslan [19] further investigated algebraic properties of Pascal-type matrices involving double factorial binomial coefficients. Hoskins and Ponzo [16] studied a class of band matrices whose entries are defined using binomial coefficients, examining their algebraic properties. Arıkan and Kılıç [1] introduced a type of non-symmetric band matrices with Gaussian q-binomial coefficients, investigating their determinants, inverses, and LU decompositions. Dalkılıç and Koçer [6] studied circulant matrices defined by q-binomial coefficients, deriving explicit formulas for eigenvalues, norms, and determinants, bridging combinatorial analysis and matrix theory.

Max matrices, introduced by Pólya and Szegő [30], form a class of structured matrices whose entries are defined by the maximum of the corresponding row and column indices; that is, $A_{\max} = [\max(i, j)]_{i,j=1}^{n}$. A more general form is given by

$$\mathcal{A}_{\max} = \left[a_{\max(i,j)}\right]_{i,j=1}^{n},$$

where $\{a_s\}_{s=1}^{n}$ is a real sequence. These matrices and their various generalizations have been extensively studied due to their rich algebraic and combinatorial properties. Various authors have investigated their determinants, inverses, eigenvalues, exponential forms, LU decompositions, Hadamard inverses, and matrix norms [5, 17, 20, 22, 28]. Kılıç and Arıkan [18] provided explicit formulas for the LU and Cholesky decompositions, as well as for the inverses of generalized Max matrices. Stuart [36] analysed a related variant called the nested symmetric matrix, which is essentially a general form of a Max matrix, offering explicit expressions for its determinant, inverse, principal minors, and its LU and QR decompositions. Since Max matrices can be regarded as a special case of join matrices, some general results obtained by Mattila and Haukkanen [22] in the context of meet and join matrices are also applicable to Max matrices. A special type of Max matrix is called the Frank matrix and is defined by [9] as

$$F = \begin{bmatrix} n & n-1 & 0 & \cdots & 0 & 0 \\ n-1 & n-1 & n-2 & \cdots & 0 & 0 \\ n-2 & n-2 & n-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & \cdots & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}$$

whose entries are characterized by the rule

$$f_{ij} = \begin{cases} n + 1 - \max(i, j), & i > j - 2 \\ 0, & \text{otherwise.} \end{cases}$$

The Frank matrix is a well-known test matrix frequently used to evaluate eigenvalue algorithms due to its spectrum containing both well-conditioned and ill-conditioned eigenvalues

[7, 38]. Numerous structural and spectral properties of the Frank matrix F have been extensively studied [7, 12]. These properties include the facts that all eigenvalues are real, positive, and occur in reciprocal pairs, that the determinant of F equals 1, its inverse is an upper Hessenberg matrix, and that F admits an LU factorization. Moreover, the characteristic polynomial of F satisfies a second-order linear recurrence relation with specific initial conditions. Varah [38] introduced a generalization of the Frank matrix and proposed a method for accurately computing its eigenstructure. Mersin et al. [27] defined generalized Frank matrix

$$F_a = \begin{bmatrix} a_n & a_{n-1} & 0 & 0 & \cdots & 0 & 0 \\ a_{n-1} & a_{n-1} & a_{n-2} & 0 & \cdots & 0 & 0 \\ a_{n-2} & a_{n-2} & a_{n-2} & a_{n-3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_2 & a_2 & a_2 & \cdots & a_2 & a_1 \\ a_1 & a_1 & a_1 & a_1 & \cdots & a_1 & a_1 \end{bmatrix}.$$

whose entries are determined by a real number sequence $\{a_n\}$. The entries of the matrix F_a is generated by the rule

$$(f_a)_{ij} = \begin{cases} a_{n+1-\max(i,j)}, & i > j-2 \\ 0, & \text{otherwise} \end{cases}$$

where a_s's $(s = 1, 2, \ldots, n)$ represent the elements of the real sequence $\{a_n\}$. The authors presented some of its properties including determinant, characteristic polynomial and inverse. The eigenvalues of the generalized Frank matrix F_a are investigated, and the number of eigenvalues within a given interval is determined using the Sturm Theorem [25]. Shi and Kızılateş [33] and Gökbaş [11] each proposed a generalization of the Frank matrix and investigated its determinant, inverse, and various matrix norms. Higham [14] studied on bidiagonal matrices including the Frank and Pascal matrices. Mersin and Bahşi [26] studied the bounds for the maximum eigenvalues of the Fibonacci-Frank and Lucas-Frank matrices, which are obtained by choosing the Fibonacci and Lucas sequences, respectively, as the sequence $\{a_n\}$ in the generalized Frank matrix F_a. The authors also obtained some results on the Euclidean and spectral norms of these matrices. Mersin [24] introduced some new generalizations of the Fibonacci-Frank and Lucas-Frank matrices and presented some of their similar properties to the paper [25]. Polatlı [29] introduced an alternative generalization of the Frank matrix by constructing a matrix whose entries are based on the minimum of the corresponding row and column indices, with values taken from an arbitrary real sequence $\{a_n\}$. Prasad and Kumari [31] investigated the fundamental properties of the generalized Frank matrix whose entries are formed by the Mersenne number sequence. Very recently, Shi and Kızılateş [34] introduced another generalization of the Frank matrix, called the geometric r-Frank matrix, and discussed its algebraic structure.

In this study, we introduce a new generalization of the classical Frank matrix, referred to as the binomial Frank matrix, whose entries are determined by binomial coefficients. We investigate several of its fundamental properties, including the determinant, inverse, adjoint matrix, LU decomposition, and various matrix norms. This matrix structure enriches the family of special matrices by establishing connections between combinatorial identities and matrix theory.

2. Main results

Definition 2.1. *The $n \times n$ matrix*

$$F_n^* = \left[f_{ij}^*\right]_{i,j=1}^n = \begin{bmatrix} \binom{n}{n} & \binom{n}{n-1} & 0 & \dots & 0 & 0 \\ \binom{n}{n-1} & \binom{n}{n-1} & \binom{n}{n-2} & \dots & 0 & 0 \\ \binom{n}{n-2} & \binom{n}{n-2} & \binom{n}{n-2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{n}{2} & \binom{n}{2} & \binom{n}{2} & \dots & \binom{n}{2} & \binom{n}{1} \\ \binom{n}{1} & \binom{n}{1} & \binom{n}{1} & \dots & \binom{n}{1} & \binom{n}{1} \end{bmatrix},$$

whose entries are given by

$$f_{ij}^* = \begin{cases} \binom{n}{n+1-\max(i,j)}, & i > j-2 \\ 0, & \text{otherwise,} \end{cases}$$

is called the binomial Frank matrix.

Theorem 2.2. *The determinant of the binomial Frank matrix F_n^* is*

$$\det(F_n^*) = n \prod_{k=2}^{n} \left[\binom{n}{k} - \binom{n}{k-1}\right].$$

Proof. We keep the nth column fixed and replace each ith column with the difference between the ith and $(i+1)$th columns, for $i = 1, 2, \dots, n-1$. Then, the determinant of the matrix F_n^* is obtained as follows

$$\det(F_n^*) = \begin{vmatrix} \binom{n}{n} - \binom{n}{n-1} & \binom{n}{n-1} & 0 & \dots & 0 & 0 \\ 0 & \binom{n}{n-1} - \binom{n}{n-2} & \binom{n}{n-2} & \dots & 0 & 0 \\ 0 & 0 & \binom{n}{n-2} - \binom{n}{n-3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \binom{n}{2} - \binom{n}{1} & \binom{n}{1} \\ 0 & 0 & 0 & \dots & 0 & \binom{n}{1} \end{vmatrix}.$$

Thus, the well-known determinant property immediately yields the desired result. □

According to Theorem 2.2, the determinant is zero when n is odd. Therefore, when n is even, F_n^* is invertible. The following theorem provides its inverse.

Theorem 2.3. *Let n is even. Then, the inverse of the matrix F_n^* is $T_n = [t_{ij}]_{i,j=1}^n$ where its entries are characterized by*

$$t_{ij} = \begin{cases} \dfrac{1}{1-n}, & i=j=1 \\ \dfrac{\binom{n}{n+2-i}}{\left[\binom{n}{n+2-i}-\binom{n}{n+1-i}\right]\left[\binom{n}{n+1-i}-\binom{n}{n-i}\right]}, & i=j\neq 1,n \\ \dfrac{-1}{\binom{n}{n+2-i}-\binom{n}{n+1-i}}, & i=j+1 \\ 0, & i>j+1 \\ (-1)^{j-i}\prod_{k=1}^{j-i} t_{ii}\dfrac{\binom{n}{n+1-i-k}}{\left(\binom{n}{n+1-i-k}-\binom{n}{n-i-k}\right)}, & i<j\neq n \\ -t_{i,n-1}, & i<j=n \\ \dfrac{n-1}{n\,(n-3)}, & i=j=n. \end{cases}$$

Proof. The proof follows directly from the equalities $F_n^* T_n = T_n F_n^* = I_n$, where I_n is the identity matrix of order n. □

Using the results from Theorems 2.2 and 2.3 in the well-known equality $Adj\,(F_n^*) = T_n \det(F_n^*)$, we obtain the following result.

Corollary 2.4. *The adjoint matrix $A_n = [a_{ij}]_{i,j=1}^n$ of the matrix F_n^* is*

$$a_{ij} = \begin{cases} n\prod_{k=2}^{n-1}\left[\binom{n}{k}-\binom{n}{k-1}\right], & i=j=1 \\ n\binom{n}{n+2-i}\prod_{\substack{k=2\\ k\neq n+2-i\\ k\neq n+1-i}}^{n}\left[\binom{n}{k}-\binom{n}{k-1}\right], & i=j\neq 1,n \\ -n\prod_{\substack{k=2\\ k\neq n+2-i}}^{n}\left[\binom{n}{k}-\binom{n}{k-1}\right], & i=j+1 \\ 0 & i>j+1 \\ (-1)^{j-i}\prod_{k=1}^{j-i} a_{ii}\dfrac{\binom{n}{n+1-i-k}}{\left[\binom{n}{n+1-i-k}-\binom{n}{n-i-k}\right]}, & i<j\neq n \\ -a_{i,n-1}, & i<j=n \\ \dfrac{n-1}{n-3}\prod_{k=2}^{n}\left[\binom{n}{k}-\binom{n}{k-1}\right], & i=j=n \end{cases}$$

for even n.

Before presenting the LU decomposition of the matrix F_n^*, recall that it expresses a square matrix as the product of a lower triangular matrix L and an upper triangular matrix U, which facilitates the computation of matrix inverses, determinants, and solutions to linear systems.

Theorem 2.5. *The entries of the matrices $L = [l_{ij}]_{i,j=1}^n$ and $U = [u_{ij}]_{i,j=1}^n$, which form the LU decomposition of the matrix F_n^*, are characterized as follows.*

For even n

$$l_{ij} = \begin{cases} 0, & i < j \\ 1, & i = j \\ \dfrac{\dbinom{n}{n+1-i}}{\dbinom{n}{n+1-j}}, & otherwise \end{cases}$$

and

$$u_{ij} = \begin{cases} 1, & i = j = 1 \\ \dfrac{\dbinom{n}{n+1-i}\left[\dbinom{n}{n+2-i} - \dbinom{n}{n+1-i}\right]}{\dbinom{n}{n+2-i}}, & i = j \neq 1 \\ \dbinom{n}{n-i}, & i = j - 1 \\ 0, & otherwise. \end{cases}$$

For odd n

$$l_{ij} = \begin{cases} 0, & i < j \\ 1, & i = j \quad and \quad i = j - 1 \quad for \quad i > \frac{n+1}{2} \\ \dfrac{\dbinom{n}{n+1-i}}{\dbinom{n}{n+1-j}}, & otherwise \end{cases}$$

and

$$u_{ij} = \begin{cases} 1, & i = j = 1 \\ \dfrac{\dbinom{n}{n+1-i}\left[\dbinom{n}{n+2-i} - \dbinom{n}{n+1-i}\right]}{\dbinom{n}{n+2-i}}, & i = j \neq 1 \quad and \quad i \leq \frac{n+1}{2} \\ \dbinom{n}{n-i}, & i = j - 1 \\ 0, & otherwise. \end{cases}$$

Proof. The matrix multiplication $F_n^* = LU$ immediately yields the desired result. □

The following theorems present certain norms of the matrix F_n^*.

Theorem 2.6. *The matrix F_n^* has the following result for its Euclidean norm*

$$\|F_n^*\|_F = \sqrt{\sum_{i=1}^{n-1} (n-i+2)\binom{n}{i}^2 + 1}.$$

Proof. The Euclidean (Frobenius) of an $m \times n$ matrix A is defined as [15]

$$\|A\|_F = \left[\sum_{i=1}^{m}\sum_{j=1}^{n} | a_{ij} |^2\right]^{\frac{1}{2}}.$$

Considering the definition of the Euclidean norm, we obtain

$$\|F_n^*\|_F^2 = \binom{n}{n}^2 + 3\binom{n}{n-1}^2 + 4\binom{n}{n-2}^2 + \ldots + (n+1)\binom{n}{1}^2,$$

which gives the desired result. □

Theorem 2.7. *An upper bound for the spectral norm of the matrix F_n^* $(n > 3)$ is as follows*

$$\|F_n^*\|_2^2 \leq \begin{cases} \left[\left(\frac{n}{2}+1\right)\binom{n}{n/2}^2+1\right]\left[\binom{n}{n/2}^2+\frac{n}{2}\right], & n \quad \textit{is even} \\ \left[\left(\frac{n+3}{2}\right)\binom{n}{(n-1)/2}^2+1\right]\left[\binom{n}{(n+1)/2}^2+\frac{n+1}{2}\right], & n \quad \textit{is odd.} \end{cases}$$

Proof. The spectral norm of an $m \times n$ matrix A is defined as

$$\|A\|_2 = \sqrt{\max_{1\leq i\leq n} \lambda_i\left(A^H A\right)},$$

where A^H is the conjugate transpose of the matrix A and $\lambda_i\left(A^H A\right)$'s are the eigenvalues of $A^H A$ [15]. Let $B = [b_{ij}]$ and $D = [d_{ij}]$ be any $m \times n$ matrices. Then, the maximum row length norm $r_1(B)$ and the maximum column length norm $c_1(D)$ are defined by

$$r_1(B) = \max_i \sqrt{\sum_j |\, b_{ij}\,|^2} \quad \text{and} \quad c_1(D) = \max_j \sqrt{\sum_i |\, d_{ij}\,|^2},$$

respectively [15]. Moreover, if the matrix $A = [a_{ij}]$ has the equality $A = B \circ D$, then

$$\|A\|_2 \leq r_1(B)\, c_1(D),$$

where $B \circ D$ is the Hadamard product of the matrices B and D, which is defined by the rule

$$B \circ D = [b_{ij} d_{ij}]$$

[15]. Using the Hadamard product, the matrix F_n^* holds the equality

$$F_n^* = B \circ D,$$

where

$$B = \begin{bmatrix} \binom{n}{n} & 1 & 0 & \cdots & 0 & 0 \\ \binom{n}{n-1} & \binom{n}{n-1} & 1 & \cdots & 0 & 0 \\ \binom{n}{n-2} & \binom{n}{n-2} & \binom{n}{n-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{n}{2} & \binom{n}{2} & \binom{n}{2} & \cdots & \binom{n}{2} & 1 \\ \binom{n}{1} & \binom{n}{1} & \binom{n}{1} & \cdots & \binom{n}{1} & \binom{n}{1} \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & \binom{n}{n-1} & 0 & \cdots & 0 & 0 \\ 1 & 1 & \binom{n}{n-2} & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & \binom{n}{1} \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}.$$

Based on the properties of the binomial function, we derive the following results. Let n be an even number. Then, the largest binomial occurs at $\binom{n}{n/2}$. Thus, the maximum row length norm of B is

$$r_1(B) = \sqrt{\left(\frac{n}{2}+1\right)\binom{n}{n/2}^2+1}$$

and maximum column length norm of D is

$$c_1(D) = \sqrt{\binom{n}{n/2}^2+\frac{n}{2}}.$$

For odd n, the largest values are obtained at $\binom{n}{(n-1)/2}$ and $\binom{n}{(n+1)/2}$, which are equal and represent the central values. Then, we have

$$r_1(B) = \sqrt{\left(\frac{n+1}{2}+1\right)\binom{n}{(n+1)/2}^2 + 1}$$

and

$$c_1(D) = \sqrt{\binom{n}{(n+1)/2}^2 + \frac{n+1}{2}},$$

which complete the proof. □

Next, we give an example to illustrate our results.

Example 2.8. *For the even case, let $n = 4$. Then, we have*

$$F_4^* = \begin{bmatrix} \binom{4}{4} & \binom{4}{3} & 0 & 0 \\ \binom{4}{3} & \binom{4}{3} & \binom{4}{2} & 0 \\ \binom{4}{2} & \binom{4}{2} & \binom{4}{2} & \binom{4}{1} \\ \binom{4}{1} & \binom{4}{1} & \binom{4}{1} & \binom{4}{1} \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 4 & 4 & 6 & 0 \\ 6 & 6 & 6 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}.$$

The determinant of the F_4^* is

$$\det(F_4^*) = 4\prod_{j=2}^{4}\left[\binom{4}{j} - \binom{4}{j-1}\right] = 48.$$

Thus, Theorem 2.2 is satisfied. The inverse of the matrix F_4^* is

$$\begin{bmatrix} -\frac{1}{3} & -\frac{2}{3} & 2 & -2 \\ \frac{1}{3} & \frac{1}{6} & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 & 1 \\ 0 & 0 & -\frac{1}{2} & \frac{3}{4} \end{bmatrix},$$

which holds Theorem 2.3. The matrices $L = [l_{ij}]_{i,j=1}^{4}$ and $U = [u_{ij}]_{i,j=1}^{4}$, which are the components of the LU decomposition of the matrix F_4^*, are

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 6 & \frac{3}{2} & 1 & 0 \\ 4 & 1 & \frac{2}{3} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & \frac{4}{3} \end{bmatrix}.$$

It is clear that the entries l_{ij} and u_{ij} are satisfy the formulas given in Theorem 2.5. The Euclidean norm of the matrix F_4^* is

$$\|F_4^*\|_F = \sqrt{\sum_{i=1}^{3}(4-i+2)\binom{4}{i}^2 + 1} = \sqrt{273} = 16.5227,$$

which is consistent with Theorem 2.6. The spectral norm of F_4^* is

$$\|F_4^*\|_2 = 15.8702.$$

The matrix F_4^* *can be written as*

$$F_4^* = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 4 & 4 & 1 & 0 \\ 6 & 6 & 6 & 1 \\ 4 & 4 & 4 & 4 \end{bmatrix}}_{B} \circ \underbrace{\begin{bmatrix} 1 & 4 & 0 & 0 \\ 1 & 1 & 6 & 0 \\ 1 & 1 & 1 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{D}.$$

Then, considering Theorem 2.7, the maximum row length norm of the matrix B *is*

$$r_1(B) = \sqrt{\left(\frac{4}{2}+1\right)\binom{4}{2}^2 + 1} = \sqrt{109}$$

and maximum column length norm of the matrix D *is*

$$c_1(D) = \sqrt{\binom{4}{2}^2 + \frac{4}{2}} = \sqrt{38}.$$

Hence, the spectral norm of F_4^* *has the following upper bound*

$$\|F_4^*\|_2 = 15.8702 \le r_1(B)\, c_1(D) = \sqrt{(109)(38)} = 64.3584.$$

To illustrate the odd dimension, consider $n = 5$*. Then,*

$$F_5^* = \begin{bmatrix} \binom{5}{5} & \binom{5}{4} & 0 & 0 & 0 \\ \binom{5}{4} & \binom{5}{4} & \binom{5}{3} & 0 & 0 \\ \binom{5}{3} & \binom{5}{3} & \binom{5}{3} & \binom{5}{2} & 0 \\ \binom{5}{2} & \binom{5}{2} & \binom{5}{2} & \binom{5}{2} & \binom{5}{1} \\ \binom{5}{1} & \binom{5}{1} & \binom{5}{1} & \binom{5}{1} & \binom{5}{1} \end{bmatrix} = \begin{bmatrix} 1 & 5 & 0 & 0 & 0 \\ 5 & 5 & 10 & 0 & 0 \\ 10 & 10 & 10 & 10 & 0 \\ 10 & 10 & 10 & 10 & 5 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}.$$

The determinant of the matrix

$$\det(F_5^*) = 0,$$

which is consistent with Theorem 2.2. Therefore, its inverse does not exist. The matrices $L = [l_{ij}]_{i,j=1}^5$ *and* $U = [u_{ij}]_{i,j=1}^5$*, which constitute the LU decomposition of the matrix* F_4^**, are given by*

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 \\ 10 & 2 & 1 & 0 & 0 \\ 10 & 2 & 1 & 1 & 0 \\ 5 & 1 & \frac{1}{2} & 1 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 5 & 0 & 0 & 0 \\ 0 & -20 & 10 & 0 & 0 \\ 0 & 0 & -10 & 10 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is clear that the entries l_{ij} *and* u_{ij} *satisfy the formulas provided in Theorem 2.5. The Euclidean norm of the matrix* F_5^* *is*

$$\|F_5^*\|_F = \sqrt{\sum_{i=1}^{4}(5-i+2)\binom{5}{i}^2 + 1} = \sqrt{1126} = 33.5559$$

as predicted by Theorem 2.6. The spectral norm of F_5^* *is calculated as*

$$\|F_5^*\|_2 = 32.2027.$$

To get a bound for the spectral norm of F_5^**, we use the Hadamard product*

$$F_5^* = B' \circ D',$$

where

$$B' = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 5 & 5 & 1 & 0 & 0 \\ 10 & 10 & 10 & 1 & 0 \\ 10 & 10 & 10 & 10 & 1 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}, \quad D' = \begin{bmatrix} 1 & 5 & 0 & 0 & 0 \\ 1 & 1 & 10 & 0 & 0 \\ 1 & 1 & 1 & 10 & 0 \\ 1 & 1 & 1 & 1 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then, using Theorem 2.7, we have

$$r_1(B') = \sqrt{\left(\frac{5+1}{2}+1\right)\left(\frac{5}{(5+1)/2}\right)^2+1} = \sqrt{401}$$

and

$$c_1(D') = \sqrt{\left(\frac{5}{(5+1)/2}\right)^2+\frac{5+1}{2}} = \sqrt{103},$$

which yield the upper bound

$$\|F_5^*\|_2 \leq r_1(B')c_1(D') = \sqrt{(401)(103)} = 203.2314.$$

Example 2.8 demonstrates the correspondence between the computed results and the principal conclusions established in this study, thereby supporting the validity of the theoretical results developed.

3. Conclusion

The present paper introduced a new generalization of the Frank matrix, called the binomial Frank matrix, and investigated several of its key properties, including the determinant, inverse, LU decomposition, and various matrix norms. The results contribute to the existing literature by offering a novel perspective on extending Frank matrices and deepening the understanding of their algebraic and analytical characteristics.

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A constructive definition of the Riemann Integral on a separable Banach Space $\mathcal{B}$ and its relationship to the Lebesgue Integral on $\mathcal{B}$, Part I

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ABSTRACT. The goal of this paper is to construct a Riemann integral on a separable Banach space which possesses all of the fundamental properties of the Riemann integral on $\mathbb{R}^n$. Let $\mathcal{B}$ represent a separable Banach space. The paper [9] presents a proof that $\mathcal{B}$ has an isomorphic, isometric embedding in $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \cdots$. In this work we will use this embedding to define a Riemann integral on special subsets of $\mathcal{B}$. Similar to the Lebesgue integral on $\mathcal{B}$ in [9], this Riemann integral has the advantage of equaling a limit of Riemann integrals on $\mathbb{R}^n$ as $n \to \infty$, thereby making this Riemann integral constructive and computationally implementable. Throughout this paper we will study how the Riemann and Lebesgue integrals on special sets in $\mathcal{B}$ are related. This comparison will demonstrate that the Riemann integral on $\mathcal{B}$ advances the Lebesgue theory on $\mathcal{B}$ slightly by being useful in defining the volume of some bounded subsets of $\mathcal{B}$ that the Lebesgue integral on $\mathcal{B}$ cannot represent.

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1. INTRODUCTION

Since 1933, mathematicians have constructed various versions of the Lebesgue integral on an infinite dimensional Banach space. These constructions include the early works of Harr [12], Weil [26], Banach [4], and Cartan [6]; and more recently the constructions of Elliott and Morse [8], Yamasaki [27], Baker [2, 3], Gill and Pantsulaia and Zachary [10], and Gill and Myers [9]. Although these aforementioned constructions indirectly include functions that one would expect be Riemann integrable, to date, there is no *direct* construction of the Riemann integral on an infinite dimensional space. This paper will fill this historical void.

In 1990, P. Muldowney [19] and R. Henstock [13] constructed a version of the generalized Riemann (HK) integral on infinite dimensional spaces. Although their formulation theoretically includes a Riemann integral as a special case, here we will adopt a different approach by using the same functions and special subsets of $\mathcal{B} \hookrightarrow \mathbb{R}^\infty$ that are used to define the Lebesgue integral on $\mathcal{B}$ in [9].

The construction presented here offers several advantages. First, the Riemann integral advances the Lebesgue theory slightly by representing the volume of an admissible subset A of $\mathcal{B}$ as the Riemann integral of χ_A when the Lebesgue integral of this singleton disagrees in value with its Riemann counterpart, and thus fails to represent the volume of A.

Second, because the Riemann integral on $\mathcal{B}$ equals a limit of Riemann integrals on $\mathbb{R}^n$ as $n \to \infty$, this limit provides a method for computation and error estimation, which we will demonstrate with an example. To highlight the practicality of this convergence, we will exhibit its ability to evaluate and also approximate an integral of a function with infinitely many variables. Riemann integrals inferred by the aforementioned constructions are not as directly computational. By using tame functions, this limit readily extends key theorems from the Riemann integral on $\mathbb{R}^n$ to the Riemann integral on $\mathcal{B}$.

Third, the Riemann integral on $\mathcal{B}$ has the pedagogical advantage of making the constructive theory on $\mathcal{B}$ more accessible to students who are not yet familiar with the Lebesgue integral, such as first year graduate students and undergraduate math majors with an understanding of real and functional analysis. Since large classes of functions that are important in physics and engineering are continuous on $\mathbb{R}^n$, their analogues on the infinite-dimensional space $\mathcal{B}$ will be accessible to beginning students in these fields since, as we will prove, a function is Riemann integrable on any special rectangle in $\mathcal{B}$ and its admissible subsets if it continuous a.e. there. Such readers will find the ideas presented here easy to grasp since most of the tools needed are already available from studying the Riemann integral on $\mathbb{R}^n$; and because the Riemann integral on $\mathcal{B}$ is defined as a limit of finite dimensional Riemann integrals, this limit extends all of the definitions and properties from the familiar Riemann integral on $\mathbb{R}^n$ to this integral on $\mathcal{B}$. In particular, the proof that a continuous function on a special rectangle in $\mathcal{B}$ is Riemann integrable holds precisely because this result holds in finite dimensions, and the proof follows in several steps from the familiar approximation property by computing a limit of upper and lower Darboux sums as the dimension approaches infinity. Like the Riemann integral on $\mathcal{B}$, the definitions and properties used to define it all approximate from below in this way.

Moreover, the construction presented here will initially proceed in a manner that is virtually identical to the integral constructions of Riemann's and Darboux's integrals in finite dimensions, and will show that their formulations are equivalent on $\mathcal{B}$. Introducing undergraduate students to the Riemann integral on $\mathcal{B}$ can provide them with a foundation that is helpful for understanding the construction of Lebesgue theory on $\mathcal{B}$ after completing a standard graduate course in real analysis.

2. PRELIMINARIES

2.1. **Separable Banach Spaces.** Let $\mathcal{B}^*$ denote the continuous dual space of the separable Banach space $\mathcal{B}$. A sequence $\{e_n\}_{n=1}^{\infty} \subset \mathcal{B}$ is called a Schauder basis (S-basis) for $\mathcal{B}$ if $\|e_n\|_{\mathcal{B}} = 1$ and for each $x \in \mathcal{B}$ there is a unique sequence of real scalars $\{x_n\}_{n=1}^{\infty}$ such that

$$x = \lim_{n\to\infty} \sum_{k=1}^{n} x_k e_k = \sum_{n=1}^{\infty} x_n e_n \tag{1}$$

where the convergence is with respect to the norm $\|\cdot\|_{\mathcal{B}}$, which implies that $\lim_{n\to\infty} x_n = 0$. Gill and Zachary used the property (1) to construct a Lebesgue measure on any Banach space with a S-basis. Starting with Banach in 1932 [4], mathematicians explored the conditions necessary for a Banach space to have a S-basis. In 1987, Szarek answered this question by demonstrating that the bounded approximation property need not imply the existence of a S-basis [23]. Consequently, a separable Banach space need not have a S-basis, so that the approach of [10] does not in general hold for such spaces.

In 1965, Gross [11] proved that every separable Banach space contains a separable Hilbert space, called a Cameron-Martin space, as a continuous dense embedding; in order to generalize Wiener's measure theory, by using the densely embedded Sobolev space $\mathbb{H}_0^1[0,1] \subset \mathcal{C}_0^1[0,1]$. In 1970, Kuelbs [16] extended Gross' inclusion to the rigging

$$\mathbb{H}_0^1[0,1] \subset \mathcal{C}_0^1[0,1] \subset L^2[0,1]. \tag{2}$$

Gill and Myers [9] used Kuelbs Lemma, the generalization of the second inclusion in (2), to extend the Lebesgue theory of [10] to any separable Banach space. Here is a proof of this lemma as presented in [9].

Theorem 2.1 (Kuelbs Lemma)**.** *Any separable Banach space $\mathcal{B}$ can be contained in a Hilbert space $\mathcal{H}$ as a continuous dense embedding.*

Proof. Since any subset of $\mathcal{B}$ as a metric space is separable [20], let $\{\mathcal{E}_n\}_{n=1}^{\infty}$ be a countable dense sequence on the unit sphere of $\mathcal{B}$, and let $\{\mathcal{E}_n^*\}_{n=1}^{\infty}$ be any fixed set of corresponding duality mappings (i.e., for each n, $\mathcal{E}_n^* \in \mathcal{J}(\mathcal{E}_n) \subset \mathcal{B}^*$ and $\mathcal{E}_n^*(\mathcal{E}_n) = \langle \mathcal{E}_n, \mathcal{E}_n^* \rangle = \|\mathcal{E}_n\|_{\mathcal{B}}^2 = \|\mathcal{E}_n^*\|_{\mathcal{B}^*}^2 = 1$). For each n, let $t_n = \frac{1}{2^n}$, and define $(u,v)_2$ as follows ($\bar{\mathcal{E}_n}$ is the complex conjugate):

$$(u,v)_2 = \sum\nolimits_{n=1}^{\infty} \tfrac{1}{2^n} \mathcal{E}_n^*(u) \bar{\mathcal{E}_n}^*(v).$$

It is clear that $(u, v)_2$ is an inner product on $\mathcal{B}$. Let $\mathcal{H}$ be the completion of $\mathcal{B}$ with respect to this inner product. It is clear that $\mathcal{B}$ is dense in $\mathcal{H}$, and

$$\|u\|_{\mathcal{H}_2}^2 = \sum_{n=1}^{\infty} \tfrac{1}{2^n} |\mathcal{E}_n^*(u)|^2 \le \sup_{n\in\mathbb{N}} |\mathcal{E}_n^*(u)|^2 = \|u\|_{\mathcal{B}}^2 ,$$

so the embedding is continuous. □

We will henceforth assume that $\mathcal{H}$ is the (infinite dimensional) Hilbert space prescribed by Kuelbs Lemma for which $\mathcal{B} \subset \mathcal{H}$, and that $\{e_n\}_{n=1}^{\infty} \subset \mathcal{H}$ is a S-basis on $\mathcal{H}$ which is complete and orthonormal. For each $x \in \mathcal{H}$ there is a unique sequence $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ for which

$$x = \sum_{n=1}^{\infty} x_n e_n \quad \text{such that} \quad \|x\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} x_n^2 < \infty. \tag{3}$$

By the inclusion of $\mathcal{B}$ in $\mathcal{H}$, every $x \in \mathcal{B}$ also has the representation (3), thereby bestowing the uniqueness of a S-basis representation to each element in a separable Banach space even if there is no such basis for this space.

2.2. **The Lebesgue theory on $\mathcal{B}$.** Here we will summarize the theory of the Lebesgue integral in [9], henceforth referred to as the constructive Lebesgue integral, in order to relate to it the constructive Riemann integral that we will develop in this work.

2.2.1. *Embedding $\mathcal{B}$ in $\mathbb{R}^\infty$ with Special Subsets of $\mathbb{R}^\infty$.* The following Definition 2.2 and Theorem 2.3 embed $\mathcal{B}$ in $\mathbb{R}^\infty$, which is the key to developing the Lebesgue measure and integral on $\mathcal{B}$.

Definition 2.2. *Let $\{a_k\}_{k=1}^{\infty} \subset \mathbb{R}$ such that $1 \ge a_k \ge a_{k+1} > 0$, $\lim_{k\to\infty} a_k = 0$, and $\sum_{k=1}^{\infty} a_k$ converges.*

(i) $J_k = [-a_k, a_k]$, *note that* $J_k \supset J_{k+1} \supset \ ... \supset \{0\}$ *and* $\lim_{k\to\infty} \lambda(J_k) = 0$.

(ii) $J^n = \prod_{k=n+1}^{\infty} J_k$.

(iii) $\mathcal{B}_J = \{(x_1, x_2, ...) \mid x_1e_1 + x_2e_2 + ... \in \mathcal{B}\}$, *equipped with the norm* $\|(x_1, x_2, ...)\|_{\mathcal{B}_J} = \|x_1e_1 + x_2e_2 + \cdots\|_{\mathcal{B}}$.

(iv) $\mathcal{B}_J^n = \mathbb{R}^n \times J^n$, *and* $\mathcal{B}_J^\infty = \bigcup_{n=1}^{\infty} \mathcal{B}_J^n$. *We shall refer to a set of the form $A_n \times J^n$ as an nth unit set. By construction, $\mathcal{B}_J^\infty \subset \mathcal{B}_J$.*

Theorem 2.3. *Define the operator $T : \mathcal{B} \to \mathcal{B}_J$ by $T(x) = (x_k)$ for $x = \sum_{k=1}^{\infty} x_k e_k$. Then in fact*

$$T : (\mathcal{B}, \|\cdot\|_{\mathcal{H}}) \to (\mathcal{B}_J, ||\cdot||_{\mathcal{B}_J})$$

is an isometric isomorphism from $\mathcal{B}$ onto $\mathcal{B}_J$, which therefore establishes $(\mathcal{B}_J, ||\cdot||_{\mathcal{B}_J})$ as a Banach space in $\mathbb{R}^\infty$.

Because $\mathcal{B}_J$ is the image of $\mathcal{B}$ in $\mathbb{R}^\infty$ under the isometric isomorphism T, we will frequently identify $\mathcal{B} \equiv \mathcal{B}_J$ and tacitly assume that $\mathcal{B}$ is this canonical image under T.

The following subsets of $\mathbb{R}^\infty$ will be useful for defining the Riemann integral on a special rectangle in $\mathcal{B}$.

Definition 2.4.

(i) $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $I^n = \prod_{i=n+1}^{\infty} I$, $I_1^n = \prod_{k=1}^{n} I$.

(ii) $\mathbb{R}_I^n = \mathbb{R}^n \times I^n$.

(iii) $\mathbb{R}_I^{\prime,\infty} = \bigcup_{n=1}^{\infty} \mathbb{R}_I^n$.

Since $\lim_{n\to\infty} x_n = 0$ for any $x = (x_1, x_2, ...) \in \mathcal{B}$, we can contain $\mathcal{B}$ in $\mathbb{R}_I^{\prime,\infty}$.

2.2.2. *Topological Structures on $\mathcal{B}$.* To study continuity on $\mathcal{B}$, we will use the following topologies.

Definition 2.5.

(i) $\mathcal{T}_{\mathbb{R}^n}$ *will denote the usual topology on $\mathbb{R}^n$, and in particular, $\mathcal{T}_{\mathbb{R}}$ will denote the usual topology on R.*

(ii) $\mathcal{T}_n = \{V : V = U \times I^n,\ U$ *is open in* $\mathbb{R}^n\}$.

(iii) $\mathcal{T}_I$ *is the topology on* $\mathbb{R}_I^{\prime,\infty} = \bigcup_{n=1}^{\infty} \mathbb{R}_I^n$ *generated by the subbase* $\bigcup_{n=1}^{\infty} \mathcal{T}_n$.

(iv) *Note that since $\mathcal{B} \subset \mathbb{R}_I^{\prime,\infty}$, we will equip $\mathcal{B}$ with the subspace topology of $\mathcal{T}_I$ on $\mathcal{B}$ and denote it as $\mathcal{T}_{\mathcal{B}}$.*

2.2.3. *The Lebesgue Measure on $\mathcal{B}$.* Here we will show how Definition 2.2 and Theorem 2.3 provide a context for defining a theory of measure that approximates from below. In [10], Gill and Zachary defined a Lebesgue measure λ_∞ on $\mathbb{R}^\infty$ that coincides with the intuitive notion of volume. Because $\lambda_\infty(\mathcal{B}_J^n) = 0$, it follows by sub-additivity that $\lambda_\infty(\mathcal{B}_J^\infty) = 0$ and therefore, as explained in [10], $\lambda_\infty(\mathcal{B}_J) = 0$; so it is not possible to construct a non-trivial Lebesgue measure $\lambda_\mathcal{B}$ on $\mathcal{B}$ directly from λ_∞. Instead, Gill and Myers defined $\lambda_\mathcal{B}$ by using a method due to Yamasaki [27].

To avoid trivializing $\lambda_\mathcal{B}$, they defined it as an increasing limit of scaled measures ν_J^n on n-th unit sets, each of which equals scaled n-dimensional Lebesgue measure λ_n in (iii) of the following definition [9].

Definition 2.6.

(i) *$\mathscr{B}[\mathcal{B}_J]$ will denote the Borel algebra on $\mathcal{B}_J$ generated by the open sets in the topology on $\mathcal{B}_J$.*

(ii) *$\mathscr{B}[\mathcal{B}] = \{T^{-1}(B) \mid B \in \mathscr{B}[\mathcal{B}_J]\}$ is the Borel algebra on $\mathcal{B}$, which is generated by the initial topology on $\mathcal{B}$: $\tau_\mathcal{B} = \{T^{-1}(G) \mid G \in \tau_J\}$.*

(iii) *Define $\bar{\nu}_k$, $\bar{\mu}_k$ on $B \in \mathscr{B}[\mathbb{R}]$ by*

$$\bar{\nu}_k(B) = \frac{\lambda(B)}{\lambda(J_k)}, \quad \bar{\mu}_k(B) = \frac{\lambda(B \cap J_k)}{\lambda(J_k)}.$$

Let B denote an elementary set in $\mathcal{B}_J$, such that $B = \left(\prod_{k=1}^\infty B_k\right) \cap \mathcal{B}_J$, where each $B_k \in \mathscr{B}[\mathbb{R}]$. Define $\bar{\nu}_J^n$ for such a set B by:

$$\bar{\nu}_J^n(B) = \prod_{k=1}^n \bar{\nu}_k(B_k) \times \prod_{k=n+1}^\infty \bar{\mu}_k(B_k). \tag{4}$$

(iv) *By Carathéodory's Theorem [24], each $\bar{\nu}_J^n$ extends to a measure ν_J^n on $(\mathcal{B}_J, \mathscr{B}[\mathcal{B}_J])$ with the property that $\{\nu_J^n\}_{n=1}^\infty$ is increasing and for $A_n \in \mathscr{B}[\mathbb{R}^n]$, $B_k \in \mathscr{B}[\mathbb{R}]$,*

$$\nu_J^n\left(\left(A_n \times \prod_{k=n+1}^\infty B_k\right) \cap \mathcal{B}_J\right) = \frac{\lambda_n(A_n)}{\prod_{k=1}^n \lambda(J_k)} \times \prod_{k=n+1}^\infty \bar{\mu}_k(B_k).$$

(v) *$\nu_J = \lim_{n\to\infty} \nu_J^n$ is a measure on $(\mathcal{B}_J, \mathscr{B}[\mathcal{B}_J])$ such that $\nu_J(\mathcal{B}_J \setminus \mathcal{B}_J^\infty) = 0$; thus ν_J is concentrated on $\mathcal{B}_J$. Also, $\nu_J(A) = \nu_J^n(A)$ if $A \subset \mathcal{B}_J^n$ and $A \in \mathscr{B}(\mathcal{B})$.*

(vi) *The Lebesgue measure $\lambda_\mathcal{B}$ on $(\mathcal{B}, \mathscr{B}[\mathcal{B}])$, is defined by*

$$\lambda_\mathcal{B}(A) = \nu_J(T(A)) \text{ for every } A \in \mathscr{B}[\mathcal{B}].$$

Remark 1. *Because the isometric isomorphism T is also measure-preserving, we will henceforth identify $\lambda_\mathcal{B} \equiv \nu_J$.*

As mentioned in the introduction, tame functions will be the main tool that will extend the properties of the Riemann integrals from $\mathbb{R}^n$ to $\mathcal{B}$, and will enable us to relate it to the constructive Lebesgue integral on $\mathcal{B}$. We define tame functions as follows.

2.2.4. *Measurable Functions on $\mathbb{R}^\infty$.* Since $\mathcal{B}_J \subset \mathbb{R}^\infty$, Gill and Myers defined measurable functions on $\mathcal{B}_J$ by using special functions on $\mathbb{R}^\infty$ as follows [9]. Part (iii) is the key to defining the Lebesgue integral on $\mathcal{B}$ as a limit.

Definition 2.7.

(i) *Let $x = (x_1, x_2, \cdots) \in \mathbb{R}^\infty$, $h = \chi_I$ where $I = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $h_n = \bigotimes_{k=n+1}^\infty h(x_k)$.*

(ii) *$\mathcal{M}^n$ will denote the class of Lebesgue measurable, real valued functions on $\mathbb{R}^n$.*

(iii) *$\mathcal{M}_I^n = \{s_n : s_n(x) = \bar{s}_n(x_1, \cdots, x_n) \otimes h_n,\ x = (x_1, x_2, \cdots) \in \mathbb{R}^\infty$ and $\bar{s}_n \in \mathcal{M}^n\}$. We will call such a function s_n a tame function.*

(iv) *A function $f : \mathcal{B} \to (-\infty, \infty)$ is $\mathscr{B}[\mathcal{B}]$-measurable if there is a sequence of tame functions $\{s_n\}$ such that $s_n \in \mathcal{M}_I^n$ for which $\lim_{n\to\infty} s_n(x) = f(x)$ for $\lambda_\mathcal{B}$ a.e. $x \in \mathcal{B}$.*

By the following theorem, the formulation of measurable function in Definition 2.7 coincides with the classical definition of measurable function as in [21].

Theorem 2.8. *A function $f : \mathcal{B} \to (-\infty, \infty)$ is $\mathscr{B}[\mathcal{B}]$-measurable if and only if for every $A \in \mathscr{B}[\mathbb{R}]$, $f^{-1}(A) \in \mathscr{B}[\mathcal{B}]$.*

2.2.5. *The Lebesgue Integral on $\mathcal{B}$.* The link between ν_J^n and λ_n in (iii) of Definition 2.6 yields the second equality in (5) for any tame function $s_n \in \mathcal{M}_I^n$ which will serve as a bridge between the Lebesgue integrals on $\mathcal{B}$ and $\mathbb{R}^n$.

$$(5)\qquad \int_{\mathcal{B}_J}\left[s_n\chi_{\mathcal{B}_J^n}\prod_{k=1}^{n}\lambda(J_k)\right]d\nu_J = \int_{\mathcal{B}_J^n}\left[s_n\prod_{k=1}^{n}\lambda(J_k)\right]d\nu_J^n = \int_{\mathbb{R}^n}\bar{s}_n d\lambda_n.$$

Note that the product $\prod_{k=1}^{n}\lambda(J_k)$ which scales the tame function s_n in the integrand of (5) effectively cancels the same product that occurs in the denominator of the measure in the first product for ν_J^n in (4). In fact, this scaling of the integrand actually converts the integral to one that is based on the measure λ_∞ for the larger space $\mathbb{R}^\infty$ (See [9] and [10]).

Furthermore, if a sequence of non-negative tame functions $\{s_n\}_{n=1}^{\infty}$ is increasing, the sequence of their integrals in the left member of (5) is also increasing, so that the following limit exists.

$$(6)\qquad \lim_{m\to\infty}\int_{\mathcal{B}_J}\left[s_m\chi_{\mathcal{B}_J^m}\prod_{k=1}^{m}\lambda(J_k)\right]d\nu_J \ \text{ exists.}$$

Equation (5) and statement (6) will motivate the ensuing definition of the Lebesgue integral on $\mathcal{B}$. To insure that this is well-defined and is not generally trivial on n-th unit sets we first define it for non-negative measurable functions on $\mathcal{B}$ as follows. Note that in order to obtain the result in (5) we will need to integrate over $\mathcal{B}_J^n$. Since the function f in $\mathcal{M}$ only needs to dominate on this set, we will scale each tame function by $\chi_{\mathcal{B}_J^n}$, and thus require that $s_n\chi_{\mathcal{B}_J^n} \le f$.

Definition 2.9. *Let $f : \mathcal{B} \to [0,\infty)$ be $\mathscr{B}[\mathcal{B}]$- measurable, and let* $\mathrm{Spt}(f) = \{f(x) \mid f(x) \neq 0\}$. *We define the Lebesgue integral of f over $\mathcal{B}$ by*

$$(7)\qquad \int_{\mathcal{B}} f d\lambda_{\mathcal{B}} = \sup\left\{\int_{\mathcal{B}_J}\left[s_n\chi_{\mathcal{B}_J^n}\prod_{k=1}^{n}\lambda(J_k)\right]d\nu_J \;\middle|\; 0 \le s_n\chi_{\mathcal{B}_J^n} \le f,\ s_n \in \mathcal{M}_I^n\right\}.$$

Theorem 2.10 is the key that expresses the integral on $\mathcal{B}$ as limit of Lebesgue integrals on $\mathbb{R}^n$.

Theorem 2.10. *Let $f : \mathcal{B} \to [0,\infty)$ be $\mathscr{B}[\mathcal{B}]$-measurable. Then there is a $\mathscr{B}[\mathcal{B}]$-measurable real-valued function $g = \lim_{n\to\infty} s_n$ with $\{s_n \in \mathcal{M}_I^n\}_{n=1}^{\infty}$ increasing, such that $f \le g$, which has the property that*

$$(8)\qquad \int_{\mathcal{B}} f d\lambda_{\mathcal{B}} = \int_{\mathcal{B}} g d\lambda_{\mathcal{B}} = \lim_{n\to\infty}\int_{\mathcal{B}_J}\left[s_n\prod_{k=1}^{n}\lambda(J_k)\right]d\nu_J = \lim_{n\to\infty}\int_{\mathbb{R}^n}\bar{s}_n d\lambda_n.$$

As a consequence of Theorem 2.10, we will tacitly assume that for any for any non-negative $f \in \mathcal{M}$ there is an increasing sequence $\{s_n\}$ of tame functions for which $\lim_{n\to\infty} s_n(x) = f(x)$ for $\lambda_{\mathcal{B}}$ a.e. $x \in \mathcal{B}$.

Finally, we will define the Lebesgue integral for a non-negative measurable function on $\mathcal{B}$ computationally as follows.

Definition 2.11. *Let $f : \mathcal{B} \to [0,\infty)$ be $\mathscr{B}[\mathcal{B}]$-measurable, and let $\{s_n\}_{n=1}^{\infty}$ be an increasing sequence of non-negative tame functions for which $\lim_{n\to\infty} s_n = f$ a.e. on $\mathcal{B}$. Then the Lebesgue integral of f on $\mathcal{B}$ is defined by*

$$\int_{\mathcal{B}} f\ d\lambda_{\mathcal{B}} = \lim_{n\to\infty}\int_{\mathcal{B}} s_n\ d\lambda_{\mathcal{B}} = \lim_{n\to\infty}\int_{\mathbb{R}^n} \bar{s}_n\ d\lambda_{\mathcal{B}}.$$

Note that Definition 2.9 insures that Definition 2.11 of the Lebesgue integral of $\mathcal{B}$ is well-defined for any sequence of non-negative tame functions converging to f a.e. on $\mathcal{B}$.

Definition 2.12. *For real-valued $f \in \mathcal{M}$ we know that $f_+, f_- \in \mathcal{M}$. Hence we may define*

$$(9)\qquad \int_{\mathcal{B}} f(x)d\lambda_{\mathcal{B}}(x) = \int_{\mathcal{B}} f_+(x)d\lambda_{\mathcal{B}}(x) - \int_{\mathcal{B}} f_-(x)d\lambda_{\mathcal{B}}(x)$$

provided that at most one of the integrals in (9) *is infinite. We say that f is integrable whenever both integrals on the right side of* (9) *are finite We define $L^1[\mathcal{B}]$ to be the class of functions in $\mathcal{M}$ satisfying $\int_{\mathcal{B}} |f| d\lambda_{\mathcal{B}} < \infty$. It follows that $f \in L^1[\mathcal{B}]$ if and only if* (9) *is finite.*

Note that by definition, any tame function $s_n \in L^1[\mathcal{B}]$.

Remark 2. *If $f \in L^1[\mathcal{B}]$, then since each of f_+ and f_- are in $\mathcal{M}^+$,* Theorem 2.10 *insures that there are functions $g_1, g_2 \in \mathcal{M}^+$ for which $g_1 = \lim_{n\to\infty}(s_1)_n$ and $g_2 = \lim_{n\to\infty}(s_2)_n$ where $\left\{(s_1)_n \in \mathcal{M}_I^{n,+}\right\}_{n=1}^{\infty}$ and $\left\{(s_2)_n \in \mathcal{M}_I^{n,+}\right\}_{n=1}^{\infty}$ are increasing and satisfy $f_- \leq g_1$, $f_+ \leq g_2$, and*

$$\int_{\mathcal{B}} f_+ d\lambda_{\mathcal{B}} = \int_{\mathcal{B}} g_1 d\lambda_{\mathcal{B}} \ , \ \int_{\mathcal{B}} f_- d\lambda_{\mathcal{B}} = \int_{\mathcal{B}} g_1 d\lambda_{\mathcal{B}}$$

so that

$$\int_{\mathcal{B}} f(x) d\lambda_{\mathcal{B}}(x) = \int_{\mathcal{B}} g_1 d\lambda_{\mathcal{B}} - \int_{\mathcal{B}} g_1 d\lambda_{\mathcal{B}}.$$

Remark 2 implies the following computational analogue to Definition 2.11 for any $f \in L^1[\mathcal{B}]$.

Theorem 2.13. *Let $f \in L^1[\mathcal{B}]$. Then there is a sequence $\{f_n\}$ of tame functions for which $\lim_{n\to\infty} f_n(x) = f(x)$ for $\lambda_{\mathcal{B}}$ a.e. $x \in \mathcal{B}$, $|f_n(x)| \leq |f(x)|$, and*

$$\int_{\mathcal{B}} f d\lambda_{\mathcal{B}}(x) = \lim_{n\to\infty} \int_{\mathcal{B}} f_n d\lambda_{\mathcal{B}} = \lim_{n\to\infty} \int_{\mathbb{R}^n} \bar{f}_n d\lambda_n.$$

As a consequence of Definition 2.11 and Theorem 2.13 many of the properties for the Lebesgue integral on $\mathbb{R}^n$, such as linearity and Lebesgue's Dominated and Monotone Convergence Theorems, will automatically hold for the Lebesgue integral on $\mathcal{B}$.

2.2.6. *Null Sets and Integrals on $\mathbb{R}^n$ and $\mathcal{B}$.* Similar to the Lebesgue theory on $\mathbb{R}^n$, the Lebesgue integral of a Lebesgue measurable function on a null set in $\mathcal{B}$ is zero.

Let R_n be a rectangle in $\mathbb{R}^n$ with sides parallel to the coordinate axes. A non-negative Lebesgue measurable function f that satisfies $\int_R f \, d\lambda_n = 0$ has the property that $f = 0$ a.e. relative to λ_n. By contrast, a zero valued Lebesgue integral on a rectangle R in $\mathcal{B}$ can have a non-negative integrand f that can be strictly positive on all of R. In essence, this can occur when f is sufficiently bounded above by a small enough bound b that this integral has a value less than $\prod_{k=1}^{\infty} b = 0$, whereas in finite dimensions the bound would be of the form $\prod_{k=1}^{m} b > 0$. Consider the following counter-example.

Counter Example 2.14. *Let $R = I^0 \cap \mathcal{B}_J^{\infty} = \left(\prod_{k=1}^{\infty} \left[-\frac{1}{2}, \frac{1}{2}\right]\right) \cap \mathcal{B}_J^{\infty}$. First, note that $\lambda_{\mathcal{B}}(R) = \infty$ by the continuity from below property for $\lambda_{\mathcal{B}}$. We have that*

$$R = I^0 \cap \mathcal{B}_J^{\infty} = I^0 \cap \left(\bigcup_{n=1}^{\infty} \mathcal{B}_J^n\right) = \bigcup_{n=1}^{\infty} (I^0 \cap \mathcal{B}_J^n) = \bigcup_{n=1}^{\infty} (I_1^n \times J^n). \tag{10}$$

Since the final union in (10) *increases and $\lambda_n\left(\prod_{k=1}^{n} \lambda(J_k)\right) = 1$, it follows that*

$$\lambda_{\mathcal{B}}(R) = \lim_{n\to\infty} \lambda_{\mathcal{B}}(I_1^n \times J^n) = \lim_{n\to\infty} \frac{\lambda_n(I_1^n)}{\prod_{k=1}^{n} \lambda(J_k)} = \lim_{n\to\infty} \frac{1}{\prod_{k=1}^{n} \lambda(J_k)} = \infty.$$

Second, observe that $\mathcal{B}_J^{\infty} \subset \ell^2$ as follows. Since $\lim_{k\to\infty} a_k = 0$, there is some $m_1 \in \mathbb{N}$ for which $a_k < 1$ whenever $k \geq m_1$. For $x = (x_1, x_2, ...) \in \mathcal{B}_J^{\infty}$, there is $m_2 \in \mathbb{N}$ such that $x \in \mathcal{B}_J^{m_2}$. Let $m = \max\{m_1, m_2\}$, so that $|x_k| < a_k < 1$ for $k \geq m$, hence $x_k^2 < a_k^2 < a_k$, and

$$\sum_{k=m}^{\infty} x_k^2 < \sum_{k=m}^{\infty} a_k^2 < \sum_{k=1}^{\infty} a_k < \infty \quad \Rightarrow \quad \sum_{k=1}^{\infty} x_k^2 < \infty.$$

Therefore $x \in \ell^2$ and $\mathcal{B}_J^{\infty} \subset \ell^2$. Thus, for any $x = (x_1, x_2, ...) \in R$ we may define $f(x) = e^{-2\pi \sum_{k=1}^{\infty} x_k^2}$. Note that $f(x) > 0$ on R. Let $s_n(x) = e^{-2\pi \sum_{k=1}^{n} x_k^2} \otimes h_n$, which *defines a decreasing sequence in $\mathcal{M}_I^{n,+}$ that satisfies $\lim_{n\to\infty} s_n(x) = f(x)$. Since $s_1 \in L^1[R]$, Lebesgue's Monotone Convergence Theorem implies that*

$$\begin{aligned}\int_R f\ d\lambda_{\mathcal{B}} &= \lim_{n\to\infty}\int_R s_n\ d\lambda_{\mathcal{B}}\\ &= \lim_{n\to\infty}\int_{I_1^n} e^{-2\pi\sum_{k=1}^n x_k^2} d\lambda_n \ \le \lim_{n\to\infty}\int_{\mathbb{R}^n} e^{-2\pi\sum_{k=1}^n x_k^2} d\lambda_n = \lim_{n\to\infty}\left(\tfrac{1}{\sqrt{2}}\right)^n = 0.\end{aligned}$$

Thus, we have shown that $\displaystyle\int_R f\ d\lambda_{\mathcal{B}} = 0$ *even though $f(x) > 0$ on R and $\lambda_{\mathcal{B}}(R) > 0$.*

3. Main Results

3.1. **Riemann Sum on a Special Rectangle in $\mathcal{B}$.** Here we will extend the familiar definitions of tagged partition and Riemann sum to a special rectangle in $\mathcal{B}$.

Definition 3.1.

(i) *An n-th unit special rectangle $R \subset \mathcal{B}$ is defined by $R = (R_n \times I^n) \cap \mathcal{B}$ where $R_n \subset \mathbb{R}^n$ is a rectangle in $\mathbb{R}^n$ with sides parallel to the coordinate axes* [15]. *We shall refer to R_n as the n-th unit part.*

(ii) *Let H_k be an m-th unit special rectangle $H_k = \left(H_{1,k}^m \times I^m\right) \cap \mathcal{B} \subsetneq R$ where $m \ge n$ and $H_{1,k}^m = H_{k1} \times \cdots \times H_{km}$ such that $0 < \lambda(H_{kj}) < 1$ for $j = 1, ..., m$. We will call such a rectangle H_k a sub-rectangle or more specifically an m-th unit sub-rectangle of the special rectangle R.*

If $m \ge n+1$, note that $H_{kj} \subsetneq I$ for $k = n+1, ..., m$.

Definition 3.2. *Let $R = (R_n \times I^n) \cap \mathcal{B}$ be an n-th unit special rectangle, and choose any $m \ge n$.*

(i) *The collection $\mathcal{P} = \left\{H_k\right\}_{k=1}^p$ where each H_k is an m-th unit sub-rectangle of R such that $R = \bigcup_{k=1}^p H_k$ and each pair of sub-rectangles only intersect at most at boundary points is called a partition of R.*

(ii) *The collection $\left\{\left(H_k, t_k\right)\right\}_{k=1}^p$ where each $t_k \in (H_{1,k}^n \times I^n) \cap \mathcal{B}$ is called a tagged partition of R.*

(iii) *The partition $\mathcal{P}'$ is a refinement of the partition $\mathcal{P}$ if every rectangle in $\mathcal{P}'$ is contained in a rectangle in $\mathcal{P}$.*

(iv) *The mesh of a partition $\mathcal{P} = \{H_k\}_{k=1}^p$, is defined as*

$$\mathrm{mesh}(\mathcal{P}) = \max\{\lambda(H_{kj}) \mid k = 1, ..., p\ ,\ j = 1, ..., m\} \text{ where } H_k = \left(\prod_{j=1}^m H_{kj} \times I^m\right) \cap \mathcal{B}.$$

(v) *Let $f :\ R \to \mathbb{R}$ be bounded. The Riemann sum for the partition in* (ii) *is defined in the usual way as* $\displaystyle\sum_{k=1}^p f(t_k)\mathrm{vol}(H_k)$ *where* $\mathrm{vol}(H_k) = \mathrm{vol}(H_{1,k}^m) \times \prod_{k=1}^\infty \lambda(I) = \mathrm{vol}(H_{1,k}^m)$. *When no tags are specified, we shall denote such a sum as $S(f, \mathcal{P})$.*

To facilitate a constructive definition of the Riemann integral on a special rectangle in $\mathcal{B}$ as a limit, Proposition 3.3 relates continuity in $\mathcal{B}$ to $\mathbb{R}^n$.

Proposition 3.3.

(i) *Let $f : R \to \mathbb{R}$ be continuous on the special rectangle $R = (R_n \times I^n) \cap \mathcal{B}$ with respect to $\mathcal{T}_{\mathcal{B}}$. Let $c \in R$. Then $f(\ \cdot\ , c_{r+1}, c_{r+2}, ...)$ is continuous on $R_n \times I_{n+1}^r$ relative to $\mathcal{T}_{\mathbb{R}^r}$ for all $r > n$.*

(ii) *For any $m \in \mathbb{N}$, $\bar{s}_m : \mathbb{R}^m \to \mathbb{R}$ is continuous on $\mathbb{R}^m$ relative to $\mathcal{T}_{\mathbb{R}^m}$ if and only if the tame function $s_m = \bar{s}_m \otimes h_m$ is continuous on $\mathcal{B}$ with respect to $\mathcal{T}_{\mathcal{B}}$.*

Proof. To prove (i), let $\epsilon > 0$, $r > n$ and choose $\mathbf{x}_r \in R_n \times I_{n+1}^r$.
Since $x = (\mathbf{x}_r, c_{r+1}, c_{r+2}, ...) \in (R_n \times I^n) \cap \mathcal{B}$ there is a subbase set $(V_m \times I^m) \cap \mathcal{B} \in \mathcal{T}_{\mathcal{B}}$ containing x where $m > r$ such that $f(z) \in (f(x) - \epsilon, f(x) + \epsilon)$ for $z \in (V_m \times I^m) \cap \mathcal{B}$. Since V_m is open in $\mathbb{R}^m$, then there is an open rectangle
$U_m = (b_1, d_1) \times \cdots \times (b_r, d_r) \times (b_{r+1}, d_{r+1}) \times \cdots \times (b_m, d_m) \subset V_m$ for which

$$(\mathbf{x}_r, c_{r+1}, ..., c_m) \in (b_1, d_1) \times \cdots \times (b_r, d_r) \times (b_{r+1}, d_{r+1}) \times \cdots \times (b_m, d_m).$$

Let $\mathbf{y}_r \in (b_1, d_1) \times \cdots \times (b_r, d_r)$ so that $y = (\mathbf{y}_r, c_{r+1}, c_{r+2}, ...) \in (V_m \times I^m) \cap \mathcal{B}$. Then $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$, and the conclusion is true.

To prove the sufficiency of (ii), let $\epsilon > 0$, and let $x = (x_1, x_2, \cdots) \in \mathcal{B}$. Then there is an open rectangle $U_m = (b_1, d_1) \times \cdots \times (b_m, c_m) \subset \mathbb{R}^m$ containing $\mathbf{x}_m = (x_1, ..., x_m)$ for which $\bar{s}_m(\mathbf{y}_m) \in (\bar{s}_m(\mathbf{x}_m) - \epsilon, \bar{s}_m(\mathbf{x}_m) + \epsilon)$ when $\mathbf{y}_m \in U_m$. First assume that $x_k \in I$ for $k = m+1, m+2, ...$, which implies that $x \in (U_m \times I^m) \cap \mathcal{B}$. Let $y = (\mathbf{y}_m, y_{m+1}, y_{m+2}, ...) \in (U_m \times I^m) \cap \mathcal{B}$ where $\mathbf{y}_m = (y_1, ..., y_m)$. Then $s_m(y) = \bar{s}_m(\mathbf{y}_m)$ and $s_m(x) = \bar{s}_m(\mathbf{x}_m)$. Since $\mathbf{y}_m \in U_m$, then

$$s_m(y) = \bar{s}_m(\mathbf{y}_m) \in \big(\bar{s}_m(\mathbf{x}_m) - \epsilon, \bar{s}_m(\mathbf{x}_m) + \epsilon\big) = \big(s(x) - \epsilon, s(x) + \epsilon\big).$$

Now suppose there is $m_1 = \min\left\{k \in \mathbb{N} |\ k > m+1 \text{ and } |x_k| > \frac{1}{2}\right\}$. Then there is also $m_2 = \max\left\{k \in \mathbb{N} |\ k > m+1 \text{ and } |x_k| > \frac{1}{2}\right\}$ since $\lim_{k\to\infty} x_k = 0$. Choose $b_{m_1}, d_{m_1} \in \mathbb{R}$ so that either $\frac{1}{2} < b_{m_1} < x_{m_1} < d_{m_1}$ or $b_{m_1} < x_{m_1} < d_{m_1} < -\frac{1}{2}$, and without loss of generality for $k = m_1 + 1, ..., m_2$ choose $b_k, d_k \in \mathbb{R}$ such that $b_k < x_k < d_k$. Thus, $x \in U = \big(U_m \times (b_{m_1}, d_{m_1}) \times \cdots \times (b_{m_2}, d_{m_2}) \times I^{m_2}\big) \cap \mathcal{B}$. For any $y \in U$ we have that $s_m(x) = 0$ and $s_m(y) = 0$ so that

$$s_m(y) \in (-\epsilon, \epsilon) = (s_m(x) - \epsilon, s_m(x) + \epsilon).$$

In either case, s_m is continuous with respect to $\mathcal{T}_{\mathcal{B}}$.

To prove the necessity of (ii), let $\mathbf{x}_m \in \mathbb{R}^m$. Choose $c \in \mathcal{B}$ such that $c_k \in I$ for $k \geq m+1$. Let $(R_m \times I^m) \cap \mathcal{B}$ be any rectangle containing $(\mathbf{x}_m, c_{m+1}, c_{m+2}, ...)$. By part (i) of this proposition $s_m(\,\cdot\,, c_{m+1}, c_{m+2}, ...) = \bar{s}_m(\cdot)$ is continuous at $\mathbf{x}_m$ relative to $\mathcal{T}_{\mathbb{R}^m}$. □

3.2. **The Riemann Integral on a Special Rectangle in $\mathcal{B}$.** Even though we will define the Riemann integral on a special rectangle in $\mathcal{B}$ in the usual way, Proposition 3.6 illustrates how tame functions can equate the Reimann integral on $\mathcal{B}$ with the Riemann integral on $\mathbb{R}^n$.

Definition 3.4. *Let $R \subset \mathcal{B}$ be an n-th unit rectangle, and let $f : R \to \mathbb{R}$ be bounded. A real number A is the Riemann integral of f on R if for every $\epsilon > 0$ there is a $\delta > 0$ such that for any tagged partition $\big\{(H_k, t_k)\big\}_{k=1}^p$ of R satisfying* $\mathrm{vol}(H_k) < \delta$, *the following inequality holds*

$$\left| \sum_{k=1}^{p} f(t_k)\mathrm{vol}(H_k) \ - \ A \right| < \epsilon.$$

If such a number A exists we say that f is Riemann integrable on R, and we shall denote this be writing $f \in \mathcal{R}[R]$. Equivalently, $f \in \mathcal{R}[R]$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that for any partition $\mathcal{P}$ of R satisfying $\mathrm{mesh}(\mathcal{P}) < \delta$, *we have that*

$$\big|S(f, \mathcal{P}) \ - \ A\big| < \epsilon$$

regardless of how $\mathcal{P}$ is tagged.

Proposition 3.5. *Let $R \subset \mathcal{B}$ be a special rectangle. If $f : R \to \mathbb{R}$ is Riemann integrable on R, the A in* Definition 3.4 *is unique, and we will therefore denote it as $(\mathcal{R})\displaystyle\int_R f\ dx$, and refer to it as the Riemann integral of f on R.*

The proof of this proposition is identical to the proof on $\mathbb{R}^n$, and shall be omitted.

Proposition 3.6. *Let $s_n = \bar{s}_n \otimes h_n$ be bounded on the rectangle $R = (R_n \times I^n) \cap \mathcal{B}$. Then s_n is Riemann integrable over R iff $\bar{s}_n$ is Riemann integrable over R_n, and*

$$(\mathcal{R})\int_R s_n\ dx = (\mathcal{R})\int_{R_n} \bar{s}_n\ d\mathbf{x}_n. \tag{11}$$

Proof. First assume that $\bar{s}_n$ is Riemann integrable over R_n, and let $\epsilon > 0$. Then there is a $\delta > 0$ such that (12) holds for any tagged partition $\{(H^n_{1,k}, \mathbf{t}_{n,k})\}_{k=1}^p$ of R_n satisfying $\mathrm{vol}(H^n_{1,k}) < \delta$

$$\left| \sum_{k=1}^{p} \bar{s}_n(\mathbf{t}_{n,k})\ \mathrm{vol}(H^n_{1,k}) - (\mathcal{R})\int_{R_n} \bar{s}_n\ d\mathbf{x}_n \right| < \epsilon. \tag{12}$$

Let

(13) $\qquad H_k = (H^n_{1,k} \times I^n) \cap \mathcal{B}$ and $t_k = (\mathbf{t}_{n,k}, 0, 0, ...) \in (H^n_{1,k} \times I^n) \cap \mathcal{B}$ for $k = 1, ..., p$.

Note that $s_n(t_k) = \bar{s}_n(\mathbf{t}_{n,k})$ and $\text{vol}(H_k) = \text{vol}(H^n_{1,k})$ so that $\{(H_k, t_k)\}_{k=1}^p$ is a tagged partition of R. Then (12) implies that

$$\left| \sum_{k=1}^{p} s_n(t_k) \ \text{vol}(H_k) - (\mathcal{R}) \int_{R_n} \bar{s}_n \ d\mathbf{x}_n \right| < \epsilon, \tag{14}$$

and thus s_n is Riemann integrable over R and (11) holds. Conversely, assume that s_n is Riemann integrable over R. Then for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left| \sum_{k=1}^{p} s_n(t_k) \ \text{vol}(H_k) - (\mathcal{R}) \int_{R} s_n \ d\mathbf{x}_n \right| < \epsilon, \tag{15}$$

for any tagged partition $\{(H_k, t_k)\}_{k=1}^p$ of R satisfying $\text{vol}(H_k) < \delta$. Let $\{(H^n_{1,k}, \mathbf{t}_{n,k})\}_{k=1}^p$ be a tagged partition of R_n satisfying $\text{vol}(H^n_{1,k}) < \delta$. Let H_k and t_k be as in (13), so that $\bar{s}_n(\mathbf{t}_{n,k}) = s_n(t_k)$ and $\text{vol}(H^n_{1,k}) = \text{vol}(H_k)$. Then $\{(H_k, t_k)\}_{k=1}^p$ is a tagged partition of R satisfying $\text{vol}(H_k) < \delta$ and (15) holds. Then

$$\left| \sum_{k=1}^{p} \bar{s}_n(\mathbf{t}_{n,k}) \ \text{vol}(H^n_{1,k}) - (\mathcal{R}) \int_{R} s_n \ dx \right| = \left| \sum_{k=1}^{p} s_n(t_k) \ \text{vol}(H_k) - (\mathcal{R}) \int_{R} s_n \ d\mathbf{x}_n \right| < \epsilon,$$

which means that $\bar{s}_n$ is Riemann integrable over R_n and (11) holds. □

The proofs of the following theorems are identical to the proofs on $\mathbb{R}^n$, so we shall omit them.

Theorem 3.7. *If $R \subset \mathcal{B}$ is a special rectangle, $a, b \in \mathbb{R}$, and $f, g : R \to \mathbb{R}$ are bounded and Riemann integrable on R, then*

(i) $(\mathcal{R}) \int_R (af + bg) \ dx \ = \ a \ (\mathcal{R}) \int_R f \ dx \ + \ b \ (\mathcal{R}) \int_R g \ dx.$

(ii) *If $f \leq g$, then* $(\mathcal{R}) \int_R f \ dx \leq (\mathcal{R}) \int_R g \ dx.$

Theorem 3.8. (The Cauchy Criterion [17]) *Let $R \subset \mathcal{B}$ be a special rectangle, and let $f : R \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}[R]$ iff for every $\epsilon > 0$ there is a $\delta > 0$ such that if $\mathcal{P}_1$ and $\mathcal{P}_2$ are any tagged partitions of R with* $\text{mesh}(\mathcal{P}_1) < \delta$ *and* $\text{mesh}(\mathcal{P}_2) < \delta$, *then* $\left| S(f, \mathcal{P}_1) \ - \ S(f, \mathcal{P}_2) \right| < \epsilon$.

The Squeeze Theorem for the Riemann Integral [5] follows from the Cauchy Criterion.

Theorem 3.9. (The Squeeze Theorem) *Let $R \subset \mathcal{B}$ be a special rectangle, and let $f : R \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}[R]$ iff for every $\epsilon > 0$ there are functions $g, h \in \mathcal{R}[R]$ for which $g(x) \leq f(x) \leq h(x)$ for all $x \in R$ and*

$$(\mathcal{R}) \int_R (h - g) \ dx < \epsilon.$$

3.3. **The Darboux Integral on a Special Rectangle in $\mathcal{B}$.** We will also define the Darboux integral on $\mathcal{B}$ in the usual way, and in Proposition 3.11 tame functions will connect the upper and lower integral sums on $\mathcal{B}$ to those on $\mathbb{R}^n$.

Definition 3.10. *Let $R \subset \mathcal{B}$ be an n-th unit special rectangle, and let $f : R \to \mathbb{R}$ be bounded. Let $\mathcal{P} = \{H_k\}_{k=1}^p$ be any partition of R. For $k = 1, ..., p$, the following, as usual, are all finite since f is bounded on $\mathbb{R}$.*

$$m_k(f) = \inf\{f(x) \mid x \in H_k\} \quad , \quad M_k(f) = \sup\{f(x) \mid x \in H_k\}$$

$$L(f, \mathcal{P}) = \sum_{k=1}^{p} m_k(f) \ \text{vol}(H_k) \quad , \quad U(f, \mathcal{P}) = \sum_{k=1}^{p} M_k(f) \ \text{vol}(H_k) \ .$$

By a standard computation, (see [22]), it follows that

(16) $\qquad L(f, \mathcal{P}) \leq L(f, \mathcal{P}')$ *and* $U(f, \mathcal{P}') \leq U(f, \mathcal{P})$ *if $\mathcal{P}'$ is a refinement of $\mathcal{P}$.*

(17) $\qquad L(f, \mathcal{P}) \leq U(f, \mathcal{P}')$ *where $\mathcal{P}$ and $\mathcal{P}'$ are any two partitions of R.*

By (17) *the following are upper and lower Darboux integrals are finite real numbers :*

$$\underline{\int_R} f\ dx \;=\; \sup\{L(f,\mathcal{P})\}\ ,\quad \overline{\int_R} f\ dx \;=\; \inf\{U(f,\mathcal{P})\} \tag{18}$$

and

$$\underline{\int_R} f\ dx \;\le \overline{\int_R} f\ dx\ .$$

We say that f *is Darboux integrable over* R*, denoted* $f \in \mathcal{D}[R]$*, if the upper and lower Darboux integrals are equal, and in this case we call this common real number the Darboux integral of* f *over* R *and write*

$$(\mathcal{D})\int_R f\ dx \;= \underline{\int_R} f\ dx \;= \overline{\int_R} f\ dx\ . \tag{19}$$

Proposition 3.11. *Let* $s_n = \bar{s}_n \otimes h_n$ *be bounded on the rectangle* $R = (R_n \times I^n) \cap \mathcal{B}$*. Then* s_n *is Darboux integrable over* R *iff* $\bar{s}_n$ *is Darboux integrable over* R_n*, and*

$$(\mathcal{D})\int_R s_n\ dx = (\mathcal{D})\int_{R_n} \bar{s}_n\ d\mathbf{x}_n\ .$$

Proof. Let $\mathcal{P} = \{H_k\}_{k=1}^p$ be any partition of R. Write $H_k = (H_{1,k}^m \times I^m)\cap\mathcal{B}$, $R_m = R_n \times I_{n+1}^m$ so that $R = (R_m \times I^m)\cap\mathcal{B}$, and let $\mathcal{P}' = \{H_{1,k}^m\}_{k=1}^r$, which is an arbitrary partition of R_n. Note that

$$\mathrm{vol}(H_k) = \mathrm{vol}(H_{1,k}^m),\ s_n = \bar{s}_n \otimes h_{n+1}^m \otimes h_m = \bar{s}_m \otimes h_m \text{ where } \bar{s}_m = \bar{s}_n \otimes h_{n+1}^m,$$
$$m_k(s_n) = m_k(\bar{s}_m)\ ,\ \text{and } M_k(s_n) = M_k(\bar{s}_m).$$

Then

$$\begin{aligned} L(s_n,\mathcal{P}) &= \sum_{k=1}^p m_k(s_n)\ \mathrm{vol}(H_k) = \sum_{k=1}^p m_k(\bar{s}_m)\ \mathrm{vol}(H_{1,k}^m) = L(\bar{s}_m,\mathcal{P}') \text{ and} \\ U(s_n,\mathcal{P}) &= \sum_{k=1}^p M_k(s_n)\ \mathrm{vol}(H_k) = \sum_{k=1}^p M_k(\bar{s}_n)\ \mathrm{vol}(H_{1,k}^m) = U(\bar{s}_m,\mathcal{P}'). \end{aligned} \tag{20}$$

Therefore

$$\underline{\int_R} s_n\ dx = \sup\{L(s_n,\mathcal{P})\} = \sup\{L(\bar{s}_m,\mathcal{P}')\} = \underline{\int_{R_m}} \bar{s}_m\ d\mathbf{x}_m = \underline{\int_{R_n}} \bar{s}_n\ d\mathbf{x}_n \tag{21}$$

and

$$\overline{\int_R} s_n\ dx = \inf\{U(s_n,\mathcal{P})\} = \inf\{U(\bar{s}_m,\mathcal{P}')\} = \overline{\int_{R_m}} \bar{s}_m\ d\mathbf{x}_m = \overline{\int_{R_n}} \bar{s}_n\ d\mathbf{x}_n. \tag{22}$$

By (21) and (22) this proposition is true. □

Theorem 3.12. (The Approximation Theorem) *Let* $R \subset \mathcal{B}$ *be a special rectangle, and let* $f : R \to \mathbb{R}$ *be bounded. Then* $f \in \mathcal{D}[R]$ *iff for every* $\epsilon > 0$ *there is a partition* $\mathcal{P}$ *of* R *such that* $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon$.

3.4. **An agreement when relating Darboux and Lebesgue integrable functions on a Special Rectangle in $\mathcal{B}$.** Recall that Definition 2.10 of the constructive Lebesgue integral of a non-negative function f on $\mathcal{B}$ requires f to dominate a tame function s_n in $\mathcal{M}_I^{n,+}$ scaled by $\mathcal{B}_J^n$. This scaling was needed to localize the support of s_n to $\mathcal{B}_J^n$ where f is defined. Yet, this scaling introduces a pathology that we will need to resolve.

If $R \subset \mathbb{R}^n$ is a special rectangle, it is well known that a Darboux integrable function on R is Lebesgue integrable to the same value. When the special rectangle R resides in an infinite dimensional space, these integrals need not agree in value due to the scaling of s_n by $\mathcal{B}_J^n$, which can admit a tame function that makes the Lebesgue integral larger than the Darboux integral. As Counter Example 3.13 will demonstrate, this can occur if the support of a Darboux integrable function f is too small to contain a special sub-rectangle of the form $(A_n \times I^n) \cap \mathcal{B}$.

Counter Example 3.13. *Let $f = \chi_I \otimes \chi_{J_2} \otimes h_2$, and $R = (I^0) \cap \mathcal{B}$. Note that the largest tame function which f can dominate when scaled according to* Definition 2.9 *is $\chi_I \otimes h_1$ since $(\chi_I \otimes h_1)\chi_{\mathcal{B}_J^1} \le f$. Therefore*

$$\int_R f \, d\lambda_{\mathcal{B}} = \int_{\mathcal{B}_J^1} \Big[(\chi_I \otimes h_1)\lambda(J_1)\Big] d\nu_J = \int_{\mathbb{R}} \chi_I \, d\lambda = \lambda(I).$$

Also, $\int_R f \, dx = \text{vol}\big((I \times J_2 \times I^2) \cap \mathcal{B}\big) = \lambda(J_2)$, and we obtain that $\int_R f \, d\lambda_{\mathcal{B}} > \int_R f \, dx$.

Note that Theorem 2.10 in 2.2.5 allows us to replace a function $f \in \mathcal{M}^+$ with a function $g \in \mathcal{M}^+$ with a larger support that can contain a special sub-rectangle of the form $(A_1 \times I^1) \cap \mathcal{B}$ since $g \ge \bar{s}_1 \otimes h_1$, which allows us to equate the Lebesgue integral of g with the Darboux integral of g. Here we will use this result to formulate a condition which will allow us to equate the Riemann and Lebesgue integrals on a special rectangle.

We will start by formulating criteria for equating the Riemann and Lebesgue integrals for a special class of tame functions. For a Lebesgue integrable tame function $s_n \ge 0$ on R, note that since $s_n \chi_{\mathcal{B}_J^n} \le s_n$, Definition 2.9 implies that

$$\int_R s_n \, d\lambda_{\mathcal{B}} \ge \int_{\mathcal{B}_J^n} \Big[s_n \prod_{k=1}^{n} \lambda(J_k)\Big] d\nu_J = \int_{R_n} \bar{s}_n \, d\lambda_n. \tag{23}$$

To obtain equality in (23), we will use the following lemma. Here we will use the following notation to denote the projection of set to a set of smaller dimension. If $A \subset \mathcal{B}$, then $\pi_1^r(A) = \{(x_1, x_2, ..., x_r) \mid (x_1, x_2, ...) \in A\}$.

Proposition 3.14. *Let $s_n \in \mathcal{M}_I^{n,+}$ have the property that $\pi_1^m\big(\text{Spt}(s_n)\big) \times I^m \subset \text{Spt}(s_n)$ within $\mathcal{B}$ for $1 \le m \le n$. Then*

$$\int_{\mathcal{B}} s_n d\lambda_{\mathcal{B}} = \int_{\mathcal{B}_J^n} \Big[s_n \prod_{k=1}^{n} \lambda(J_k)\Big] d\nu_J = \int_{R_n} \bar{s}_n \, d\lambda_n. \tag{24}$$

Proof. First, note that in accordance with the definition of the integral in (7) s_n itself is admissible since $s_n \in \mathcal{M}_I^{n,+}$ and $s_n \chi_{\mathcal{B}_J^n \cup \text{Spt}(s_n)} \le s_n$. Let $s_m \in \mathcal{M}_I^{m,+}$ such that $s_m \chi_{\mathcal{B}_J^m \cup \text{Spt}(s_n)} \le s_n$. If $m \ge n$, then

$$\int_{\mathcal{B}_J^n} \Big[s_n \prod_{k=1}^{n} \lambda(J_k)\Big] d\nu_J = \int_{\mathcal{B}_J^m} \Big[s_n \prod_{k=1}^{m} \lambda(J_k)\Big] d\nu_J \ge \int_{\mathcal{B}_J^m} \Big[s_m \prod_{k=1}^{m} \lambda(J_k)\Big] d\nu_J. \tag{25}$$

Now consider $n > m$. Suppose there is some $\mathbf{x}_m \in \text{Spt}(\bar{s}_m)$ such that $\mathbf{x}_m \notin \pi_1^m(\text{Spt}(s_n))$. Let $x_k \in J_k$ for $k \ge m+1$, and note that $x = (\mathbf{x}_m, x_{m+1}, x_{m+2}, ...) \in \mathcal{B}_J^m$. Also, since each such $x_k \in I$ for $k \ge m+1$ then $s_m(x) > 0$. We then have that

$$0 < s_m(x) = s_m(x)\chi_{\mathcal{B}_J^m \cup \text{Spt}(s_n)} \le s_n(x),$$

which contradicts that fact that $s_n(x) = 0$ since $x \notin \text{Spt}(s_n)$. Thus, $\text{Spt}(\bar{s}_m) \subset \pi_1^m(\text{Spt}(s_n))$, which means that

$$\text{Spt}(s_m) = \text{Spt}(\bar{s}_m) \times I^m \subset \pi_1^m(\text{Spt}(s_n)) \times I^m \subset \text{Spt}(s_n). \tag{26}$$

Thus, (26) implies that if $s_m(x) > 0$, then $x \in \text{Spt}(s_n)$. Now let $x \in \mathcal{B}_J^n$ and suppose that $s_m(x) > 0$. Then we have that

$$s_m(x) = s_m(x)\chi_{\mathcal{B}_J^m \cup \text{Spt}(s_n)}(x) \le s_n(x),$$

and so

$$\int_{\mathcal{B}_J^m} \Big[s_m \prod_{k=1}^{m} \lambda(J_k)\Big] d\nu_J = \int_{\mathcal{B}_J^n} \Big[s_m \prod_{k=1}^{n} \lambda(J_k)\Big] d\nu_J \le \int_{\mathcal{B}_J^n} \Big[s_n \prod_{k=1}^{n} \lambda(J_k)\Big] d\nu_J. \tag{27}$$

By (25) and (27), equation (24) is true. ☐

Corollary 3.15. *As a consequence of* Proposition 3.14 *it follows that if the sequence $\{s_n\}_{n=1}^{\infty}$ is increasing where each $s_n \in \mathcal{M}_I^{n,+}$, then in fact each s_n has the property that in the space $\mathcal{B}$,*

$$\pi_1^m\big(\text{Spt}(s_n)\big) \times I^m \subset \text{Spt}(s_n) \text{ for } 1 \le m \le n \tag{28}$$

and so equation (24) is true for each such s_n in this sequence.

Proof. Since the sequence $\{s_n\}_{n=1}^{\infty}$ is increasing, it follows for $1 \le m < n$ that

$$\text{Spt}(\bar{s}_m) \times I_{m+1}^n \subset \text{Spt}(\bar{s}_n). \tag{29}$$

Let $(\mathbf{x}_m, x_{m+1}, ..., x_n) \in \pi_1^m\big(\text{Spt}(\bar{s}_n)\big) \times I_{m+1}^n$. Then $\mathbf{x}_m \in \pi_1^m(\text{Spt}(\bar{s}_n))$ and $(x_{m+1}, ..., x_n) \in I_{m+1}^n$. By definition of cross product, $\pi_{m+1}^n(\text{Spt}(\bar{s}_m \times I_{m+1}^n) = I_{m+1}^n$, so that $(x_{m+1}, ..., x_n) \in \pi_{m+1}^n(\text{Spt}(\bar{s}_m \times I_{m+1}^n)$. Also by (29), since

$$\pi_{m+1}^n\big(\ \text{Spt}(\bar{s}_m) \times I_{m+1}^n\ \big) \subset \pi_{m+1}^n\big(\ \text{Spt}(\bar{s}_n)\ \big),$$

then $(x_{m+1}, ..., x_n) \in \pi_{m+1}^n(\text{Spt})(\bar{s}_n)$; and thus $(\mathbf{x}_m, x_{m+1}, ..., x_n) \in \text{Spt}(\bar{s}_n)$, so that

$$\pi_1^m\big(\text{Spt}(\bar{s}_n)\big) \times I_{m+1}^n \ \subset\ \text{Spt}(\bar{s}_n)\ \Rightarrow$$

$$\pi_1^m\big(\text{Spt}(\bar{s}_n)\big) \times I^m = \pi_1^m\big(\text{Spt}(\bar{s}_n)\big) \times I_{m+1}^n \times I^n \subset \text{Spt}(\bar{s}_n) \times I^n = \text{Spt}(s_n);$$

and the inclusion property (28) holds for this sequence. □

Recall that Theorem 2.10 justifies the assumption that for any for any non-negative $f \in \mathcal{M}$ there is an increasing sequence $\big\{s_n \in \mathcal{M}_I^{n,+}\big\}_{n=1}^{\infty}$ of tame functions for which $\lim_{n\to\infty} s_n(x) = f(x)$ for $\lambda_{\mathcal{B}}$ a.e. $x \in \mathcal{B}$. Indeed, it may actually be convenient to replace the integrand $f \in \mathcal{M}^+$ in $\int_{\mathcal{B}} f\ d\lambda_{\mathcal{B}}$ with a function $g \in \mathcal{M}^+$ that has a larger support. Theorem 2.10 will serve as our blueprint for formulating the following agreement, which will enable us to equate the Lebesgue integral of an admissible function f with a Darboux integral of f in some contexts. In light of Theorem 2.10 we may invoke the following agreement to insure this equality.

Agreement 3.16.

(i) *If $f \ge 0$ is Darboux integrable on a special rectangle R, we will assume that there is a sequence $\{s_r \in \mathcal{M}_I^{r,+}\}_{r=1}^{\infty}$ of Riemann integrable tame functions for which $0 \le s_1 \le s_2 \le \cdots \le f$ such that* $\text{Spt}(\bar{s}_1) \supset [a, b]$ *where $a < b$. By* Corollary 3.15 *the inclusion* (28) *holds for each s_r.*

(ii) *For a more general Darboux integrable function f we may write $f = f_+ - f_-$ since f_+ and f_- are also Riemann-integrable by* Corollary 3.23. Remark 2 *and* Theorem 2.13 *provide the justification for assuming that* (i) *applies to f_+ and f_-.*

Remark 3.19 presents a general setting where Agreement 3.16 can hold.

3.5. Relationship Between the Riemann and Darboux Integrals on a Special Rectangle in $\mathcal{B}$.

Proposition 3.17. *Let $s_n = \bar{s}_n \otimes h_n$ be bounded on the rectangle $R = (R_n \times I^n) \cap \mathcal{B}$. Then the following are equivalent.*

(a) *s_n is Riemann integrable on R.*
(b) *$\bar{s}_n$ is Riemann integrable on R_n.*
(c) *s_n is Darboux integrable on R.*
(d) *$\bar{s}_n$ is Darboux integrable on R_n.*

Furthermore, if any of (a) *through* (d) *hold, then the first three equalities in* (30) *hold and also $s_n \in L^1[R]$. If* Agreement 3.16 *holds for s_n, then the final equality in* (30) *also holds.*

$$(\mathcal{R})\int_R s_n\ dx \ =\ (\mathcal{R})\int_{R_n} \bar{s}_n\ d\mathbf{x}_n \ =\ (\mathcal{D})\int_{R_n} \bar{s}_n\ d\mathbf{x}_n \ =\ (\mathcal{D})\int_R s_n\ dx \ =\ \int_R s_n\ d\lambda_{\mathcal{B}}\ . \tag{30}$$

Proof. Proposition 3.6 and Proposition 3.11 imply that (a) through (d) are equivalent since (b) and (d) are equivalent. If (b) or (d) hold, then $\bar{s}_n \in L^1[\mathbb{R}^n]$, which implies $s_n \in L^1[\mathcal{B}]$. Also, since in such a case

$$(\mathcal{R})\int_{R_n} \bar{s}_n\ d\mathbf{x}_n = (\mathcal{D})\int_{R_n} \bar{s}_n\ d\mathbf{x}_n.$$

equation (30) also holds. □

Remark 3. *Let $\mathcal{R}[R_n \subset \mathbb{R}^n]$ and $\mathcal{R}[R \subset \mathcal{B}]$ respectively denote the class of real-valued functions for which each one is respectively Riemann integrable on some special rectangle $R_n \subset \mathbb{R}^n$ and $R \subset \mathcal{B}$. As a consequence of equation* (30) *in* Proposition 3.17, *$\mathcal{R}[R_m \subset \mathbb{R}^m] \hookrightarrow \mathcal{R}[R \subset \mathcal{B}]$ as an embedding, for all $m \in \mathbb{N}$ and all $R_n \subset \mathbb{R}^n$. By contrast, $\mathcal{R}[R_m \subset \mathbb{R}^m] \hookrightarrow \mathcal{R}[R_n \subset \mathbb{R}^n]$ only for $m = 1, 2, ..., n-1$. Thus, $\mathcal{R}[R \subset \mathcal{B}]$ is a much larger class of functions than $\mathcal{R}[R_n \subset \mathbb{R}^n]$.*

We will now define two special sequences of tame step functions to study the relationships between the Riemann, Darboux, and Lebesgue integrals on a special rectangle in $\mathcal{B}$.

Definition 3.18. *Let $R \subset \mathcal{B}$ be an n-th unit special rectangle, let $f : R \to \mathbb{R}$ be bounded, and let f be Darboux integrable on R. Let $\{H_k\}_{k=1}^{k_r}$ be a partition of the special rectangle $(R_n \times I^n) \cap \mathcal{B}$ for $r \geq n$ where each $H_k = (H_{1,k}^r \times I^r) \cap \mathcal{B}$, and let $f : R \to \mathbb{R}$ be bounded. Set $\bar{H}_1 = H_1 \;\;, \; \bar{H}_2 = H_2 \setminus H_1 \;\;, ...,$
$\bar{H}_p = H_p \setminus (H_1 \cup \cdots \cup H_{p-1})$ so that $R = \bigcup_{k=1}^{k_r} \bar{H}_k \;,\; \bar{H}_i \cap \bar{H}_j = \emptyset$, and $\operatorname{vol}(\bar{H}_k) = \operatorname{vol}(H_k)$. For each $r \geq n$ define*

$$
\begin{aligned}
L_r(f) &= \left[\sum_{k=1}^{k_r} m_{k,r}(f)\chi_{\bar{H}_{1,k}^r}\right] \otimes h_r \;=\; \bar{L}_r(f) \otimes h_r \\
U_r(f) &= \left[\sum_{k=1}^{k_r} M_{k,r}(f)\chi_{\bar{H}_{1,k}^r}\right] \otimes h_r \;=\; \bar{U}_r(f) \otimes h_r.
\end{aligned} \tag{31}
$$

By Proposition 3.17 *the tame step functions in* (31) *are Riemann and Darboux integrable on R since $\bar{L}_r$ and $\bar{U}_r$ are Riemann integrable on $R_r = R_n \times I_{n+1}^r$ and by* Proposition 3.17 *we have that*

$$
\begin{aligned}
\int_R L_r(f)\ dx &= \int_{R_r} \left[\sum_{k=1}^{k_r} m_{k,r}(f)\chi_{\bar{H}_{1,k}^r}\right] d\mathbf{x}_r = \sum_{k=1}^{k_r} m_{k,r}(f)\ \operatorname{vol}(H_{1,k}^r) = L(f, \mathcal{P}_r), \\
\int_R U_r(f)\ dx &= \int_{R_r} \left[\sum_{k=1}^{k_r} M_{k,r}(f)\chi_{\bar{H}_{1,k}^r}\right] d\mathbf{x}_r = \sum_{k=1}^{k_r} M_{k,r}(f)\ \operatorname{vol}(H_{1,k}^r) = U(f, \mathcal{P}_r).
\end{aligned} \tag{32}
$$

Also, note that

$$
L_r(f)(x) \leq f(x) \leq U_r(f)(x) \text{ for all } x \in R. \tag{33}
$$

Proposition 3.19. *Let R and $(R_1 \times I^1) \cap \mathcal{B}$ be special rectangles where $R = (R_n \times I^n) \cap \mathcal{B}$ such that $R \supset (R_1 \times I^1) \cap \mathcal{B}$.*

(i) *Let $f \geq 0$ be Darboux integrable on R. Suppose that there is some non-degenerate interval $[a, b]$ of R_1 such that $\inf \{f(x) |\ x \in [a, b] \times I^1\} > 0$. Then there exists a sequence of partitions $\{\mathcal{P}_r\}_{r=n+1}^{\infty}$ of R for which*

$$L_{n+1}(f) \leq L_{n+2}(f) \leq \cdots \leq f$$

where each L_r for $r > n$ satisfies the inclusion relation in Proposition 3.14.

(ii) *If $f = f_+ - f_-$ we will assume that each of f_+ and f_- satisfy condition* (i),*which is justified by* Remark 2 *and* Theorem 2.13.

Proof. First let $\{\mathcal{P}'_r\}_{r=1}^{\infty}$ be a sequence of partitions of $R_1 \times I^1$ given by $\mathcal{P}'_r = \{H_{r,k}\}_{k=1}^{k_r}$: $r \geq 1$ is a partition where each sub-rectangle $H_{r,k} = \big(H_{r,k,1} \times H_{r,k,2} \times \cdots \times H_{r,k,r} \times I^r\big) \cap \mathcal{B}$ for which $L'_1(f)(x) > 0$ for some $x \in H_{r,j,1}$ for some j. Then in fact $L_1(f)'(x) > 0$ for all $x \in H_{r,j,1}$, and Agreement 3.16 applies to the sequence $\{L_r(f)'\}_{r=1}^{\infty}$ of Darboux integrable tame functions since we have that $L_1(f)' \leq L_2(f)' \leq \cdots \leq f$ such that $\operatorname{Spt}(\bar{L}_1(f)') \supset H_{r,j,1}$ where $H_{r,j,1}$ is a non-degenerate interval.

Now form a sequence of partitions $\{\mathcal{P}_r\}_{r=n+1}^{\infty}$ of R where each $\mathcal{P}_r$ adjoins $\mathcal{P}'_r$. Then since $L_r(f) \geq L_r(f)'$ for $r = n + 1, n + 2, ...$ it follows that each L_r for $r > n$ satisfies the inclusion relation in Proposition 3.14. ☐

Proposition 3.20. *Let $R \subset \mathcal{B}$ be an n-th unit special rectangle, and let $f : R \to [0, \infty)$ be bounded and Darboux integrable on R.*

(i) *By* (23), (32) *and* Proposition 3.17,

$$
\begin{aligned}
\int_R L_r(f)\ d\lambda_{\mathcal{B}} &\geq \int_R L_r(f)\ dx \;= L(f, \mathcal{P}_r) \text{ and} \\
\int_R U_r(f)\ d\lambda_{\mathcal{B}} &\geq \int_R U_r(f)\ dx \;= U(f, \mathcal{P}_r).
\end{aligned} \tag{34}
$$

(ii) *Now also assume that the conditions in* Proposition 3.19 *apply to* R *and* f. *Since* $L_r(f) \le U_r(f)$, *it follows that each* $U_r(f)$ *also satisfies the inclusion relation in* Proposition 3.14, and in this case we have that

$$\begin{aligned} \int_R L_r(f)\ d\lambda_{\mathcal{B}} &= \int_R L_r(f)\ dx \ = L(f, \mathcal{P}_r) \ \textit{and} \\ \int_R U_r(f)\ d\lambda_{\mathcal{B}} &= \int_R U_r(f)\ dx \ = U(f, \mathcal{P}_r). \end{aligned} \tag{35}$$

The following proof of the equivalence between the Riemann and Darboux integrals on a special rectangle in $\mathcal{B}$ is based on the equivalence proof in [5] for such integrals in finite dimensions.

Theorem 3.21. *Let* $f : R \to \mathbb{R}$ *be bounded on the special rectangle* $R = (R_n \times I^n) \cap \mathcal{B}$. *Then* f *is Riemann integrable on* R *if and only if* f *is Darboux integrable on* R *and in either case*

$$(\mathcal{R}) \int_R f\ dx \ = \ (\mathcal{D}) \int_R f\ dx. \tag{36}$$

Proof. Assume that f is Riemann integrable on R. Let $\{\mathcal{P}_r\}_{r=1}^{\infty}$ be a sequence of partitions of R denoted $\mathcal{P}_r = \{H_{r,k}\}_{k=1}^{k_r}$ where $\mathcal{P}_{r+1}$ is a refinement of $\mathcal{P}_r$ such that $\lim_{r\to\infty} \mathrm{mesh}(\mathcal{P}_r) = 0$. Then

$$\lim_{r\to\infty} L(f, \mathcal{P}_r) = \underline{\int_R} f\ dx \quad \text{and} \quad \lim_{r\to\infty} U(f, \mathcal{P}_r) = \overline{\int_R} f\ dx. \tag{37}$$

Let $\epsilon > 0$. Then there is $\delta > 0$ such that (38) holds for any tagged partition $\{(H_k, t_k)\}_{k=1}^{r}$ satisfying $\mathrm{mesh}(\mathcal{P}) < \delta$.

$$\left| S(f, \mathcal{P}) - (\mathcal{R}) \int_R f\ dx \right| < \tfrac{\epsilon}{2}. \tag{38}$$

There is $N \in \mathbb{N}$ such that $\mathrm{mesh}(\mathcal{P}_r) < \delta$ for $r \ge N$. For each such r and $k = 1, ..., k_r$ there is $t_{k,r} \in H_{k,r}$ such that

$$f(t_{k,r}) < m_{k,r}(f) + \tfrac{\epsilon}{2\mathrm{vol}(R)}. \tag{39}$$

For any tagged partition $\{(H_{k,r}\ ,\ t_{k,r})\}_{k=1}^{k_r}$ satisfying $r \ge N$, $\mathrm{vol}(H_{k,r}) < \delta$ which by (38) implies that

$$\left| \sum_{k=1}^{k_r} f(t_{k,r})\mathrm{vol}(H_{k,r}) - (\mathcal{R}) \int_R f\ dx \right| < \tfrac{\epsilon}{2}. \tag{40}$$

Therefore

$$\begin{aligned} \left| (\mathcal{R}) \int_R f\ dx \ - \ L(f, \mathcal{P}_r) \right| \le \ & \left| (\mathcal{R}) \int_R f\ dx \ - \ \sum_{k=1}^{k_r} f(t_{k,r})\mathrm{vol}(H_{k,r}) \right| \\ & + \left| \sum_{k=1}^{k_r} f(t_{k,r})\mathrm{vol}(H_{k,r}) \ - \ \sum_{k=1}^{k_r} m_{k,r}(f)\mathrm{vol}(H_{k,r}) \right| \\ < \epsilon & \end{aligned}$$

by (39) and (40). Thus

$$(\mathcal{R}) \int_R f\ dx \ = \ \lim_{r\to\infty} L(f, \mathcal{P}_r) = \underline{\int_R} f\ dx$$

and similarly

$$(\mathcal{R}) \int_R f\ dx \ = \ \lim_{r\to\infty} U(f, \mathcal{P}_r) = \overline{\int_R} f\ dx,$$

which together imply that f is Darboux integrable and (36) is true.

Now assume that f is Darboux integrable on R, and let $\epsilon > 0$. There is a partition $\mathcal{P}_\epsilon = \{H_k\}_{k=1}^{k_\epsilon}$ of R such that $\left| U(f, \mathcal{P}_\epsilon) \ - \ L(f, \mathcal{P}_\epsilon) \right| < \epsilon$ where each $H_k = (H_{1,k}^r \times I^r) \cap \mathcal{B}$ and

$r \geq n$. Let the Riemann integrable step functions $L_r(f)$ and $U_r(f)$ be as in (31) and (32). Note that $U_r(f) - L_r(f) \in \mathcal{R}[R]$ by Theorem 3.7 part (ii). Then

$$L_r(f)(x) \leq f(x) \leq U_r(f)(x) \tag{41}$$

and

$$(\mathcal{R}) \int_R (U_r(f) - L_r(f)) \; dx = U(f, \mathcal{P}_r) \; - \; L_r(f)(f, \mathcal{P}_r) < \epsilon. \tag{42}$$

By the Squeeze Theorem (Theorem 3.9) equation (42) implies that $f \in \mathcal{R}[R]$. By part (i) of Theorem 3.7 the inequalities in (41) and (32) imply that

$$L(f, \mathcal{P}_r) = (\mathcal{R}) \int_R L_r(f) \; dx \leq (\mathcal{R}) \int_R f \; dx \leq (\mathcal{R}) \int_R U_r(f) \; dx = U(f, \mathcal{P}_r). \tag{43}$$

By assumption,

$$L(f, \mathcal{P}_r) \leq (\mathcal{D}) \int_R f \; dx \leq U(f, \mathcal{P}_r). \tag{44}$$

Statements (43) and (44) together imply that

$$\left| (\mathcal{R}) \int_R f \; dx \; - \; (\mathcal{D}) \int_R f \; dx \right| \leq U(f, \mathcal{P}_r) - L(f, \mathcal{P}_r) < \epsilon$$

and (36) is true. □

Because the Riemann and Darboux integrals on a special rectangle in $\mathcal{B}$ are equivalent, we shall henceforth adopt the usual convention and refer to each of them as the Riemann integral and drop the prefix for each.

The following additional properties of the Riemann integral on $\mathcal{B}$ have standard proofs (see [25]), and will be useful for computation.

Theorem 3.22. *Let $f : R \to \mathbb{R}$ be Riemann integrable on the rectangle $R = (R_n \times I^n) \cap \mathcal{B}$. Then $|f|$ is Riemann integrable on R and* $\left| \int_R f \; dx \right| \leq \int_R |f| \; dx$.

Corollary 3.23. *It follows from* Theorem 3.22 *that f_- and f_+ are Riemann integrable on a special rectangle in $\mathcal{B}$ if f is since* $f_- = \dfrac{|f| - f}{2}$ *and* $f_+ = \dfrac{|f| + f}{2}$.

3.6. **Relationship Between the Riemann and Lebesgue Integrals on a Special Rectangle in $\mathcal{B}$.** We will first extend the following well known condition for Riemann integrability on a special rectangle in $\mathcal{B}$ in order to demonstrate the advantage of being able to express this integral as a limit in finite-dimensional space where the condition already holds.

Theorem 3.24. *Let $f : (R, \mathcal{T}_{\mathcal{B}}) \to (\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ be bounded and continuous on the special rectangle $R = (R_n \times I^n) \cap \mathcal{B}$. Then f is Riemann integrable on R.*

Proof. By the continuity of f on R relative to $\mathcal{T}_{\mathcal{B}}$, f is continuous at each point $(\mathbf{x}_r, 0, 0, ...) \in R$ where $r > n$. Since $f(\mathbf{x}_r, 0, 0, ...)$ is continuous on $R_n \times I^r_{n+1}$ relative to $\mathcal{T}_{\mathbb{R}^r}$ by Proposition 3.3, there is a partition $\mathcal{P}''_r = \{H_{r,k}\}_{k=1}^{k_r}$ of $R_n \times I^r_{n+1}$ for which

$$U(f, \mathcal{P}''_r) - L(f, \mathcal{P}''_r) < \tfrac{1}{r} \; . \tag{45}$$

By equation (20) in Proposition 3.11,

$$U(f, \mathcal{P}'_r) = U(f, \mathcal{P}''_r) \text{ and } L(f, \mathcal{P}'_r) = L(f, \mathcal{P}''_r) \text{ where } \mathcal{P}'_r = \{(H_{r,k} \times I^r) \cap \mathcal{B}\}_{k=1}^{k_r}.$$

By (46), we obtain that

$$U(f, \mathcal{P}'_r) - L(f, \mathcal{P}'_r) < \tfrac{1}{r}. \tag{46}$$

For each $r = n+1, n+2, ...$ refine $\mathcal{P}'_r$ into a partition $\mathcal{P}_r$ so that $\lim_{r \to \infty} \text{mesh}(\mathcal{P}_r) = 0$. Then

$$U(f, \mathcal{P}_r) - L(f, \mathcal{P}_r) \leq U(f, \mathcal{P}'_r) - L(f, \mathcal{P}'_r) < \tfrac{1}{r},$$

which implies that

$$\lim_{r \to \infty} U(f, \mathcal{P}_r) - \lim_{r \to \infty} L(f, \mathcal{P}_r) \leq \lim_{r \to \infty} \tfrac{1}{r} = 0.$$

By (37) in Theorem 3.21, we obtain

$$\overline{\int_R} f\ dx - \underline{\int_R} f\ dx = 0,$$

and f is Riemann integrable on R. □

Recall from Counter Example 3.13 that the Riemann and Lebesgue integrals need not agree in value on any special rectangle R in $\mathcal{B}$. By assuming the conditions in Proposition 3.19 we will prove that they agree in value in a setting that could be useful for equating a Lebesgue integral with a Riemann integral. We will establish this relationship by modifying a standard proof such as in [15].

Theorem 3.25. *Let $f : R \to \mathbb{R}$ be bounded on the special rectangle $R = (R_n \times I^n) \cap \mathcal{B}$ such that the conditions in* Proposition 3.19 *apply to f and R. Then*

(i) *If f is Riemann integrable on R and* Agreement 3.16 *applies to f, then f is Lebesgue integrable on R and*
$\int_R f\ dx = \int_R f\ d\lambda_{\mathcal{B}}.$

(ii) *f is Riemann integrable on R iff f is continuous $\lambda_{\mathcal{B}}$-a.e. on R with respect to $\mathcal{T}_{\mathcal{B}}$.*

Proof. As in the proof of Theorem 3.21 let $\{\mathcal{P}_r\}_{r=n}^{\infty}$ be a sequence of partitions of R denoted $\mathcal{P}_r = \{H_{r,k}\}_{k=1}^{k_r}$ where each $H_{k,r} = (H_{1,k}^r \times I^r) \cap \mathcal{B}$, and $\mathcal{P}_{r+1}$ is a refinement of $\mathcal{P}_r$ such that $\lim_{r\to\infty} \text{mesh}(\mathcal{P}_r) = 0$ and these partitions are as in Proposition 3.19. Then

$$\lim_{r\to\infty} L(f, \mathcal{P}_r) = \underline{\int_R} f\ dx \quad \text{and} \quad \lim_{r\to\infty} U(f, \mathcal{P}_r) = \overline{\int_R} f\ dx\ . \tag{47}$$

Since $\{L_r(f)\}_{r=1}^{\infty}$ increases and $\{U_r(f)\}_{r=1}^{\infty}$ decreases, then the limits

$$L(x) = \lim_{r\to\infty} L_r(f)(x) \quad \text{and} \quad U(x) = \lim_{r\to\infty} U_r(f)(x) \tag{48}$$

each exist and are in $\mathcal{M}$. Also, since $m_{k,r}(f) \le f \le M_{k,r}(f)$ where $x \in H_{k,r}$ for each $k = 1, ..., k_r$, the following hold for all $x \in R$:

$$L_r(f)(x) \le L(x) \le f(x) \le U(x) \le U_r(f)(x)\ . \tag{49}$$

First assume that $f \ge 0$. Respectively by (35) in Proposition 3.20 we obtain that

$$\int_R L_r(f)\ d\lambda_{\mathcal{B}} = \int_R L_r(f)\ dx = L(f, \mathcal{P}_r),$$
$$\int_R U_r(f)\ d\lambda_{\mathcal{B}} = \int_R U_r(f)\ dx = U(f, \mathcal{P}_r).$$

By (48) Lebesgue's Monotone Convergence Theorem implies that (50) and (51) hold.

$$\int_R U\ d\lambda_{\mathcal{B}} = \lim_{r\to\infty} \int_R U_r(f)\ d\lambda_{\mathcal{B}} = \lim_{r\to\infty} U(f, \mathcal{P}_r) = \overline{\int_R} f\ dx \tag{50}$$

$$\int_R L\ d\lambda_{\mathcal{B}} = \lim_{r\to\infty} \int_R L_r(f)\ d\lambda_{\mathcal{B}} = \lim_{r\to\infty} L(f, \mathcal{P}_r) = \underline{\int_R} f\ dx\ . \tag{51}$$

More generally, Corollary 3.23 insures that for $f = f^+ - f^-$ (50) and (51) hold by the case where $f \ge 0$ by tacitly assuming that (ii) of Proposition 3.19 holds. If f is Riemann integrable on R, then (49), (50) and (51) imply that

$$\int_R L\ d\lambda_{\mathcal{B}} = \int_R U\ d\lambda_{\mathcal{B}} = \int_R f\ dx, \tag{52}$$

and so $\int_R (U - L)\ d\lambda_{\mathcal{B}} = 0$. Although it is still possible that $U - L > 0$ on a set of non-zero measure with respect to $\lambda_{\mathcal{B}}$ as demonstrated by Counter example 2.14, we can still establish that $U - L = 0$ a.e. $\lambda_{\mathcal{B}}$ by using the fact that (50), (51), and (52) imply that

$$\lim_{r\to\infty} \|U_r(f) - L_r(f)\|_{L^1[R]} = 0. \tag{53}$$

By (53) there is a subsequence $\{U_{r_k} - L_{r_k}\}_{k=1}^{\infty}$ for which $\lim_{k\to\infty}(U_{r_k} - L_{r_k}) = 0$ a.e. with respect to $\lambda_{\mathcal{B}}$. By (48) we therefore have that

$$U - L = \lim_{r\to\infty}(U_r(f) - L_r(f)) = \lim_{k\to\infty}(U_{r_k} - L_{r_k}) = 0 \quad \text{a.e. with respect to } \lambda_{\mathcal{B}},$$

hence $U = f = L$ $\lambda_{\mathcal{B}}$ a.e. by (49), and so (i) is true by (50) and (51).

Let $E_1 = \{x \in R \,|U(x) \neq L(x)\}$, $E_2 = \{x \in \partial(H_{k,r}) \mid r \in \mathbb{N}, 1 \leq k \leq k_r\}$, and $E = E_1 \cup E_2$. Choose $x \in R \setminus E$ and let $\epsilon > 0$. Then since $U(x) = L(x)$ equation (48) implies that there is $r \in \mathbb{N}$ such that $U_r(f)(x) - L_r(f)(x) < \epsilon$. Since also $x \notin \partial(H_{k,r})$ for $k = 1, ..., k_r$ then there is some $V = (V_r \times I^r) \cap \mathcal{B} \in \mathcal{T}_{\mathcal{B}}$ such that $x \in V$ and $U_r(f) - L_r(f)$ is constant on V. Then

$$U_r(f)(y) - L_r(f)(y) = U_r(f)(x) - L_r(f)(x) < \epsilon \text{ for } y \in V, \tag{54}$$

and by (49) and (54) we obtain

$$-\epsilon < L_r(f)(x) - U_r(f)(x) \leq f(y) - f(x) \leq U_r(f)(x) - L_r(f)(x) < \epsilon$$

for such y and thus $f(V) \subset (f(x) - \epsilon, f(x) + \epsilon) \in \mathcal{T}_{\mathbb{R}}$. Therefore f is continuous $\lambda_{\mathcal{B}}$ a.e. on R with respect to $\mathcal{T}_{\mathcal{B}}$ since $\lambda_{\mathcal{B}}(E) = 0$.

Conversely, assume that f is continuous on $R \setminus E'$ with respect to $\mathcal{T}_{\mathcal{B}}$ where $\lambda_{\mathcal{B}}(E') = 0$. Let $x \in R \setminus (E' \cup E_1)$, and let $\epsilon > 0$. Since f is continuous at x, there is $V = (V_s \times I^s) \cap \mathcal{B} \in \mathcal{T}_{\mathcal{B}}$ such that $x \in V$ and $|f(y) - f(x)| < \frac{\epsilon}{4}$ for $y \in V$. Choose r so large that $x \in H_{k,r} \subset V$ for some $k \in \{1, ..., k_r\}$. Then there are $x_1, x_2 \in H_{k,r}$ such that

$$f(x_1) < m_{k,r}(f) + \tfrac{\epsilon}{4} \;=\; L_r(f)(x_1) + \tfrac{\epsilon}{4} \;=\; L_r(f)(x) + \tfrac{\epsilon}{4} \leq L(x) + \tfrac{\epsilon}{4}, \tag{55}$$

$$f(x_2) > M_{k,r}(f) - \tfrac{\epsilon}{4} \;=\; U_r(f)(x_2) - \tfrac{\epsilon}{4} \;=\; U_r(f)(x) - \tfrac{\epsilon}{4} \geq U(x) - \tfrac{\epsilon}{4}. \tag{56}$$

By the continuity of f at x,

$$|f(x_2) - f(x_1)| \leq |f(x_2) - f(x)| + |f(x_1) - f(x)| < \tfrac{\epsilon}{2} \tag{57}$$

so that by (55), (56), and (57)

$$U(x) - L(x) < |f(x_2) - f(x_1) + \tfrac{\epsilon}{2}| \leq |f(x_2) - f(x_1)| + \tfrac{\epsilon}{2} < \epsilon.$$

Therefore $U(x) = L(x)$ for $\lambda_{\mathcal{B}}$ a.e. $x \in R$; thus by (50) and (51) f is Riemann integrable on R. □

We can extend the definition of the Riemann integral to more general sets contained in a special rectangle in the usual way.

Definition 3.26. *Let $A \subset \mathcal{B} \cap R$ where R is a special rectangle, and let ∂A denote the boundary of A. Let $f : R \to \mathbb{R}$ be Riemann integrable on R. If ∂A has $\lambda_{\mathcal{B}}$ measure zero, then since χ_A is continuous everywhere on R except for ∂A, so that χ_A and $f\chi_A$ are continuous on R $\lambda_{\mathcal{B}}$ a.e. with respect to $\mathcal{T}_{\mathcal{B}}$; then $f\chi_A \in \mathcal{R}[R]$ by* Theorem 3.25. *If these conditions hold we say that f is Riemann integrable on A, we write $f \in \mathcal{R}[A]$, and we define*

$$\int_A f \, dx \;=\; \int_R f\chi_A \, dx.$$

Even though the Riemann and Lebesgue integrals on a special rectangle in $\mathcal{B}$ can differ in value, we will use Theorem 3.25 to extend the usual condition for Riemann integrability and its converse beyond (ii) of this theorem to *any* special Rectangle R in $\mathcal{B}$. By Definition 3.26, this condition also extends to any admissible subset of R.

Theorem 3.27. *Let $f : R \to \mathbb{R}$ be bounded on the special rectangle R in $\mathcal{B}$. Then the following hold.*

(i) *f is Riemann integrable on R if and only if f is continuous $\lambda_{\mathcal{B}}$-a.e. on R with respect to $\tau_{\mathcal{B}}$.*

(ii) *If f is Riemann integrable on R, then f is Lebesgue integrable on R.*

Proof. To prove (i), denote $R = ([b_1, c_1] \times \cdots \times [b_n, c_n] \times I^n) \cap \mathcal{B}$, and let R' be any special rectangle in $\mathcal{B}$ which includes R and the non-degenerate interval $R'' = ([a, b_1] \times I^1) \cap \mathcal{B}$. Define $F = f + \chi_{R''}$, and note that the conditions in Proposition 3.19 hold for F and R'. Let f be Riemann integrable on R. Since f is Riemann integrable on R' part (ii) of Theorem 3.25 holds and f is continuous $\lambda_{\mathcal{B}}$-a.e. on R with respect to $\tau_{\mathcal{B}}$. Thus, f has this same continuity condition on R.

Now assume that f is continuous $\lambda_{\mathcal{B}}$-a.e. on R with respect to $\tau_{\mathcal{B}}$. Since f has this continuity condition on all of R', part (ii) of Theorem 3.25 holds and f is Riemann integrable on R'. Thus f is Riemann integrable on the sub-rectangle R of R' (the proof of this last fact is standard).
Part (ii) holds since by part (i), the Riemann integrability of f on R implies that f is continuous $\lambda_{\mathcal{B}}$-a.e. on R with respect to $\tau_{\mathcal{B}}$, which means that $f \in \mathcal{M}$. Since f is also bounded on R, then $f \in L^1[R]$. □

3.7. **Application of the Riemann integral on $\mathcal{B}$ to Volume.** If a subset A of $\mathcal{B}$ satisfies the criterion in Definition 3.26, we can represent its volume in $\mathcal{B}$ constructively in the usual way as

$$\text{vol}(A) = \int_R \chi_A \, dx, \tag{58}$$

where R is any special rectangle that contains A. Recall from Counter Example 3.13 that

$$\text{vol}\big((I \times J_2 \times I^2) \cap \mathcal{B}\big) = \int_R \chi_{I\times J_2 \times I^2} \, dx = \lambda(J_2) < 1 = \lambda(I) = \int_R \chi_{I\times J_2\times I^2} \, d\lambda_{\mathcal{B}} \tag{59}$$

Where R is an special rectangle in $\mathcal{B}$ that includes A. Thus as (59) demonstrates, the Riemann integral on a special rectangle in $\mathcal{B}$ can represent the volume of a class of subsets in $\mathcal{B}$ that have a sufficiently nice boundary in the sense of Definition 3.26 when the Lebesgue integral fails to equal the volume of such sets. In this way, the Riemann integral advances the Lebesgue theory in [9] slightly.

3.8. **Computing the Riemann integral on a special rectangle in $\mathcal{B}$.** We will now demonstrate the computational advantage of defining a Riemann integral on $\mathcal{B}$ constructively as a limit of finite-dimensional integrals (resulting from integals of tame functions) by using this limit to approximate a Riemann integral of a function on a special rectangle in $\mathcal{B}$ that is not tame.

3.8.1. *Convergence of Integrals.*

Remark 4. *Note that in the proof of part* (i) *of* Theorem 3.25 *we developed sequences of Riemann integrable tame step functions* $\{L_r(f) \in \mathcal{M}_I^r\}_{r=1}^{\infty}$ *for which* $L_r(f) = \bar{L}_r \otimes h_r$ *and*

$$\lim_{r\to\infty} L_r(f)(x) = f(x) \qquad \textit{uniformly for } \lambda_{\mathcal{B}} \textit{ a.e. } x \in R.$$

For any special rectangle $R \subset \mathcal{B}$ *and* $f \in \mathcal{R}[R]$, (32) *and* (37) imply that

$$\int_R f \, dx = \lim_{r\to\infty} \int_R L_r(f) \, dx = \lim_{r\to\infty} \int_{R_m} \bar{L}_r(f) \, d\mathbf{x}_r.$$

We can generalize this result to the following theorem.

Theorem 3.28. *Let the real valued function f be bounded on the special rectangle* $R = (R_n \times I^n) \cap \mathcal{B}$.

(i) *If f is Riemann integrable on R where the conditions in* Proposition 3.19 *apply to f and R, then there exists a sequence of Riemann integrable tame functions* $\{s_m \in \mathcal{M}_I^m\}_{m=1}^{\infty}$ *for which* $\lim_{m\to\infty} s_m(x) = f(x)$ *uniformly for* $\lambda_{\mathcal{B}}$ *a.e.* $x \in R$.
(ii) (The Uniform Convergence Theorem for the Riemann Integral) *If* $\{s_m \in \mathcal{M}_I^m\}_{m=1}^{\infty}$ *is a sequence of Riemann integrable tame functions on R, and* $\lim_{m\to\infty} s_m = f$ *uniformly then f is Riemann integrable on R.*

If either of (i) *or* (ii) *hold, then the following hold where* $R_m = R_n \times I_{n+1}^m$.

$$\int_R f \, dx \; = \; \lim_{m\to\infty} \int_R s_m \, dx \; = \lim_{m\to\infty} \int_{R_m} \bar{s}_m \, d\mathbf{x}_m \; .$$

Proof. Part (i) is true by Remark 4. The proof for part (ii) is the same as for $\mathbb{R}^n$. □

Example 3.29. *We will use the Uniform Convergence Theorem for the Riemann Integral to compute the Riemann integral over $R \subset \mathcal{B}$ for a class of functions $f : R \to \mathbb{R}$ which are not tame. Let* $\sum_{k=1}^{\infty} b_k \, t^k$ *be a power series with a radius of convergence of* $1 \le r \le \infty$. *Note that for* $t = 1$, $\sum_{k=1}^{\infty} b_k$ *converges, so* $\lim_{k\to\infty} b_k = 0$

For $x = (x_1, x_2, ...) \in R = (R_n \times I^n) \cap \mathcal{B}$, *define* $f(x) = \sum_{k=1}^{\infty} b_k \; x_k^k$. *There is some*

$m_1 \in \mathbb{N}$ for which $|\sqrt[k]{|b_k|}x_k| \le \frac{1}{2} < r$ when $k \ge m_1$. Since $|b_k\ x_k^k| \le \frac{1}{2^k}$ and the geometric series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges, the Weirstrass M-test implies that the polynomials $\bar{s}_m(x_1, ..., x_m) = \sum_{k=1}^{m} b_k\ x_k^k$ converge to f uniformly. Because $\lim_{m\to\infty} s_m(x) = f(x)$ *uniformly for $x \in R$ where $s_m = \bar{s}_m \otimes h_m$, then $f \in \mathcal{R}[R]$ and*

$$(60)\quad \int_R \sum_{k=1}^{\infty} \left(b_k\ x_k^k\right) dx = \lim_{m\to\infty} \int_R \left(\sum_{k=1}^{m} b_k\ x_k^k\right) \otimes h_m\ dx = \lim_{m\to\infty} \int_{R_n \times I_{n+1}^m} \left(\sum_{k=1}^{m} b_k\ x_k^k\right) d\mathbf{x}_m.$$

To demonstrate how the limit (60) of Riemann integrals in finite dimensions can approximate the integral in the left member of (60), consider the case where $f(x) = \sum_{k=0}^{\infty} \frac{x_{k+1}^k}{k!}$ and $R = [0,1] \times [0,1] \times [0,1] \times [0,1] \times I^4$. We obtain

(61)

$$\int_R \left(\sum_{k=0}^{\infty} \frac{x_{k+1}^k}{k!}\right) dx = \lim_{m\to\infty} \int_I \cdots \int_I \int_0^1 \int_0^1 \int_0^1 \int_0^1 \left(\sum_{k=0}^{m} \frac{x_{k+1}^k}{k!}\right) dx_1\ dx_2\ dx_3\ dx_4\ dx_5 \cdots dx_m$$

$$= \lim_{m\to\infty} \int_I \cdots \int_I \int_0^1 \int_0^1 \int_0^1 \int_0^1 \left(1 + x_2 + \frac{x_3^2}{2} + \frac{x_4^3}{6}\right) dx_1\ dx_2\ dx_3\ dx_4\ dx_5 \cdots dx_m$$

$$(62)\quad + \lim_{m\to\infty} \int_I \cdots \int_I \int_0^1 \int_0^1 \int_0^1 \int_0^1 \left(\sum_{k=4}^{m} \frac{x_{k+1}^k}{k!}\right) dx_1\ dx_2\ dx_3\ dx_4\ dx_5 \cdots dx_m.$$

We can use the absolute integrability property to bound the limit in (62) *by bounding the integrand as follows*

$$\left|\sum_{k=4}^{\infty} \frac{x_{k+1}^k}{k!}\right| \le \sum_{k=4}^{\infty} \frac{|x_{k+1}|^k}{k!} \le \sum_{k=4}^{\infty} \frac{1}{2^k\ k!} = \sqrt{e} - \sum_{k=0}^{3} \frac{1}{2^k\ k!}\ <\ 0.05\ .$$

Thus the integral in the left member (61) *is approximately $\frac{41}{24}$ with an error of at most* 0.05.

3.8.2. *Convergence of Riemann Sums.* Let $R = (R_n \times I^n) \cap \mathcal{B}$ be a special rectangle. As in the proof of Theorem 3.21 let $\{\mathcal{P}_r\}_{r=1}^{\infty}$ be a sequence of partitions of R denoted $\mathcal{P}_r = \{H_{r,k}\}_{k=1}^{k_r}$ where each $H_{k,r} = (H_{1,k}^r \times I^r) \cap \mathcal{B}$, $\mathcal{P}_{r+1}$ is a refinement of $\mathcal{P}_r$ such that $\lim_{r\to\infty} \mathrm{mesh}(\mathcal{P}_r) = 0$. In this proof we showed that

$$(63)\quad \int_R f\ dx = \lim_{r\to\infty} L(f, \mathcal{P}_r) \ \text{ and } \ \int_R f\ dx = \lim_{r\to\infty} U(f, \mathcal{P}_r).$$

Since $L(f, \mathcal{P}_r) \le S(f, \mathcal{P}_r) \le U(f, \mathcal{P}_r)$, (63) implies that

$$\int_R f\ dx = \lim_{r\to\infty} S(f, \mathcal{P}_r) = \lim_{r\to\infty} \sum_{k=1}^{k_r} f(t_k) \mathrm{vol}(H_{r,k}) = \lim_{r\to\infty} \sum_{k=1}^{k_r} f(\mathbf{t}_{r,k}, 0, 0, ...) \mathrm{vol}(H_{1,k}^r)\ ,$$

where we can choose $t_k = (\mathbf{t}_{r,k}, 0, 0, ...)$ with $\mathbf{t}_{r,k} \in H_{1,k}^r$.

4. Conclusion

This work shows that a sequence of convergent tame functions is the primary tool used to construct the Riemann integral over $\mathcal{B}$ and admissible subsets of $\mathcal{B}$, and to compute integrals over these sets. For instance, The sequences of tame functions $L_r(f)$ and $U_r(f)$ were the key to proving the equivalence between the Darboux and Riemann integrals and to establishing a modified version of the usual relationship between the Riemann and Lebesgue integrals on $\mathcal{B}$, which extends the connection between these two integrals from finite to infinite dimensions.

The Riemann integral on a special rectangle in $\mathcal{B}$ can provide the initial structure for defining a Henstock-Kurzweil type integral on $\mathcal{B}$, and for using it to potentially advance the theory of constructive analysis by developing tools for studying highly oscillatory functions on infinite dimensions. Also, an integral on $\mathcal{B}$ may be useful for extending recent advancements in constructive and functional analysis such as the following. In [7] Castro and Guerra define a Quadratic Phase Fourier Transform on $\mathbb{R}^n$ that may be expandable to infinite dimensions. The general Hardy type operators that Ho and Yee define on local generalized Morrey spaces in [14] using real-valued Lebesgue measurable functions on intervals in $(0, \infty)$ may be suitable for Riemann integrable or Lebesgue measurable functions on special subsets of $\mathcal{B}$. Since Antipova defined the inverse for a multidimensional Mellin transform [1], the results which Milgram and Hughes obtain from integrals on $\mathbb{R}$ in [18] may be extendable to a Riemann integral on all of $\mathcal{B}$. Indeed, tame functions should be able to lift such recent advances as these from finite to infinite dimensions by a convergence process that parallels the way in which they extended the Riemann integral from $\mathbb{R}^n$ to $\mathcal{B}$.

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Geometry of curves and equivalence of integrable nonlinear systems: the Kairat-Kuralay-Myrzakulov-Shynaray equation

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Abstract. The Kairat-Kuralay-Myrzakulov-Shynaray equation is studied as an integrable model, relevant to nonlinear wave phenomena in hydrodynamics, optics, quantum mechanics, and plasma physics. Integrable reductions and Lax pairs are derived, and the system's geometric interpretation is discussed. Exact solutions with periodic, exponential, and rational profiles, including Jacobi elliptic forms, are obtained and visualized through three-dimensional and contour plots. The results enhance the understanding of the geometric structure of integrable systems and nonlinear wave dynamics in physical applications.

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1. Introduction

Nonlinear differential equations (NDEs), and in particular nonlinear partial differential equations (PDEs), play a fundamental role in mathematics, physics, and several other fields of science, describing a wide range of complex natural phenomena and processes. Among them, integrable nonlinear PDEs are of particular interest because they allow for precise analysis, exhibit rich dynamics, and possess localized wave so- lutions known as solitons. In recent decades, numerous methods for studying integrable sys- tems have been developed. Significant progress was achieved with the discovery of the inverse scattering transform method (1967), which laid the foundation for soliton theory[1]-[3] The most well-known models — the Korteweg–de Vries equation, the nonlinear Schr$o$dinger equation, the sine-Gordon equation, and the Heisenberg model — have gained widespread acceptance due to their ability to clearly describe a wide range of physical phenomena using localized solutions [4]-[5]. Subsequently, integrable multidimensional extensions of soliton models [6]-[8] attracted the attention of researchers, stimulating the development of new analytical approaches. Modern research focuses both on finding new integrable systems and constructing their exact solutions. In particular, equations with fractional derivatives [9]-[11]are actively studied, allowing one to model phenomena with memory effects, nonlocality, and anomalous dynamics. Furthermore, integrable systems with self-consistent potentials [12], as well as their various modifications and generalizations [13]-[14], are of considerable interest. One representative of this class is the Kairat-Kuralay-Myrzakulov-Shynaray system [15], which admits a Lax pair with a nonzero spectral parameter. At the same time, the field of constructing exact solutions to nonlinear PDEs using modern analytical and symbolic methods is developing. Approaches such as the first integral method, the exp-function method, the modified extended tanh function method, the auxiliary equation method, and others have been proposed [16]-[23]. The obtained exact solutions not only allow a deeper understanding of nonlinear physical processes but also play an important role in testing and validating numerical schemes. Thus, the problem of finding new integrable systems and their exact solutions remains one of the central topics of modern mathematical physics and the theory of nonlinear equations.

2. The Kairat-Kuralay-Myrzakulov-Shynaray equation

Consider the well-known Kairat-Kuralay-Myrzakulov-Shynaray (KKMS) equation. The KKMS equation reads as (see for example, [18] and references therein):

$$q_t - \frac{1}{b}uq_x + \frac{\beta}{b}qq_t - \beta r_t = 0, \tag{1}$$

$$r_t - \frac{1}{b}ur_x + \frac{\beta}{b}rq_t - \frac{\beta}{4ab}q_{xxt} = 0, \tag{2}$$

$$u_x + 0.5\beta q_t = 0, \tag{3}$$

where a, b, β are real constants, (q, r, u) are some functions of (x, t). Note that the KKMS equation (1)-(3) is integrable. Its Lax representation is given by [18]

$$\Phi_x = U_1\Phi, \tag{4}$$

$$\Phi_t = V_1\Phi = \frac{1}{1-\beta\lambda}B\Phi, \tag{5}$$

where

$$U_1 = \begin{pmatrix} 0 & a \\ b\lambda^2 + q\lambda + r & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & a \\ r & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{6}$$

$$B = B_2\lambda^2 + B_1\lambda + B_0, \tag{7}$$

$$B_2 = (a_{ij}) = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} = u\Sigma, \quad B_1 = (b_{ij}) = \begin{pmatrix} 0 & 0 \\ b^{-1}uq & 0 \end{pmatrix} = b^{-1}uq\Sigma, \tag{8}$$

$$B_0 = (c_{ij}) = \begin{pmatrix} \frac{\beta}{4b}q_t & ab^{-1}u \\ b^{-1}ur + \frac{\beta}{4ab}q_{xt} & -\frac{\beta}{4b}q_t. \end{pmatrix} = -\frac{u_x}{2b}\sigma_3 + \frac{u}{b}Q - \frac{u_{xx}}{2ab}\Sigma. \tag{9}$$

The compatibility condition $\Phi_{xt} = \Phi_{tx}$ of the linear equations (4)-(5) that is

$$(1-\beta\lambda)U_t - B_x + [U, B] = 0 \tag{10}$$

gives the KKMS equation (1)-(3). As the integrable equation, the KKMSE (1)-(3) has the N-soliton solution, infinite number of conservation laws, Hamiltonian structure and so on.

3. Integrable motion of curves induced by the KKMS equation

The aim of this section is to present the geometric formulation of the KKMS equation in terms of curves and to find its geometrical equivalent counterpart.

We start from the differential geometry of space curves. In this subsection, we consider the integrable motion of space curves induced by the KKMS equation. As usual, let us consider a smooth space curve $\gamma(x,t) : [0, X] \times [0, T] \to R^3$ in R^3. Let x is the arc length of the curve at each time t. In differential language, such curve is given by the Frenet-Serret equation (FSE). The FSE and its temporal counterpart look like

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_x = C\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}_t = G\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \tag{11}$$

where $\mathbf{e}_j$ are the unit tangent vector ($j = 1$), principal normal vector ($j = 2$) and binormal vector ($j = 3$) which given by $\mathbf{e}_1 = \gamma_x, \quad \mathbf{e}_2 = \frac{\gamma_{xx}}{|\gamma_{xx}|}, \quad \mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2$, respectively. Here

$$C = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \tau \\ -\kappa_2 & -\tau & 0 \end{pmatrix} = -\tau L_1 + \kappa_2 L_2 - \kappa_1 L_3 \in so(3),$$

$$G = \begin{pmatrix} 0 & \omega_3 & \omega_2 \\ -\omega_3 & 0 & \omega_1 \\ -\omega_2 & -\omega_1 & 0 \end{pmatrix} = -\omega_1 L_1 + \omega_2 L_2 - \omega_3 L_3 \in so(3),$$

where τ, κ_1, κ_2 are the "torsion", "geodesic curvature" and "normal curvature" of the curve, respectively; ω_j are some functions. Note that L_j are basis elements of $so(3)$ algebra and have the forms

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

They satisfy the following commutation relations

$$[L_1, L_2] = L_3, \quad [L_2, L_3] = L_1, \quad [L_3, L_1] = L_2.$$

In the following, we need also in the basis elements of $su(2)$ algebra. They have the forms

$$e_1 = \frac{1}{2i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \frac{1}{2i}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where the Pauli matrices have the form

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These elements satisfy the following commutation relations

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.$$

Note that the Pauli matrices obey the following commutation relations

$$[\sigma_1, \sigma_2] = 2i\sigma_3, \quad [\sigma_2, \sigma_3] = 2i\sigma_1, \quad [\sigma_3, \sigma_1] = 2i\sigma_2$$

or

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k.$$

The well-known isomorphism between the Lie algebras $su(2)$ and $so(3)$ means the following correspondence between their basis elements $L_j \leftrightarrow e_j$. Using this isomorphism let us construct the following two matrices

$$U = -\tau e_1 + \kappa_2 e_2 - \kappa_1 e_3 = -\frac{1}{2i}\begin{pmatrix} \kappa_1 & \tau + i\kappa_2 \\ \tau - i\kappa_2 & -\kappa_1 \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & -u_{11} \end{pmatrix},$$
$$V = -\omega_1 e_1 + \omega_2 e_2 - \omega_3 e_3 = -\frac{1}{2i}\begin{pmatrix} \omega_3 & \omega_1 + i\omega_2 \\ \omega_1 - i\omega_2 & -\omega_3 \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & -v_{11} \end{pmatrix},$$

where

$$u_{11} = 0.5ik_1, \quad u_{12} = 0.5i(\tau + ik_2), \quad u_{21} = 0.5i(\tau - ik_2),$$
$$v_{11} = 0.5i\omega_3, \quad v_{12} = 0.5i(\omega_1 + i\omega_2), \quad v_{21} = 0.5i(\omega_1 - i\omega_2).$$

Hence we obtain

$$\kappa_1 = -2iu_{11}, \quad \kappa_2 = -(u_{12} - u_{21}), \quad \tau = -i(u_{12} + u_{21}),$$
$$\omega_1 = -i(v_{12} + v_{21}), \quad \omega_2 = -(v_{12} - v_{21}), \quad \omega_3 = -2iv_{11}.$$

The compatibility condition of the equations (10)-(11) reads as

$$C_t - G_x + [C, G] = U_t - V_x + [U, V] = 0$$

or in elements

$$\kappa_{1t} - \omega_{3x} - \kappa_2\omega_1 + \tau\omega_2 = 0,$$
$$\kappa_{2t} - \omega_{2x} + \kappa_1\omega_1 - \tau\omega_3 = 0,$$
$$\tau_t - \omega_{1x} - \kappa_1\omega_2 + \kappa_2\omega_3 = 0.$$

We now assume that

$$\kappa_1 = 0, \quad \kappa_2 = -(a - b\lambda^2 - q\lambda - r), \quad \tau = -i(a + b\lambda^2 + q\lambda + r),$$

and

$$\omega_1 = -i[\lambda^2(a_{12} + a_{21}) + \lambda(b_{12} + b_{21}) + (c_{12} + c_{21})],$$
$$\omega_2 = -[\lambda^2(a_{12} - a_{21}) + \lambda(b_{12} - b_{21}) + (c_{12} - c_{21})],$$
$$\omega_3 = -2i[\lambda^2 a_{11} + \lambda b_{11} + c_{11}],$$

where $(a_{ij}) = B_2, \quad (b_{ij}) = B_1, \quad (c_{ij}) = B_0$ as follows from our notations (8)-(9). Then it is not difficult to verify that Eqs.(1)-(3) give us the following equations for q, r, u:

$$q_t - \frac{1}{b}uq_x + \frac{\beta}{b}qq_t - \beta r_t = 0,$$
$$r_t - \frac{1}{b}ur_x + \frac{\beta}{b}rq_t - \frac{\beta}{4ab}q_{xxt} = 0,$$
$$u_x + 0.5\beta q_t = 0.$$

It is nothing but the KKMS equation (1)-(3). Finally we note that as follows from (12), the corresponding space curve is with the zero curvature $\kappa_1 = 0$.

4. Exact travelling wave solutions

In modern mathematical and theoretical physics, one of the most important topics is to find the exact solutions of nonlinear differential equations. In this section, we want find exact traveling wave solutions of the KKMS equation. To find the traveling wave solutions of the KKMS equation (1)-(3), we consider the following transformations

$$q(x,t) = q(\zeta), \quad r(x,t) = r(\zeta), \quad u(x,t) = u(\zeta), \quad \zeta = \mu x + \nu t,$$

where μ, ν are some real constants. Then we have

$$q_x = \mu q', q_{xx} = \mu^2 q'', q_{xxt} = \mu^2 \nu q''', r_t = \nu r', r_x = \mu r', u_x = \mu u',$$

where $f' = \frac{df}{d\zeta}$. Plugging these expressions into the KKMSE (1)-(3) we obtain

$$\nu q' - \frac{\mu}{b} u q' + \frac{2\beta\nu}{b} q q' - \beta\nu r' = 0, \tag{12}$$

$$\nu r' - \frac{\mu}{b} u r' + \frac{2\beta\nu}{b} r q' - \frac{\beta\mu^2\nu}{2ab} q''' = 0, \tag{13}$$

$$\mu u' + 0.5\beta\nu q' = 0. \tag{14}$$

From Eq.(14) we get

$$u = c_1 - \frac{\beta\nu}{2\mu} q,$$

where $c_1 = const$ is an integration constant. Then Eqs.(12)-(13) take the forms

$$\nu q' - \frac{\mu}{b}(c_1 - \frac{\beta\nu}{2\mu} q) q' + \frac{2\beta\nu}{b} q q' - \beta\nu r' = 0, \tag{15}$$

$$\nu r' - \frac{\mu}{b}(c_1 - \frac{\beta\nu}{2\mu} q) r' + \frac{2\beta\nu}{b} r q' - \frac{\beta\mu^2\nu}{2ab} q''' = 0. \tag{16}$$

From (15), we obtain

$$c_2\beta\nu + \nu q - \frac{\mu}{b} c_1 q + \frac{\beta\nu}{2b} q^2 + \frac{\beta\nu}{b} q^2 - \beta\nu r = 0,$$

where $c_2 = const$ is an integration constant. Hence we get

$$c_2\beta\nu + (\nu - \frac{\mu}{b} c_1) q + (\frac{\beta\nu}{2b} + \frac{\beta\nu}{b}) q^2 - \beta\nu r = 0,$$

or

$$c_2\beta\nu + c_3\beta\nu q + c_4\beta\nu q^2 - \beta\nu r = 0,$$

where

$$c_3 = \frac{1}{\beta\nu}(\nu - \frac{\mu}{b} c_1), \quad c_4 = \frac{1}{2\beta\nu}(\frac{\beta\nu}{b} + \frac{2\beta\nu}{b}) = \frac{3}{2b}, \quad c_2 = const.$$

The last equation gives

$$r = c_2 + c_3 q + c_4 q^2. \tag{17}$$

Hence we have

$$c_3\beta\nu q' + 2c_4\beta\nu q q' - \beta\nu r' = 0, \tag{18}$$

or

$$r' = c_3 q' + 2c_4 q q'. \tag{19}$$

Let us rewrite Eq.(16) as

$$c_3\beta\nu r' + \frac{\beta}{b} q r' + \frac{2\beta\nu}{b} r q' - \frac{\beta\mu^2\nu}{2ab} q''' = 0,$$

Hence, using (17) and (19), finally, we get

$$c_3\beta\nu(c_3 q' + 2c_4 q q') + \frac{\beta}{b} q(c_3 q' + 2c_4 q q') + \frac{2\beta\nu}{b}(c_2 + c_3 q + c_4 q^2) q' - \frac{\beta\mu^2\nu}{2ab} q''' = 0.$$

Integrating this equation one times, we obtain the following equation

(20) $$q'' = c_5q^3 + c_6q^2 + c_7q + c_8,$$

where $c_8 = const$ is a constant of integration and

$$\begin{aligned} c_5 &= \frac{2ab}{3\beta\mu^2\nu}(\frac{2c_4\beta}{b} + \frac{2\beta\nu c_4}{b}), \\ c_6 &= \frac{2ab}{2\beta\mu^2\nu}(\frac{2c_4\beta}{b} + \frac{2\beta\nu c_4}{b}), \\ c_7 &= \frac{2ab}{2\beta\mu^2\nu}(c_3^2\beta\nu + \frac{2\beta\nu c_2}{b}) = \frac{2ab}{2\mu^2}(c_3^2 + \frac{2c_2}{b}), \\ c_8 &= const. \end{aligned}$$

Hence we have

$$2q'q'' = (c_5q^3 + c_6q^2 + c_7q + c_8)2q',$$

so that

(21) $$q'^2 = c_9q^4 + c_{10}q^3 + c_{11}q^2 + c_{12}q + c_{13},$$

where

$$\begin{aligned} c_9 &= \frac{1}{2}c_5, \\ c_{10} &= \frac{2}{3}c_6, \\ c_{11} &= c_7, \\ c_{12} &= 2c_8, \\ c_{13} &= const. \end{aligned}$$

Let us introduce a new function $p(x,t) = p(\zeta)$ as

$$q = c_{14}p + c_{15},$$

where c_{ij} are constants. Then Eq.(20) takes the form

$$c_{14}^2p'' = c_5(c_{14}p + c_{15})^3 + c_6(c_{14}p + c_{15})^2 + c_7(c_{14}p + c_{15}) + c_8,$$

or

$$p'' = c_{16}p^3 + c_{17}p^2 + c_{18}p + c_{19},$$

where

$$\begin{aligned} c_{16} &= \frac{c_5c_{14}^3}{c_{14}^2} = c_5c_{14}, \\ c_{17} &= \frac{3c_5c_{14}^2 + c_6c_{14}^2}{c_{14}^2} = 3c_5 + c_6, \\ c_{18} &= \frac{3c_5c_{14}c_{15}^2 + 2c_6c_{14}c_{15} + c_7c_{14}}{c_{14}^2} = \frac{3c_5c_{15}^2 + 2c_6c_{15} + c_7}{c_{14}}, \\ c_{19} &= \frac{c_5c_{15}^3 + c_6c_{15}^2 + c_7c_{15} + c_8}{c_{14}^2}. \end{aligned}$$

Now we assume that

(22) $$p'' = 2mp^3 - (1+m)p$$

or

(23) $$p'^2 = (1-p^2)(1-mp^2),$$

where

$$m = 0.5c_{16} = 0.5c_5c_{14},\ c_{17} = c_{19} = 0,\ c_{18} = -(1+m) = -(1 + 0.5c_5c_{14}).$$

These expressions give us

$$c_6 = -3c_5, \quad c_8 = -(c_5c_{15}^3 - 3c_5c_{15}^2 + c_7c_{15}).$$

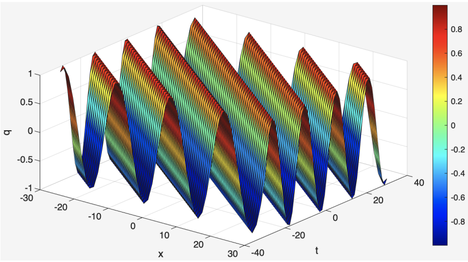
It is well-known that the equations (22)-(23) have the following solution

$$p(\zeta, m) = sn(\zeta, m),$$

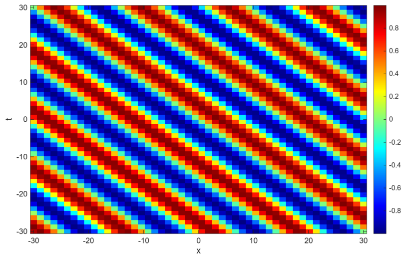
where $sn(\zeta, m)$ is the Jacobi elliptic function.

4.1. **Jacobi elliptic function solution.** The previous results give us the following solution of the KKMSE in terms of the Jacobi elliptic function as

$$\begin{aligned} q &= c_{14} sn(\zeta, m) + c_{15}, \\ r &= c_2 + c_3[c_{14} sn(\zeta, m) + c_{15}] + c_4[c_{14} sn(\zeta, m) + c_{15}]^2, \\ u &= c_1 - \frac{\beta\nu}{\mu}[c_{14} sn(\zeta, m) + c_{15}]. \end{aligned}$$



(A)

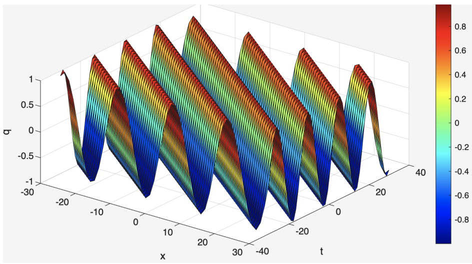


(B)

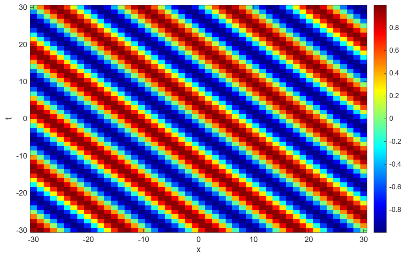
FIGURE 1. (a) The 3D plot and (b) contour plot of the Jacobi elliptic function solution. Parameters: $\beta = -1,\ \mu = 0.2,\ \nu = 0.2, c_1 = 0, c_2 = 0, c_3 = 0.5, c_{14} = 0.5$, and $c_{15} = 0.3$.

4.2. **Trigonometric function solution.** Using the well-known properties of the Jacobi elliptic function $sn(\zeta, m)$: $sn(\zeta, 0) = \sin(\zeta)$, we obtain the following trigonometric function solution of the KKMSE:

$$\begin{aligned} q &= c_{14} \sin(\zeta) + c_{15}, \\ r &= c_2 + c_3[c_{14} \sin(\zeta) + c_{15}] + c_4[c_{14} \sin(\zeta) + c_{15}]^2, \\ u &= c_1 - \frac{\beta\nu}{\mu}[c_{14} \sin(\zeta) + c_{15}]. \end{aligned}$$



(A)

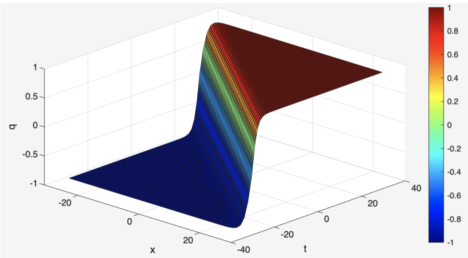


(B)

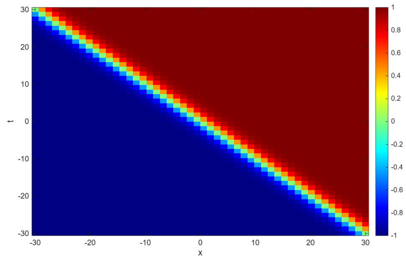
FIGURE 2. (a) The 3D plot and (b) contour plot of the Trigonometric function solution. Parameters: $\beta = -1,\ \mu = 0.2,\ \nu = 0.2, c_1 = 0, c_2 = 0, c_3 = 0.5, c_{14} = 0.5$, and $c_{15} = 0.3$.

4.3. **Soliton solution.** Now we use the following properties of the Jacobi elliptic function $sn(\zeta, m)$: $sn(\zeta, 1) = \tanh(\zeta)$. The corresponding trigonometric function solution of the KKMSE reads as:

$$\begin{aligned} q &= c_{14} \tanh(\zeta) + c_{15}, \\ r &= c_2 + c_3[c_{14} \tanh(\zeta) + c_{15}] + c_4[c_{14} \tanh(\zeta) + c_{15}]^2, \\ u &= c_1 - \frac{\beta\nu}{\mu}[c_{14} \tanh(\zeta) + c_{15}]. \end{aligned}$$



(A)



(B)

FIGURE 3. (a) The 3D plot and (b) contour plot of the Soliton solution. Parameters: $\beta = -1,\ \mu = 0.2,\ \nu = 0.2, c_1 = 0, c_2 = 0, c_3 = 0.5, c_{14} = 0.5$, and $c_{15} = 0.3$.

4.4. **Rational solution.** To find the rational solution of the KKMSE, we consider the following expression for the function $q = \frac{k}{\zeta} + l$. Then the rational solution of the KKMSE is given by

$$
\begin{aligned}
q &= \frac{k}{\zeta} + l, \\
r &= c_2 + c_3[\frac{k}{\zeta} + l] + c_4[\frac{k}{\zeta} + l]^2, \\
u &= c_1 - \frac{\beta\nu}{\mu}[\frac{k}{\zeta} + l].
\end{aligned} \tag{24}
$$

where k, l are some new constants. To fix constants k, l, we use Eq.(**??**) that is

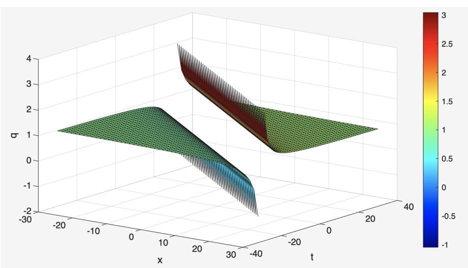
$$q^{''} = c_5 q^3 + c_6 q^2 + c_7 q + c_8, \tag{25}$$

Subctituting (24) into Eq.(25), we get

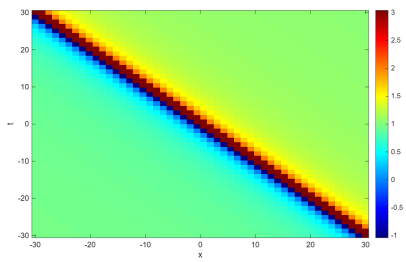
$$
\begin{aligned}
y^{-3} &: \ 2k = c_5 k^3, \\
y^{-2} &: \ 0 = 3c_5 k^2 l + c_6 k^2, \\
y^{-1} &: \ 0 = 3c_5 k l^2 + 2klc_6 + c_7 k, \\
y^{-0} &: \ 0 = c_5 l^3 + c_6 l^2 + c_7 l + c_8.
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
k &= \pm\sqrt{\frac{2}{c_5}}, \\
l &= -\frac{c_6}{3c_5}, \\
c_7 &= -(3c_5 l^2 + 2lc_6), \\
c_8 &= -(c_5 l^3 + c_6 l^2 + c_7 l).
\end{aligned}
$$



(A)



(B)

FIGURE 4. (a) The 3D plot and (b) contour plot of the rational solution. Parameters: $\beta = -1,\ \mu = 0.2,\ \nu = 0.2, c_1 = 0, c_2 = 0, c_3 = 0.5, c_{14} = 0.5$, and $c_{15} = 0.3$.

5. Hamiltonian structure

From (20) and (21) follow that

$$\begin{aligned} q^{'} &= z, \\ z^{'} &= c_5q^3 + c_6q^2 + c_7q + c_8, \end{aligned}$$

or

$$\begin{aligned} q^{'} &= \frac{\delta H}{\delta z}, \\ z^{'} &= -\frac{\delta H}{\delta q}. \end{aligned}$$

Here H has the form

$$H = \int [\frac{1}{2}z^2 - (\frac{1}{4}c_5q^4 + \frac{1}{3}c_6q^3 + \frac{1}{2}c_7q^2 + c_8q)]ds.$$

6. Conclusion

This paper examines the KKMS equation, a relatively recently proposed equation that has already attracted the attention of a number of researchers. Using the traveling-wave method, new exact solutions were obtained that expand the existing solution structure and complement existing results for this model. The obtained solutions demonstrate the effectiveness of the traveling-wave method in analyzing modern nonlinear integrable systems and confirm its significance for further study of models with a rich mathematical structure. The obtained results can be used both for a deeper theoretical understanding of the dynamics of the KKMS equation and for subsequent numerical analy- sis and applied modeling of nonlinear waves. Thus, this study contributes to the development of the theory of integrable systems and opens prospects for further study of equations of this class, in particular their multidimensional generalizations, fractional-order solutions, and con- nections with other models of nonlinear mathematical physics.

Acknowledgments

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Bijective mapping in Pythagorean triples

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Abstract. Dawson constructed a Pythagorean ring $\langle P,\ \oplus,\ \circ\rangle$ and identified a bijective mapping $\varphi : P \longrightarrow \mathbb{Z}\times\mathbb{Z}$. In this note, we have established an extension of the bijective mapping $\varphi : P \longrightarrow \mathbb{Z}^2$ in Pythagorean triples to the bijective mapping $\varphi : P^2 \longrightarrow \mathbb{Z}^4$ and further generalized it for $\varphi : P^n \longrightarrow \mathbb{Z}^{2n}$.

2020 Mathematics Subject Classifications: 05C15, 11D09, 13M05, 14G50

Keywords: Pythagorean Ring, Injective, Surjective, Bijective

1. Introduction

A Pythagorean Triple (PT) is a triple of positive integers $(a,\ b,\ c)$, which satisfies the Pythagorean equation

$$a^2+b^2=c^2,$$

where c represents the length of the hypotenuse, a and b represent the lengths of the other two sides (called legs) of a right triangle. In other words a Pythagorean triple represents the lengths of the sides of a right triangle where all the three sides have integer lengths. We say a Pythagorean triple $(a,\ b,\ c)$ is primitive if the numbers $a,\ b$ and c are pairwise co-prime [7]. Several formulas have been establish which generate Pythagorean triples, such as in [1, 2, 3, 6].

Grytczuk [5] and Wojtowicz [8] gave constructions of Pythagorean rings with new operations on P_n, different from [4].

In [4], Dawson gave a construction of a Pythagorean ring

$$P=\{<a,b,c>\in\mathbb{Z}^3 : a^2+b^2=c^2\}.$$

In this ring, he defined the operations $\oplus$ and $\circ$ by an isomorphism $\varphi : P \longrightarrow \mathbb{Z}\times\mathbb{Z}$ given by

$$\varphi\big(\langle a,\ b,\ c\rangle\big)=\big(c-b,\ d'(\langle a,\ b,\ c\rangle)\big) \qquad (1)$$

where

$$d'\big(\langle a,\ b,\ c\rangle\big)=\begin{cases}\frac{a}{r'(c-b)}, & if\ c-b\ even,\ n\neq 0\\ \frac{\frac{a}{r'(c-b)}-1}{2}, & if\ c-b\ odd\\ b, & if\ c-b=0\end{cases}.$$

and r' is defined as $r' : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ by $r'(x)=2^{b_0}p_1^{b_1}p_2^{b_2}...p_m^{b_m}$ where x has prime factorization of $x=2^{a_0}p_1^{a_1}p_2^{a_2}...p_m^{a_m}$,

$$b_k=\left\lceil\frac{a_k}{2}\right\rceil \quad k=1,\ 2,\ ...,\ m$$

$$b_0=\begin{cases}0, & if\ x\ is\ odd\\ \left\lceil\frac{a_0+1}{2}\right\rceil, & x\ is\ even\end{cases}$$

Dawson also established that $\langle P,\ \oplus,\ \circ\rangle$ is a commutative ring with identity where $\oplus$ and $\circ$ are operations on P defined by

$$\begin{aligned}\langle a,\ b,\ c\rangle\oplus\langle e,\ f,\ g\rangle &= \varphi^{-1}\Big(\varphi\big(\langle a,\ b,\ c\rangle\big)+\varphi\big(\langle e,\ f,\ g\rangle\big)\Big)\\ &= \varphi^{-1}\Big(\big(c-b,\ d'\langle a,\ b,\ c\rangle\big)+\big(g-f,\ d'\langle e,\ f,\ g\rangle\big)\Big)\\ &= \varphi^{-1}\Big(c-b+g-f,\ d'\langle a,\ b,\ c\rangle+d'\langle e,\ f,\ g\rangle\Big)\end{aligned}$$

so that,

$$\langle a,\ b,\ c\rangle\oplus\langle e,\ f,\ g\rangle=\begin{cases}\langle h,\ \frac{h^2-n^2}{2n},\ \frac{h^2+n^2}{2n}\rangle & for\ n\neq 0,\ is\ even\\ \langle k,\ \frac{k^2-n^2}{2n},\ \frac{k^2+n^2}{2n}\rangle & for\ n\ odd\\ \langle 0,\ j,\ j\rangle & for\ n=0\end{cases}$$

where

$$\begin{aligned}h &= \big[d'(\langle a,\ b,\ c\rangle)+d'(\langle e,\ f,\ g\rangle)\big]r'(n)\\ n &= c-b+g-f\\ k &= \big[2\big[d'(\langle a,\ b,\ c\rangle)+d'(\langle e,\ f,\ g\rangle)\big]+1\big]r'(n)\\ j &= d'(\langle a,\ b,\ c\rangle)+d'(\langle e,\ f,\ g\rangle).\end{aligned}$$

In addition

$$\langle a,\ b,\ c\rangle\circ\langle e,\ f,\ g\rangle=\begin{cases}\langle h,\ \frac{h^2-n^2}{2n},\ \frac{h^2+n^2}{2n}\rangle & for\ n\neq 0,\ even\ is\ odd\\ \langle k,\ \frac{k^2-n^2}{2n},\ \frac{k^2+n^2}{2n}\rangle & for\ n\ odd\\ \langle 0,\ j,\ j\rangle & for\ n=0\end{cases}$$

where

$$\begin{aligned}h &= \big[d'(\langle a,\ b,\ c\rangle)d'(\langle e,\ f,\ g\rangle)\big]r'(n)\\ n &= (c-b)(g-f)\\ k &= \big[2\big[d'(\langle a,\ b,\ c\rangle)d'(\langle e,\ f,\ g\rangle)\big]+1\big]r'(n)\\ j &= d'(\langle a,\ b,\ c\rangle)d'(\langle e,\ f,\ g\rangle).\end{aligned}$$

In this note, we have established an extension of the bijective mapping $\varphi: P\longrightarrow\mathbb{Z}^2$ in Pythagorean triples to the bijective mapping $\varphi: P^2\longrightarrow\mathbb{Z}^4$ and further generalized it for $\varphi: P^n\longrightarrow\mathbb{Z}^{2n}$.

2. The Isomorphism $\varphi: P^2\longrightarrow\mathbb{Z}^4$

Let us begin by proving the following result by Dawson which is stated but not proved in [4].

Theorem 2.1. *The mapping $\varphi: P\longrightarrow\mathbb{Z}^2$ given by $\varphi\big(\langle a,\ b,\ c\rangle\big)=\big(c-b,\ d'\langle a,\ b,\ c\rangle\big)$ is bijective.*

Proof. To show that $\varphi: P\longrightarrow\mathbb{Z}^2$ is bijective, we prove that φ is both injective and surjective. We first show that φ is injective:

$$\begin{aligned}Ker\varphi &= \big\{\langle a,\ b,\ c\rangle\in P\mid\varphi(\langle a,\ b,\ c\rangle)=(0,\ 0)\big\}\\ &= \big\{\langle a,\ b,\ c\rangle\in P\mid(c-b,\ d'(\langle a,\ b,\ c\rangle))=(0,\ 0)\big\}\\ &= \big\{\langle a,\ b,\ c\rangle\in P\mid c-b=0;\ d'(\langle a,\ b,\ c\rangle)=0\big\}\\ &= \big\{\langle a,\ b,\ c\rangle\in P\mid c-b=0;\ d'(\langle a,\ b,\ c\rangle)=b=0\big\}\\ &= \big\{\langle a,\ b,\ c\rangle\in P\mid b=c=0\big\}\\ &= \big\{\langle 0,\ 0,\ 0\rangle\big\}\end{aligned}$$

Hence injective.

To show that φ is surjective, we show that every element of $\mathbb{Z}^2$ has a unique pre-image in P.
Indeed for each $(a,\ b) \in \mathbb{Z}^2$, we have that $\varphi^{-1}(a,\ b) = \big(\langle\sqrt{a^2+2ab},\ b,\ a+b\rangle\big)$.

□

In this section we show that $\varphi : P^2 \longrightarrow \mathbb{Z}^4$ is a bijection where P^2 is defined by

$$P^2 = \{\big(\langle a,\ b,\ c\rangle,\ \langle a',\ b',\ c'\rangle\big) \mid a^2+b^2=c^2,\ a'^2+b'^2=c'^2\}.$$

We define the operations $\oplus$ and $\circ$ in $\varphi : P^2 \longrightarrow \mathbb{Z}^4$ by the following Proposition

Proposition 2.2. *The mapping $\varphi : P^2 \longrightarrow \mathbb{Z}^4$ given by $\varphi\big(\langle a,\ b,\ c\rangle,\ \langle a',\ b',\ c'\rangle\big) = \big(c-b,\ d'\langle a,\ b,\ c\rangle,\ c'-b',\ d'\langle a',\ b',\ c'\rangle\big)$ is bijective. Moreover, $\langle P^2,\ \oplus,\ \circ\rangle$ is a commutative ring with identity where $\oplus$ and $\circ$ are operations in P^2 defined by*

$$\big(\langle a,\ b,\ c\rangle,\ \langle a',\ b',\ c'\rangle\big) \oplus \big(\langle e,\ f,\ g\rangle,\ \langle e',\ f',\ g'\rangle\big) = \varphi^{-1}\Big(\varphi\big(\langle a,\ b,\ c\rangle,\ \langle a',\ b',\ c'\rangle\big) + \varphi\big(\langle e,\ f,\ g\rangle,\ \langle e',\ f',\ g'\rangle\big)\Big)$$

and

$$\big(\langle a,\ b,\ c\rangle,\ \langle a',\ b',\ c'\rangle\big) \circ \big(\langle e,\ f,\ g\rangle,\ \langle e',\ f',\ g'\rangle\big) = \varphi^{-1}\Big(\varphi\big(\langle a,\ b,\ c\rangle,\ \langle a',\ b',\ c'\rangle\big) \cdot \varphi\big(\langle e,\ f,\ g\rangle,\ \langle e',\ f',\ g'\rangle\big)\Big)$$

Proof. To show that $\varphi : P^2 \longrightarrow \mathbb{Z}^4$ is bijective, we prove that φ is both injective and surjective.
We first show that φ is injective:

$$
\begin{aligned}
&Ker\varphi\\
&= \big\{\big(\langle a,\ b,\ c\rangle,\ \langle a',\ b',\ c'\rangle\big) \in P^2 \mid \varphi\big(\langle a,\ b,\ c\rangle,\ \langle a',\ b',\ c'\rangle\big) = (0,\ 0,\ 0,\ 0)\big\}\\
&= \big\{\big(\langle a,\ b,\ c\rangle,\ \langle a',\ b',\ c'\rangle\big) \in P^2 \mid \big(c-b,\ d'\langle a,\ b,\ c\rangle,\ c'-b',\ d'\langle a',\ b',\ c'\rangle\big) = (0,\ 0,\ 0,\ 0)\big\}\\
&= \big\{\big(\langle a,\ b,\ c\rangle,\ \langle a',\ b',\ c'\rangle\big) \in P^2 \mid c-b=0;\ d'(\langle a,\ b,\ c\rangle)=0;\ c'-b'=0;\ d'(\langle a',\ b',\ c'\rangle)=0\big\}\\
&= \big\{\big(\langle a,b,c\rangle,\ \langle a',b',c'\rangle\big) \in P^2 \mid c-b=0; d'(\langle a,b,c\rangle)=b=0; c'-b'=0; d'(\langle a',b',c'\rangle)=b'=0\big\}\\
&= \big\{\big(\langle a,\ b,\ c\rangle,\ \langle a',\ b',\ c'\rangle\big) \in P^2 \mid b=c=b'=c'=0\big\}\\
&= \big\{\big(\langle 0,\ 0,\ 0\rangle,\ \langle 0,\ 0,\ 0\rangle\big)\big\}
\end{aligned}
$$

Hence injective.

To show that φ is surjective, we show that every element of $\mathbb{Z}^4$ has a unique pre-image in P^2.
Indeed for each $(a,\ b,\ a',\ b') \in \mathbb{Z}^4$, we have that

$$\varphi^{-1}(a,\ b,\ a',\ b') = \big(\langle\sqrt{a^2+2ab},\ b,\ a+b\rangle,\ \langle\sqrt{a'^2+2a'b'},\ b',\ a'+b'\rangle\big).$$

Next we show that P^2 is a ring:

If $\big(\langle a,\ b,\ c\rangle,\ \langle a',\ b',\ c'\rangle\big),\ \big(\langle e,\ f,\ g\rangle,\ \langle e',\ f',\ g'\rangle\big) \in P^2$, then

$$
\begin{aligned}
&\big(\langle a,\ b,\ c\rangle,\ \langle a',\ b',\ c'\rangle\big) \oplus \big(\langle e,\ f,\ g\rangle,\ \langle e',\ f',\ g'\rangle\big)\\
&= \begin{cases}
\big(\langle h,\ \frac{h^2-n^2}{2n},\ \frac{h^2+n^2}{2n}\rangle,\ \langle h',\ \frac{h'^2-n'^2}{2n'},\ \frac{h'^2+n'^2}{2n'}\rangle\big) & for\ n,\ n' \neq 0,\ is\ even\\
\big(\langle k,\ \frac{k^2-n^2}{2n},\ \frac{k^2+n^2}{2n}\rangle,\ \langle k',\ \frac{k'^2-n'^2}{2n'},\ \frac{k'^2+n'^2}{2n'}\rangle\big) & for\ n,\ n'\ odd\\
\big(\langle 0,\ j,\ j\rangle,\ \langle 0,\ j',\ j'\rangle\big) & for\ n=n'=0
\end{cases}
\end{aligned}
$$

where

$$\begin{aligned} h &= \big[d'(\langle a,\ b,\ c\rangle)+d'(\langle e,\ f,\ g\rangle)\big]r'(n) \\ h' &= \big[d'(\langle a',\ b',\ c'\rangle)+d'(\langle e',\ f',\ g'\rangle)\big]r'(n') \\ n &= c-b+g-f \\ n' &= c'-b'+g'-f' \\ k &= \big[2\big[d'(\langle a,\ b,\ c\rangle)+d'(\langle e,\ f,\ g\rangle)\big]+1\big]r'(n) \\ k' &= \big[2\big[d'(\langle a',\ b',\ c'\rangle)+d'(\langle e',\ f',\ g'\rangle)\big]+1\big]r'(n') \\ j &= d'(\langle a,\ b,\ c\rangle)+d'(\langle e,\ f,\ g\rangle) \\ j' &= d'(\langle a',\ b',\ c'\rangle)+d'(\langle e',\ f',\ g'\rangle). \end{aligned}$$

In addition

$$\begin{aligned} &\big(\langle a,\ b,\ c\rangle,\ \langle a',\ b',\ c'\rangle\big)\circ\big(\langle e,\ f,\ g\rangle,\ \langle e',\ f',\ g'\rangle\big) \\ &= \begin{cases} \big(\langle h,\ \frac{h^2-n^2}{2n},\ \frac{h^2+n^2}{2n}\rangle,\ \langle h',\ \frac{h'^2-n'^2}{2n'},\ \frac{h'^2+n'^2}{2n'}\rangle\big) & for\ n,\ n'\neq 0,\ is\ even \\ \big(\langle k,\ \frac{k^2-n^2}{2n},\ \frac{k^2+n^2}{2n}\rangle,\ \langle k',\ \frac{k'^2-n'^2}{2n'},\ \frac{k'^2+n'^2}{2n'}\rangle\big) & for\ n,\ n'\ odd \\ \big(\langle 0,\ j,\ j\rangle,\ \langle 0,\ j',\ j'\rangle\big) & for\ n=n'=0 \end{cases} \end{aligned}$$

where

$$\begin{aligned} h &= \big[d'(\langle a,\ b,\ c\rangle)d'(\langle e,\ f,\ g\rangle)\big]r'(n) \\ h' &= \big[d'(\langle a',\ b',\ c'\rangle)d'(\langle e',\ f',\ g'\rangle)\big]r'(n') \\ n &= (c-b)(g-f) \\ n' &= (c'-b')(g'-f') \\ k &= \big[2\big[d'(\langle a,\ b,\ c\rangle)d'(\langle e,\ f,\ g\rangle)\big]+1\big]r'(n) \\ k' &= \big[2\big[d'(\langle a',\ b',\ c'\rangle)d'(\langle e',\ f',\ g'\rangle)\big]+1\big]r'(n') \\ j &= d'(\langle a,\ b,\ c\rangle)d'(\langle e,\ f,\ g\rangle) \\ j' &= d'(\langle a',\ b',\ c'\rangle)d'(\langle e',\ f',\ g'\rangle). \end{aligned}$$

The additive identity in $\langle P^2,\ \oplus,\ \circ\rangle$ is $\big(\langle 0,\ 0,\ 0\rangle,\ \langle 0,\ 0,\ 0\rangle\big)$.

The multiplicative identity in $\langle P^2,\ \oplus,\ \circ\rangle$ is $\big(\langle 3,\ 4,\ 5\rangle,\ \langle 3,\ 4,\ 5\rangle\big)$.

The additive inverse $I(\langle a,\ b,\ c\rangle,\ \langle a',\ b',\ c'\rangle)$ of $(\langle a,\ b,\ c\rangle,\ \langle a',\ b',\ c'\rangle)$ is given by

$$I(\langle a,\ b,\ c\rangle,\ \langle a',\ b',\ c'\rangle)=\begin{cases} \big(\langle a,\ -b,\ -c\rangle,\ \langle a',\ -b',\ -c'\rangle\big) & if\ c-b,\ c'-b'\neq 0,\ is\ even \\ \big(\langle h,\ \frac{h^2-m^2}{2m},\ \frac{h^2+m^2}{2m}\rangle,\ \langle h',\ \frac{h'^2-m'^2}{2m'},\ \frac{h'^2+m'^2}{2m'}\rangle\big) & if\ c-b,\ c'-b'\ odd \\ \big(\langle 0,\ -b,\ -c\rangle,\ \langle 0,\ -b',\ -c'\rangle\big) & if\ c-b=c'-b'=0 \end{cases}$$

where $h=a-2r'(c-b)(c'-b')$ and $m=(b-c)(b'-c')$.

The units in $\langle P^2,\ \oplus,\ \circ\rangle$ are $\big(\langle 3,\ 4,\ 5\rangle,\ \langle 3,\ 4,\ 5\rangle\big)$; $\big(\langle -3,\ -4,\ -5\rangle,\ \langle -3,\ -4,\ -5\rangle\big)$; $\big(\langle -1,\ 0,\ 1\rangle,\ \langle -1,\ 0,\ 1\rangle\big)$; $\big(\langle 1,\ 0,\ -1\rangle,\ \langle 1,\ 0,\ -1\rangle\big)$; $\big(\langle 3,\ 4,\ 5\rangle,\ \langle -3,\ -4,\ -5\rangle\big)$; $\big(\langle -3,\ -4,\ -5\rangle,\ \langle 3,\ 4,\ 5\rangle\big)$; $\big(\langle 3,\ 4,\ 5\rangle,\ \langle -1,\ 0,\ 1\rangle\big)$; $\big(\langle -1,\ 0,\ 1\rangle,\ \langle 3,\ 4,\ 5\rangle\big)$; $\big(\langle 3,\ 4,\ 5\rangle,\ \langle 1,\ 0,\ -1\rangle\big)$; $\big(\langle 1,\ 0,\ -1\rangle,\ \langle 3,\ 4,\ 5\rangle\big)$; $\big(\langle -3,\ -4,\ -5\rangle,\ \langle 1,\ 0,\ -1\rangle\big)$; $\big(\langle 1,\ 0,\ -1\rangle,\ \langle -3,\ -4,\ -5\rangle\big)$; $\big(\langle -3,\ -4,\ -5\rangle,\ \langle -1,\ 0,\ 1\rangle\big)$; $\big(\langle -1,\ 0,\ 1\rangle,\ \langle -3,\ -4,\ -5\rangle\big)$; $\big(\langle 1,\ 0,\ -1\rangle,\ \langle -1,\ 0,\ 1\rangle\big)$; and $\big(\langle -1,\ 0,\ 1\rangle,\ \langle 1,\ 0,\ -1\rangle\big)$.

□

The above propositionosition can be extended for any finite number of triples as shown in the following propositionosition.
Let $\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n=\{\langle a',\ b',\ c'\rangle,\ \langle a'',\ b'',\ c''\rangle,\ ...,\ \langle a^n,\ b^n,\ c^n\rangle\}$ and $\{\langle e^i,\ f^i,\ g^i\rangle\}_{i=1}^n=\{\langle e',\ f',\ g'\rangle,\ \langle e'',\ f'',\ g''\rangle,\ ...,\ \langle e^n,\ f^n,\ g^n\rangle\}$. Also let $\{c^i-b^i,\ d'\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n=\{c'-b',\ d'\langle a',\ b',\ c'\rangle,\ c''-b'',\ d'\langle a'',\ b'',\ c''\rangle,\ ...,\ c^n-b^n,\ d'\langle a^n,\ b^n,\ c^n\rangle\}$ and $\{g^i-f^i,\ d'\langle e^i,\ f^i,\ g^i\rangle\}_{i=1}^n=\{g'-f',\ d'\langle e',\ f',\ g'\rangle,\ g''-f'',\ d'\langle e'',\ f'',\ g''\rangle,\ ...,\ g^n-f^n,\ d'\langle e^n,\ f^n,\ g^n\rangle\}$.

We then have the following propositionerty:

Proposition 2.3. *The mapping* $\varphi : P^n \longrightarrow \mathbb{Z}^{2n}$ *given by* $\varphi\Big(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n\Big) = (\{c^i - b^i,\ d'\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n)$ *is bijective. Moreover,* $\langle P^n,\ \oplus,\ \circ\rangle$ *is a commutative ring with identity where* $\oplus$ *and* $\circ$ *are operations in* P^n *defined by*

$$(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n) \oplus (\{\langle e^i,\ f^i,\ g^i\rangle\}_{i=1}^n) = \varphi^{-1}\Big(\varphi(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n) + \varphi(\{\langle e^i,\ f^i,\ g^i\rangle\}_{i=1}^n)\Big)$$

and

$$(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n) \circ (\{\langle e^i,\ f^i,\ g^i\rangle\}_{i=1}^n) = \varphi^{-1}\Big(\varphi(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n) \cdot \varphi(\{\langle e^i,\ f^i,\ g^i\rangle\}_{i=1}^n)\Big)$$

Proof. To show that $\varphi : P^n \longrightarrow \mathbb{Z}^{2n}$ is bijective, we prove that φ is both injective and surjective.
We first show that φ is injective:

$$\begin{aligned}
Ker\varphi &= \big\{(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n) \in P^n \mid \varphi(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n) = (0,\ 0,\ 0,\ ...,\ 0)\big\} \\
&= \big\{(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n) \in P^n \mid (\{c^i - b^i,\ d'\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n) = (0,\ 0,\ 0,\ ...,\ 0)\big\} \\
&= \big\{(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n) \in P^n \mid \{c^i - b^i = 0,\ d'\langle a^i,\ b^i,\ c^i\rangle = 0\}_{i=1}^n\big\} \\
&= \big\{(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n) \in P^n \mid \{c^i - b^i = 0,\ d'\langle a^i,\ b^i,\ c^i\rangle = b^i = 0\}_{i=1}^n\big\} \\
&= \big\{(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n) \in P^n \mid \{b^i = c^i = 0\}_{i=1}^n\big\} \\
&= \big\{(\langle 0,\ 0,\ 0\rangle,\ \langle 0,\ 0,\ 0\rangle,\ ...,\ \langle 0,\ 0,\ 0\rangle)\big\}.
\end{aligned}$$

Hence injective.

To show that φ is surjective, we show that every element of $\mathbb{Z}^{2n}$ has a unique pre-image in P^n.
Indeed for each $(\{a^i,\ b^i\}_{i=1}^n) \in \mathbb{Z}^{2n}$, we have that

$$\varphi^{-1}(\{a^i,\ b^i\}_{i=1}^n) = (\{\langle\sqrt{a^{i2} + 2a^ib^i},\ b^i,\ a^i + b^i\rangle\}_{i=1}^n).$$

Next we show that P^n is a ring.

If $(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n),\ (\{\langle e^i,\ f^i,\ g^i\rangle\}_{i=1}^n) \in P^n$, then

$$(\{\langle a^i, b^i, c^i\rangle\}_{i=1}^n) \oplus (\{\langle e^i, f^i, g^i\rangle\}_{i=1}^n) = \begin{cases} (\{\langle h^i,\ \frac{h^{i2}-m^{i2}}{2m^i},\ \frac{h^{i2}+m^{i2}}{2m^i}\rangle\}_{i=1}^n) & for\ m^i \neq 0,\ is\ even \\ (\{\langle k^i,\ \frac{k^{i2}-m^{i2}}{2m^i},\ \frac{k^{i2}+m^{i2}}{2m^i}\rangle\}_{i=1}^n) & for\ m^i\ odd \\ (\{\langle 0,\ j^i,\ j^i\rangle\}) & for\ m^i = 0 \end{cases}$$

where

$$\begin{aligned}
h^i &= [d'(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n) + d'(\{\langle e^i,\ f^i,\ g^i\rangle\}_{i=1}^n)]r'(m^i) \\
m^i &= \{c^i - b^i + g^i - f^i\}_{i=1}^n \\
k^i &= [2[d'(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n) + d'(\{\langle e^i,\ f^i,\ g^i\rangle\}_{i=1}^n)] + 1]r'(m^i) \\
j^i &= d'(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n) + d'(\{\langle e^i,\ f^i,\ g^i\rangle\}_{i=1}^n)
\end{aligned}$$

In addition

$$(\{\langle a^i, b^i, c^i\rangle\}_{i=1}^n) \circ (\{\langle e^i, f^i, g^i\rangle\}_{i=1}^n) = \begin{cases} (\{\langle h^i,\ \frac{h^{i2}-m^{i2}}{2m^i},\ \frac{h^{i2}+m^{i2}}{2m^i}\rangle\}_{i=1}^n) & for\ m^i \neq 0,\ is\ even \\ (\{\langle k^i,\ \frac{k^{i2}-m^{i2}}{2m^i},\ \frac{k^{i2}+m^{i2}}{2m^i}\rangle\}_{i=1}^n) & for\ m^i\ odd \\ (\{\langle 0,\ j^i,\ j^i\rangle\}) & for\ m^i = 0 \end{cases}$$

where

$$\begin{aligned} h &= \left[d'\big(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n\big)d'\big(\{\langle e^i,\ f^i,\ g^i\rangle\}_{i=1}^n\big)\right]r'(m^i) \\ m^i &= \left\{(c^i-b^i)(g^i-f^i)\right\}_{i=1}^n \\ k^i &= \Bigg[2\left[d'\big(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n\big)d'\big(\{\langle e^i,\ f^i,\ g^i\rangle\}_{i=1}^n\big)\right]+1\Bigg]r'(m^i) \\ j^i &= d'\big(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n\big)d'\big(\{\langle e^i,\ f^i,\ g^i\rangle\}_{i=1}^n\big). \end{aligned}$$

The additive identity in $\langle P^n,\ \oplus,\ \circ\rangle$ is $\big(n\{\langle 0,\ 0,\ 0\rangle\}\big) = \big(\langle 0,\ 0,\ 0\rangle,\ ...,\ \langle 0,\ 0,\ 0\rangle\big)$.

The multiplicative identity in $\langle P^n,\ \oplus,\ \circ\rangle$ is $\big(\{\langle 3^i,\ 4^i,\ 5^i\rangle\}_{i=1}^n\big)$.

The additive inverse $I\big(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n\big)$ of $\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n$ is given by

$$I\big(\{\langle a^i,\ b^i,\ c^i\rangle\}_{i=1}^n\big) = \begin{cases} \big(\{\langle a^i,\ -b^i,\ -c^i\rangle\}_{i=1}^n\big) & if\ \{c^i-b^i\}\neq 0,\ is\ even \\ \big(\{\langle h^i,\ \frac{h^{i2}-m^{i2}}{2m^i},\ \frac{h^{i2}+m^{i2}}{2m^i}\rangle\}_{i=1}^n\big) & if\ \{c^i-b^i\}\ odd \\ \big(\{\langle 0,\ -b^i,\ -c^i\rangle\}_{i=1}^n\big) & if\ \{c^i-b^i\} = 0 \end{cases}$$

where $h^i = a^i - 2r' \cap \{c^i - b^i\}_{i=1}^n$ and $m^i = \cap\{b^i - c^i\}_{i=1}^n$.

The units in $\langle P^n,\ \oplus,\ \circ\rangle$ are:

$\big(\{\langle 3^i,\ 4^i,\ 5^i\rangle\}_{i=1}^n\big),\ \big(\{\langle 3^i,\ 4^i,\ 5^i\rangle\}_{i=1}^{n-1}\langle -3',\ -4',\ -5'\rangle\big),\ ...,\ \big(\{\langle -3^i,\ -4^i,\ -5^i\rangle\}_{i=1}^n\big)$;
$\big(\{\langle 3^i,\ 4^i,\ 5^i\rangle\}_{i=1}^{n-1}\langle -1',\ 0',\ 1'\rangle\big),\ \big(\{\langle 3^i,\ 4^i,\ 5^i\rangle\}_{i=1}^{n-2}\langle -1'',\ 0'',\ 1''\rangle\big),\ ...,\ \big(\{\langle -1^i,\ 0^i,\ 1^i\rangle\}_{i=1}^n\big)$;
$\big(\{\langle 3^i,\ 4^i,\ 5^i\rangle\}_{i=1}^{n-1}\langle 1',\ 0',\ -1'\rangle\big),\ \big(\{\langle 3^i,\ 4^i,\ 5^i\rangle\}_{i=1}^{n-2}\langle 1'',\ 0'',\ -1''\rangle\big),\ ...,\ \big(\{\langle 1^i,\ 0^i,\ -1^i\rangle\}_{i=1}^n\big)$;
$\big(\{\langle -3^i,\ -4^i,\ -5^i\rangle\}_{i=1}^{n-1}\langle -1',\ 0',\ 1'\rangle\big),\ ...,\ \big(\{\langle -3',\ -4',\ -5'\rangle\langle -1^i,\ 0^i,\ 1^i\rangle\}_{i=1}^{n-1}\big)$;
$\big(\{\langle -3^i,\ -4^i,\ -5^i\rangle\}_{i=1}^{n-1}\langle 1',\ 0',\ -1'\rangle\big),\ ...,\ \big(\{\langle -3',\ -4',\ -5'\rangle\langle 1^i,\ 0^i,\ -1^i\rangle\}_{i=1}^{n-1}\big)$;
$\big(\{\langle 1^i,\ 0^i,\ -1^i\rangle\}_{i=1}^{n-1}\langle -1',\ 0',\ 1'\rangle\big),\ ...,\ \big(\{\langle 1',\ 0',\ -1'\rangle\langle -1^i,\ 0^i,\ 1^i\rangle\}_{i=1}^{n-1}\big)$. □

3. Conclusion

Dawson identified $\varphi : P \longrightarrow \mathbb{Z}^2$ to be an isomorphism on the Pythagorean ring $\langle P,\ \oplus,\ \circ\rangle$. We have investigated and generalized this mapping to $\varphi : P^n \longrightarrow \mathbb{Z}^{2n}$. It can be shown that the total number of units in $\langle P^n,\ \oplus,\ \circ\rangle$ is 4^n.

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Respecting Cauchy equivalence classes

Anthony J. Ripa

Abstract. In the standard construction of real numbers as Cauchy equivalence classes of rational sequences, only the basic arithmetic operations are defined to respect these classes, leaving other functions susceptible to pathologies, such as discontinuities or undefined points. The standard solution to resolving pathologies is to reintroduce limits, thereby complicating a convergence-based construction. This paper proposes a unified approach where all functions respect Cauchy equivalence classes, ensuring they are well-defined at all points without redundant limits. By applying rational functions to sequence representatives and assigning the resulting equivalence class, we eliminate pathologies such as removable discontinuities. Examples, including arithmetic operations, difference quotients and Riemann sums, demonstrate the method's efficacy. Compared to alternative fixes like extended number systems, our approach offers simplicity and coherence, suggesting a principled construction of real analysis.

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1. Introduction

The real numbers are traditionally constructed as equivalence classes of Cauchy sequences of rational numbers, where two sequences are equivalent if their difference converges to zero. Arithmetic operations—addition, subtraction, multiplication, and division—are defined by applying their rational counterparts pointwise to sequence representatives, ensuring the result is independent of the chosen representatives. However, all other functions are defined directly on the real numbers without this constraint, leading to two categories: functions that respect Cauchy equivalence classes and those that do not.

Non-respecting functions often exhibit pathologies, such as being undefined at points of interest or having removable discontinuities. For instance, the function $f(x) = \frac{x}{x}$ is undefined at $x = 0$ despite an intuitive value of 1, requiring the limit $\lim_{x\to 0} f(x) = 1$ to resolve it. This reliance on limits reintroduces complexity into a system initially built on convergence, undermining the elegance of the Cauchy construction.

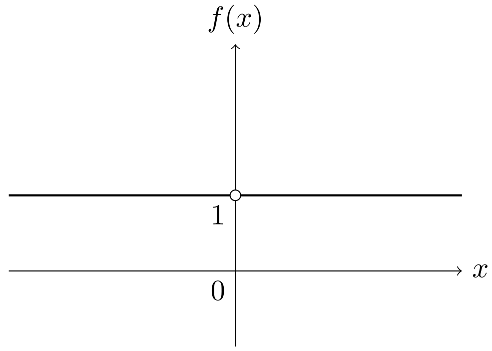

Figure 1. A traditional graph of x/x vs x, with undesirable hole at $x = 0$.

We propose that all functions on the real numbers should respect Cauchy equivalence classes, mirroring the arithmetic operations. Under this framework, a real function $f(x)$ is defined by applying its rational precursor to representatives of x's equivalence class, yielding a consistent output class when defined. This eliminates the need for limits, directly deriving values like $f(0) = 1$ for $f(x) = \frac{x}{x}$. We explore this approach's implications, compare it to existing remedies, and illustrate its application through examples, advocating for a streamlined real analysis free of unnecessary complications.

2. Motivation

The real numbers are conventionally constructed as equivalence classes of Cauchy sequences of rational numbers, where two sequences (q_n) and (r_n) are equivalent if their difference $|q_n - r_n|$ converges to zero as $n \to \infty$. The four fundamental arithmetic operations—addition, subtraction, multiplication, and division—are defined to respect these equivalence classes. For instance, given real numbers X and Y with representatives $(q_n) \in X$ and $(r_n) \in Y$, the sum $X + Y$ is the equivalence class of $(q_n + r_n)$, where addition is applied pointwise in $\mathbb{Q}$. This ensures that the operation's output is consistent regardless of the chosen representatives, aligning seamlessly with the Cauchy construction.

In contrast, all other functions on the real numbers are defined as mappings between real numbers without requiring that they respect these equivalence classes. This creates a dichotomy: functions like the arithmetic operations that inherently respect Cauchy equivalence classes, and those that do not. Non-respecting functions frequently exhibit pathological behaviors, such as being undefined at specific points or possessing removable discontinuities. Consider the function $f(x) = \frac{x}{x}$, which is undefined at $x = 0$ despite equaling 1 for all $x \neq 0$. Classically, this is resolved by introducing the limit $\lim_{x\to 0} f(x) = 1$, reintroducing a layer of complexity into a framework built to encapsulate convergence through Cauchy sequences.

This reliance on limits is particularly inconvenient for functions critical to calculus, such as difference quotients. For example, the difference quotient $\frac{f(x+h)-f(x)}{h}$ is undefined at $h = 0$, precisely where it is needed to define derivatives. To circumvent this, higher-order tools like the limit operator are employed, computing $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. While effective, this approach—defining reals via convergence, then obscuring it for most functions, only to reintroduce limits—lacks elegance and unity. If all functions respected Cauchy equivalence classes, such complications would vanish. For $f(x) = \frac{x}{x}$, we would directly derive $f(0) = 1$, eliminating the need for limits.

The classical framework also permits nuisance functions that complicate analysis. Consider a function equal to $\sqrt{x}$ everywhere except at $x = 7$, where it takes the value 9 or is undefined. Such functions, with removable discontinuities or singularities, demand sophisticated methods—limits, equivalence classes for functions, or ad hoc adjustments—to handle them properly. These pathologies stem directly from functions not respecting the underlying equivalence class structure of the real numbers.

Efforts to address this awkwardness have led to various alternative fixes, each introducing its own complexities. One approach defines equivalence classes for functions, grouping a well-behaved function (e.g., $\sqrt{x}$) with its corrupted variants (e.g., altered at $x = 7$) based on agreement almost everywhere. Another employs higher-order operators, such as a "tilde" operator, transforming a function f into $\widetilde{f}$ by removing idiosyncrasies like removable singularities. Neighborhood-based methods redefine function values via behavior in a punctured neighborhood, akin to limits but avoiding pointwise evaluation.

Extended number systems offer further workarounds, particularly for difference quotients. Dual numbers adjoin a nilpotent element ε where $\varepsilon^2 = 0$, allowing derivatives like $\frac{f(x+\varepsilon)-f(x)}{\varepsilon}$ to be computed algebraically. For $f(x) = x^2$, this yields $\frac{(x+\varepsilon)^2-x^2}{\varepsilon} = \frac{2x\varepsilon+\varepsilon^2}{\varepsilon} = \frac{2x\varepsilon+0}{\varepsilon} = \frac{2x\varepsilon}{\varepsilon} = 2x$, avoiding limits. However, dual numbers require redefining all functions and face issues with theorems not transferring (e.g., non-zero elements with zero squares). Hyperreals, incorporating infinitesimals and infinite numbers via ultrafilters or model theory, preserve first-order real properties but rely on non-constructive axioms and limit transfer to first-order statements. For the same derivative, using an infinitesimal ϵ, $\frac{(x+\epsilon)^2-x^2}{\epsilon} = 2x + \epsilon$, and a standard part operator $std(2x + \epsilon)$ yields $2x$. Surreals extend this further but amplify complexity. These systems typically replicate limit-based results without surpassing them, failing to model singular objects like the Dirac delta function.

While these fixes mitigate some nuisance functions, they introduce significant overhead and rarely exceed the power of traditional limits. In contrast, this paper proposes a fundamental solution: all functions on the real numbers should respect Cauchy equivalence classes, akin to arithmetic operations. By defining a function F via a rational precursor f applied to sequence representatives—assigning the resulting equivalence class when defined—we ensure F is well-defined and continuous across $\mathbb{R}$. For $f(x) = \frac{x}{x}$, the range of f applied to any representative of $X = 0$ (e.g., $\{\frac{1}{n}\}$) yields $\{1, 1, \ldots\} \in [1]$, so $F(0) = 1$ directly. This eliminates pathologies and limits, aligning function definitions with the real numbers' foundational structure.

This approach promises a unified, intuitive real analysis, free from the artificial complexities of non-respecting functions. Subsequent sections formalize this method, illustrate its application, and compare it to existing remedies, highlighting its simplicity and efficacy.

3. Related Work

There are many constructions that attempt to bake continuity in by design. Intuitionist analysis [1] gives up the law of the excluded middle to ensure continuity, or to be precise no total constructible function is provably discontinuous. In Computable Analysis [2] and Domain Theory [3], every computable real function is continuous. Prior work in category theory introduced the notion of *Cauchy continuous maps*: for example, Kent et al. (1979) [4] study *sequential Cauchy spaces* in which the morphisms are exactly those functions f satisfying $x_n \sim y_n \implies f(x_n) \sim f(y_n)$ for Cauchy sequences (x_n), (y_n). Our slight modification is that we only require $f(x_n) \sim f(y_n)$ when both are defined.

The above approaches share the central idea of this paper. The novelty of the present work lies in applying this philosophy *within classical real analysis* as a unifying principle to simplify definitions and exclude pathology. Unlike Brouwer's intuitionism, we do not abandon classical logic; unlike computable analysis, we do not restrict to effectively calculable functions. Instead, we formally require of every real function the same extensionality that addition and multiplication enjoy. By doing so, we aim to remove the need for ϵ–δ limit arguments in establishing basic properties – continuity is automatic by construction – and to rule out counterintuitive examples (like discontinuous or multi-valued mappings) since these cannot be defined without violating Cauchy equivalence. This approach appears unprecedented in the literature in its explicit generality: while previous frameworks imposed similar restrictions for philosophical or practical reasons, none have proposed a blanket re-definition of "function on $\mathbb{R}$" in standard analysis to *always* mean a Cauchy-continuous (extensional) function. Thus, our work can be seen as unifying and extending those earlier insights into a single principle that sits at the foundations of real analysis, offering a new lens to view limits and functions through the prism of Cauchy equivalence.

Each of the related lines of research above reinforces the feasibility of our approach. In intuitionistic logic settings, one already sees a "universe" of reals with no discontinuous functions. In computable and constructive analysis, one sees that any concretely specifiable function automatically respects Cauchy equivalence. The present work bridges these insights back to classical analysis: it suggests that we can adopt the constraint "define everything on representatives and prove independence from the representative" as a general design principle. In doing so, we inherit the benefits (no pathologies, no separate limit processes needed to define function values) while remaining in the comfortable setting of classical set-theoretic reals. This connection to prior art provides strong evidence of consistency and suggests that many standard results of analysis will carry over smoothly – but it also highlights the novel impact of our thesis, which is to reformulate the concept of a real function itself in a way that sidesteps the usual pitfalls and logical detours of ϵ–δ limits. The following sections will build on this foundation, demonstrating that such a reformation is not only theoretically sound but also advantageous for the clarity and robustness of real analysis.

4. Notation

4.1. **Constants.** In case of ambiguity, we will bracket real constants, as in [1]. By default, an unbracketed 1 is a rational number.

4.2. **Cases.** For the most part, we will capitalize real functions, as in F. By default, an uncapitalized f is a rational function. Similarly for variables: x is a rational variable, and X,Y are real variables.

4.3. **Operators.** We will use a superscript c (for Cauchy) lift operator. This operator will transform a rational f into a particular corresponding F. The bulk of this paper is about this transformation. For example, $F = {}^{c}f$. Similarly, by default $+$ is the rational plus. ${}^{c}+$ is the corresponding real plus.

We will use an overscript tilde $\tilde{}$ for the completion operator, which transforms a function F into a corresponding $\widetilde{F}$ without removable discontinuities. Unlike the Cauchy operator, which we will use as part of our method, we use tilde descriptively (e.g. Thm 5.3 tilde on right means left is complete).

4.4. **Function.** For functions, $f(\cdot)$ will refer to standard function application. To specify term-wise application we write $f[\cdot]$.

5. Methods

5.1. Standard Construction.

Definition 5.1 (Standard construction of Cauchy sequences and $\mathbb{R}$)**.**

A Cauchy sequence $(q_n)_{n\in\mathbb{N}}$ *in* $\mathbb{Q}$ *satisfies:*

- $\forall\epsilon > 0, \exists N : m, n > N \implies |q_m - q_n| < \epsilon.$

Two sequences (q_n) *and* (r_n) *are equivalent if:*

- $\forall\epsilon > 0, \exists N : n > N \implies |q_n - r_n| < \epsilon.$

The set $\mathbb{R}$ *is:*

- *the quotient set of Cauchy sequences by the equivalence relation.*

5.2. Our Method.

5.2.1. *One Dimensional Case.*

Definition 5.2 (Our construction: Cauchy lift operator - 1-dimensional)**.**

For rational partial function f: $\mathbb{Q} \rightharpoonup \mathbb{Q}$*, we define its* Cauchy-lift ${}^c f$*:* $\mathbb{R} \rightharpoonup \mathbb{R}, X \mapsto Y$ *s.t.*

- $\forall x \in X, f[x]\downarrow \implies f[x] \in Y.$

Remark 1. *In other words,* ${}^c f$ *action on real* X *is defined by* f*'s action on rational sequences representatives* $x \in X$*.*

If f *is undefined for some representatives,* ${}^c f$ *remains well-defined as long as the defined cases consistently yield sequences in* Y*. This leverages the equivalence class structure to extend* f *naturally across* $\mathbb{R}$*, avoiding artificial undefinedness (e.g. removable discontinuities).*

Theorem 5.3 (Cauchy-lifts have no removable discontinuities)**.**

Let $f : \mathbb{Q} \rightharpoonup \mathbb{Q}$ *be a partial rational function, and let* ${}^c f : \mathbb{R} \rightharpoonup \mathbb{R}$ *be its Cauchy-lift. Then*

$$ {}^c f \;=\; \widetilde{{}^c f}. $$

Proof sketch. If ${}^c f(X)$ is defined, then $\widetilde{{}^c f}(X) = {}^c f(X)$.

If ${}^c f(X)$ is undefined and the limit $L = \lim_{Y\to X} {}^c f(Y)$ existed, then by density of $\mathbb{Q}$ and the definition of the lift one can choose rationals $x_n \to X$ with $f(x_n)$ defined and $f(x_n) \to L$. The rational Cauchy sequence (x_n) represents X, and $(f(x_n))$ is Cauchy representing L, so by the lift-definition ${}^c f(X) = L$—a contradiction. Hence no such limit exists when ${}^c f(X)$ is undefined, and $\widetilde{{}^c f}(X)$ is also undefined.

Therefore ${}^c f = \widetilde{{}^c f}$. □

5.2.2. *k-Dimensional Case.*

Definition 5.4 (Our construction: Cauchy lift operator - k-dimensional)**.**

For rational partial function $f : \mathbb{Q}^k \rightharpoonup \mathbb{Q}$*, we define its* Cauchy lift ${}^c f : \mathbb{R}^k \rightharpoonup \mathbb{R}, X^k \mapsto Y$ *s.t.*

- $\forall x \in X^k, f[x]\downarrow \implies f[x] \in Y.$

Remark 2. *In the 1-d case, we apply* f *to* $x \in X$*.*

In the k-d case, we apply f *to k-tuples* $(x_1, x_2, \ldots, x_k) \in (X_1, X_2, \ldots, X_k)$*.*

6. Examples

In this section, we illuminate the method by incorporating detailed examples, demonstrating how various functions can be constructed to respect Cauchy equivalence classes. Each example illustrates the application of a rational function to representatives of real numbers (as equivalence classes), ensuring consistency and well-definedness across all real inputs.

6.1. **Arithmetic Functions.**

6.1.1. *Unary Functions.* Here we consider the unary rational functions negation $x \mapsto -x$ and reciprocation $x \mapsto /x$.

Negation

Consider the rational unary function negation, denoted by $-$. We construct the real function ${}^c-(X)$ from the rational $-(x)$ and verify its behavior. By definition, ${}^c-(X)$ exists if and only if there is a Cauchy equivalence class $Y \in \mathbb{R}$ such that for all representatives $x \in X$, if $-(x)$ is defined, then $-(x) \in Y$.

Example 6.1. *Given a real number $X \in \mathbb{R}$, take representative $x \in X$, where $x = \{c_1, c_2, c_3, \ldots\}$ with $c_i \in \mathbb{Q}$. The negation $-x$ is defined pointwise: $-x = \{-c_1, -c_2, -c_3, \ldots\}$. Since negation is always defined for rationals and the negation of a Cauchy sequence is itself a Cauchy sequence, there exists an equivalence class $Y \in \mathbb{R}$ such that $\{-c_1, -c_2, -c_3, \ldots\} \in Y$. Thus, ${}^c-(X) = Y$.*

Remark 3. *This is the 1^{st} basic unary arithmetic function $-$ which is defined in the standard Cauchy construction. Our construction yields identical results.*

Reciprocation

Reciprocation, denoted ${}^c/(X)$, is more delicate due to division by zero. It exists if and only if there is a $Y \in \mathbb{R}$ such that for all $x \in X$, if $/(x)$ is defined, then $/(x) \in Y$.

Example 6.2. *For $X \in \mathbb{R}$, take $x = \{c_1, c_2, c_3, \ldots\} \in X$, with $c_i \in \mathbb{Q}$. Pointwise, $/x = \{/c_1, /c_2, /c_3, \ldots\}$, defined whenever $c_i \neq 0$ for all i. If $X = [0]$ (the class containing the zero sequence), consider $x = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \in X$, yielding $\{1, 2, 3, \ldots\}$, which diverges. Similarly all other representatives also diverge (or are otherwise undefined). Thus, ${}^c/(X)$ is undefined when $X = [0]$. For $X \neq [0]$, we can choose x with $c_i \neq 0$ for all i (since $X \neq [0]$ implies sequences not converging to zero), and $\{/c_1, /c_2, /c_3, \ldots\}$ is Cauchy, defining Y. Hence, ${}^c/(X) = Y$ when $X \neq [0]$, matching standard real reciprocation.*

Remark 4. *This is the 2^{nd} basic unary arithmetic function $/$ which is defined in the standard Cauchy construction. Our construction yields identical results.*

6.1.2. *Binary Functions.*

Addition

Consider the rational binary function addition, denoted by $+$. We construct the real function ${}^c+(X_1, X_2)$ from the rational $+(x_1, x_2)$ and verify its behavior. By definition, ${}^c+(X_1, X_2)$ exists if and only if there is a Cauchy equivalence class $Y \in \mathbb{R}$ such that for all representatives $x_1 \in X_1$, $x_2 \in X_2$, if $+(x_1, x_2)$ is defined, then $+(x_1, x_2) \in Y$.

Example 6.3. *Given two real numbers $X_1, X_2 \in \mathbb{R}$, take representatives $x_1 \in X_1$ and $x_2 \in X_2$, where $x_1 = \{c_1, c_2, c_3, \ldots\}$ and $x_2 = \{C_1, C_2, C_3, \ldots\}$ with $c_i, C_i \in \mathbb{Q}$. The sum $x_1 + x_2$ is defined pointwise: $x_1+x_2 = \{c_1, c_2, c_3, \ldots\}+\{C_1, C_2, C_3, \ldots\} = \{c_1+C_1, c_2+C_2, c_3+C_3, \ldots\}$. Since addition is always defined for rationals and the sum of two Cauchy sequences is itself a Cauchy sequence, there exists an equivalence class $Y \in \mathbb{R}$ such that $\{c_1 + C_1, c_2 + C_2, c_3 + C_3, \ldots\} \in Y$. Thus, ${}^c+(X_1, X_2) = Y$.*

Remark 5. *This is the 1^{st} basic binary arithmetic function $+$ which is defined in the standard Cauchy construction. Our construction yields identical results.*

Subtraction and Multiplication

The constructions for subtraction and multiplication, denoted ${}^c-(X_1, X_2)$ and ${}^c*(X_1, X_2)$, proceed similarly. Using the rational operations $-$ and $*$, we apply them pointwise to representatives $x_1 \in X_1$ and $x_2 \in X_2$. The resulting sequences are Cauchy, defining the respective real operations, mirroring the addition case.

Remark 6. *This is the 2^{nd} and 3^{rd} basic binary arithmetic functions $-, \times$ which are defined in the standard Cauchy construction. Our construction yields identical results.*

Division

Division, denoted ${}^c/(X_1, X_2)$, is more delicate due to division by zero. It exists if and only if there is a $Y \in \mathbb{R}$ such that for all $x_1 \in X_1$, $x_2 \in X_2$, if $/(x_1, x_2)$ is defined, then $/(x_1, x_2) \in Y$.

Example 6.4. *For $X_1, X_2 \in \mathbb{R}$, take $x_1 = \{c_1, c_2, c_3, \ldots\} \in X_1$ and $x_2 = \{C_1, C_2, C_3, \ldots\} \in X_2$, with $c_i, C_i \in \mathbb{Q}$. Pointwise, $x_1/x_2 = \{c_1/C_1, c_2/C_2, c_3/C_3, \ldots\}$, defined whenever $C_i \neq 0$ for all i. If $X_2 = [0]$ (the class containing the zero sequence), consider $x_1 = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \in X_1$ and $x_2 = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \in X_2 = [0]$, yielding $\{1, 1, 1, \ldots\} \in [1]$, but another $x_3 = \{1, \frac{1}{4}, \frac{1}{9}, \ldots\} \in X_2$ gives $\{1, 2, 3, \ldots\}$, which diverges. Thus, ${}^c/(X_1, X_2)$ is undefined when $X_2 = [0]$, by uniqueness. For $X_2 \neq [0]$, we can choose x_2 with $C_i \neq 0$ for all i (since $X_2 \neq [0]$ implies sequences not converging to zero), and $\{c_1/C_1, c_2/C_2, c_3/C_3, \ldots\}$ is Cauchy, defining Y. Hence, ${}^c/(X_1, X_2) = Y$ when $X_2 \neq [0]$, matching standard real division.*

Remark 7. *This is the 4^{th} basic binary arithmetic function $\div$ which is defined in the standard Cauchy construction. The main difference is that instead of manually ruling out 0 denominators, our construction merely fails to remove the discontinuity.*

6.2. **Power Functions.**

6.2.0. *Power preliminaries.*

Lemma 6.5 (Odd–denominator density)**.** *The set $\{p/q \in \mathbb{Q} : q \text{ odd}\}$ is dense in $\mathbb{R}$.*

Proof sketch. Given $a < b$, choose an odd q large and take $p = \lfloor qa \rfloor, \lfloor qa \rfloor + 1, \ldots$ so that $p/q \in (a, b)$. □

Lemma 6.6 (Perfect–power refinement)**.** *Let $X > [0]$ in $\mathbb{R}$ and let $(y_n = p_n/q_n) \to Y$ be rationals in lowest terms. There exist $s_n \in \mathbb{Q}$ with $x_n := s_n^{q_n} \in \mathbb{Q}$ and $x_n \to X$.*

Proof sketch. Approximate X^{1/q_n} by rationals s_n (e.g. continued fractions or truncations). Then $s_n^{q_n} \to X$. □

6.2.1. *Unary Integer Powers $x \mapsto x^n$.*

For a fixed integer $n \in \mathbb{Z}$, consider the unary rational function $f_n(x) = x^n$, and its Cauchy lift ${}^c f_n$.

Proposition 6.7 (Domain and form of the lift)**.** *For each fixed $n \in \mathbb{Z}$:*

- *(i) If $n > 0$, then ${}^c f_n : \mathbb{R} \to \mathbb{R}$ is total and coincides with the classical real map $X \mapsto X^n$ (e.g., ${}^c f_1 = \mathrm{id}$, ${}^c f_2(X) \geq [0]$, etc.).*
- *(ii) If $n = 0$, then f_0 is the rational partial constant $x \mapsto 1$ (undefined at $x = 0$), and ${}^c f_0$ is everywhere defined with ${}^c f_0(X) = [1]$. This mirrors the earlier $x \mapsto x/x$ example.*
- *(iii) If $n < 0$ (write $n = -m$ with $m \in \mathbb{N}$), then ${}^c f_n$ agrees with $X \mapsto ({}^c f_m(X))^{-1}$ on its domain and is* undefined *at $X = [0]$. For $X \neq [0]$, the lift is well-defined and equals the classical $X \mapsto X^n$.*

Example 6.8 (Three quick instances)**.**
(a) $n = 0$. *For any X, pick a representative with no zeros; then $f_0[x] = \{1, 1, \ldots\} \in [1]$, so ${}^c f_0(X) = [1]$.*
(b) $n = 2$. *For $X = [e]$, take $x = \{2, 2.7, 2.71, \ldots\}$; then $f_2[x] = \{2^2, 2.7^2, 2.71^2, \ldots\} = \{4, 7.29, 7.3441, \ldots\} \in [e^2]$, hence ${}^c f_2([e]) = [e^2]$.*
(c) $n = -1$. *For $X \neq [0]$, ${}^c f_{-1}(X)$ is the lifted reciprocal and agrees with classical $1/X$. At $X = [0]$, the lift is undefined.*

The unary rational power function $x \mapsto x^0$ at $x = 0$ is an indeterminate form. The unary real power function ${}^c(x \mapsto x^0)$ at $x = 0$ is $[1]$, because ${}^c(x \mapsto x^0)$ is the real function $X \mapsto [1]$. The Cauchy lift is immune to indeterminate forms. Furthermore, the lift is robust to the syntax used to generate the form. The rational unary function $x \mapsto x/x$ is identical to $x \mapsto x^0$. Therefore, ${}^c(x \mapsto x/x)$ at $X = [0]$ is also $[1]$. The same is true for the Cauchy lift of the unary rational piecewise function $x \mapsto \begin{cases} 1, & x \neq 0 \end{cases}$.

Remark 8. *The standard construction of the reals $\mathbb{R}$ is merely a field (i.e. +,-,*,/). The power function is traditionally defined on $\mathbb{R}$, after the fact, with $X \mapsto X^0$ having a removable discontinuity. This contrasts with our construction where ${}^c(x \mapsto x^0)$ is defined for all $\mathbb{R}$.*

6.2.2. *Unary Integer Roots $x \mapsto \sqrt[n]{x}$.*

By Lemma 6.6 with the fixed exponent $1/n$, for $X \geq [0]$ (if n even) there exist perfect nth powers $x_k = r_k^n \to X$. Then ${}^c\sqrt[n]{X} = [\lim r_k]$, and independence of representatives follows from the lift.

Example 6.9 (Square Root, $n = 2$). *For $X = [2]$ (where $n = 2$ is even and $X \geq [0]$), the decimal expansion of $\sqrt{2}$ is approximately 1.41421356.... Form $1, 1.4, 1.41, 1.414, \ldots$, then $1^2, 1.4^2, 1.41^2, 1.414^2, \ldots \in [2]$. Applying $\sqrt{\cdot}$ pointwise gives $1, 1.4, 1.41, 1.414, \ldots \in [\sqrt{2}]$.*

Example 6.10 (Cube Root, $n = 3$). *For $X = [2]$ (where $n = 3$ is odd, defined for all X), the decimal expansion of $\sqrt[3]{2}$ is approximately 1.25992105.... Form $1, 1.2, 1.25, 1.259, \ldots$, then $1^3, 1.2^3, 1.25^3, 1.259^3, \ldots \in [2]$. Applying $\sqrt[3]{\cdot}$ pointwise gives $1, 1.2, 1.25, 1.259, \ldots \in [\sqrt[3]{2}]$.*

Example 6.11 (Fourth Root, $n = 4$). *For $X = [3]$ (even n, $X \geq [0]$), approximate $\sqrt[4]{3} \approx 1.31607401\ldots$. Form $1, 1.3, 1.31, 1.316, \ldots$, then $1^4, 1.3^4, 1.31^4, 1.316^4, \ldots \in [3]$. Applying $\sqrt[4]{\cdot}$ pointwise gives $1, 1.3, 1.31, 1.316, \ldots \in [\sqrt[4]{3}]$.*

Remark 9. *Pointwise roots may fail on a fixed representative; the lift uses* some *defining representatives. Lemma 6.6 supplies them uniformly.*

Thus, ${}^c\sqrt[n]{X} = Y$ is well-defined for all valid X, extending naturally to all positive integer $n \geq 2$. For variable n, this could be treated as a binary operation ${}^c\sqrt[N]{X}$ in §6.2.4, applying similar pointwise lifting to pairs of representatives.

6.2.3. *Unary Rational Powers $x \mapsto x^m$.*

We now generalize to unary rational powers, recovering §6.2.1 and §6.2.2 as special cases.

Proposition 6.12 (Domain and form of the lift for rational powers). *Let $f_m = x \mapsto x^m$. For each fixed $m = p/q \in \mathbb{Q}$ in lowest terms:*

(i) The domain of ${}^c f_m$ is $\mathbb{R}$ if q is odd (defined for all X), or $\mathbb{R}_{\geq 0}$ if q is even (undefined for $X < [0]$).
(ii) At $X = [0]$:
- *If $p > 0$, then ${}^c f_m(0) = [0]$.*
- *If $p = 0$, then ${}^c f_m(0) = [1]$.*
- *If $p < 0$, then ${}^c f_m(0)$ is undefined.*

(iii) For $X \neq [0]$, ${}^c f_m(X)$ coincides with the classical real power X^m.

Proof sketch. Write $m = p/q$ in lowest terms. Fix $X \in \mathbb{R}$. If q is odd, no sign restriction is required; if q is even, assume $X \geq [0]$. By Lemma 6.6, there exist rationals s_n with $x_n := s_n^q \in \mathbb{Q}$ and $x_n \to X$. Then $x_n^m = (s_n^q)^{p/q} = s_n^p$ is a rational sequence that is Cauchy; its Cauchy class depends only on X (and agrees with the classical value when defined). At $X = [0]$ the three p-sign cases agree with the unary-integer discussion (§6.2.1): $p > 0$ gives $[0]$, $p = 0$ gives $[1]$, $p < 0$ is undefined. □

Example 6.13.

(a) $m = \frac{3}{2}$, $X = [4]$: $X^{3/2} = \left(X^3\right)^{1/2} = [64]^{1/2} = [8]$.
(b) $m = -\frac{1}{3}$, $X = [-8]$: q odd, so $X^{1/3} = [-2]$ is defined and $X^{-1/3} = [-1/2]$.
(c) $m = \frac{1}{2}$, $X = [-1]$: even root of a negative base; undefined in $\mathbb{R}$.
(d) $m = -\frac{3}{2}$, $X = [0]$: negative exponent at 0; undefined.
(e) $m = 0$, any X: $X^0 = [1]$.
(f) $m = \frac{2}{3}$, $X = [\pi]$: $X^{\frac{2}{3}} = [\pi^2]^{1/3} = [\pi^{2/3}]$.

6.2.4. *Binary Real Powers* $(x,y)\mapsto x^y$.

We now extend the unary power constructions to the binary real exponentiation as the Cauchy lift of a partial rational precursor on $\mathbb{Q}^2$. This naturally generalizes Sections 6.2.1–6.2.3, reducing to those cases when the exponent is fixed as an integer or rational constant.

Definition 6.14 (Rational precursor)**.** *Define* $\mathrm{pow}:\mathbb{Q}^2\rightharpoonup\mathbb{Q}$ *by*

$$\mathrm{pow}(x,y)=\begin{cases} x^m & \text{if } y=m\in\mathbb{Z} \text{ and } (x\neq 0 \text{ if } m<0),\\ (x^{1/q})^p & \text{if } y=\frac{p}{q} \text{ (lowest terms)},\ q\in\mathbb{N}^+,\ \text{and } \begin{cases} x \text{ is a perfect } q\text{-th power in } \mathbb{Q},\\ x\geq 0 \text{ if } q \text{ is even,}\end{cases}\\ \text{undefined} & \text{otherwise.}\end{cases}$$

Its Cauchy–lift ${}^c\mathrm{pow}:\mathbb{R}^2\rightharpoonup\mathbb{R}$ *is given by the k-ary lift definition.*

Proposition 6.15 (Domain and agreement)**.** *For* $X,Y\in\mathbb{R}$*:*

(i) If $X>[0]$*, then* ${}^c\mathrm{pow}(X,Y)=[X^Y]$ *for all* Y*.*
(ii) If $X=[0]$*, then* ${}^c\mathrm{pow}([0],Y)=[0]$ *for* $Y>[0]$ *and is undefined for* $Y\leq[0]$*.*
(iii) If $X<[0]$*, then* ${}^c\mathrm{pow}(X,Y)$ *is defined iff* $Y\in\mathbb{Z}$ *or* $Y=[p/q]$ *with* q *odd, and equals the classical value when defined.*

Proof sketch.
(i) Pick $y_n=p_n/q_n\to Y$ in lowest terms and refine bases by Lemma 6.6 to $x_n=s_n^{q_n}\to X$. Thus we obtain rational pairs (x_n,y_n) with $x_n=s_n^{q_n}\to X$ and $y_n=p_n/q_n\to Y$ (lowest terms), giving

$$\mathrm{pow}(x_n,y_n)=\left(x_n^{1/q_n}\right)^{p_n}=s_n^{p_n}.$$

For $X>[0]$, the classical map $(x,y)\mapsto x^y$ is continuous on $(0,\infty)\times\mathbb{R}$, so $s_n^{p_n}\to X^Y$. Hence $(s_n^{p_n})$ is Cauchy with class $[X^Y]$. By the lift definition, ${}^c\mathrm{pow}(X,Y)=[X^Y]$.

(ii) Follows from unary integer powers and reciprocals: ${}^c\mathrm{pow}([0],Y)=[0]$ for $Y>[0]$; undefined otherwise.

(iii) Use Lemma 6.5 to approximate Y by rationals of odd denominator when needed; apply the same refinement as in (i) and the unary/roots cases. □

Example 6.16.

(a) $X=[2]$, $Y=[\sqrt{2}]$*: choose* $y_n=p_n/q_n\to\sqrt{2}$ *and* $x_n=s_n^{q_n}\to 2$*; then* $\mathrm{pow}(x_n,y_n)=s_n^{p_n}\to 2^{\sqrt{2}}$*, hence* ${}^c\mathrm{pow}([2],[\sqrt{2}])=[2^{\sqrt{2}}]$.
(b) $X=[-8]$, $Y=[\frac{2}{3}]$*:* $q=3$ *odd,* ${}^c\sqrt[3]{X}=[-2]$*, so* ${}^c\mathrm{pow}(X,Y)=(-2)^2=[4]$.
(c) $X=[0]$, $Y=[\frac{1}{2}]$*:* ${}^c\mathrm{pow}([0],[\frac{1}{2}])=[0]$.
(d) $X=[0]$, $Y=[0]$*: undefined (conflicting representative behavior).*
(e) $X=[-1]$, $Y=[\sqrt{2}]$*: undefined in* $\mathbb{R}$ *(irrational exponent on a negative base).*

Remark 10 (Continuity and agreement with the classical $(x,y)\mapsto x^y$)**.** *For* $X>[0]$ *the lifted map* $(X,Y)\mapsto{}^c\mathrm{pow}(X,Y)$ *is continuous on* $(0,\infty)\times\mathbb{R}$ *and agrees with the classical real power. For* $X\leq[0]$ *it matches the classical real domain exactly (items (ii)–(iii) of Proposition 6.15). Recall the unary case* $X^0=[1]$ *remains valid everywhere (as in §6.2.1), whereas the genuinely binary* ${}^c\mathrm{pow}([0],[0])$ *is undefined—reflecting the difference between fixing first (unary case) versus approaching* $([0],[0])$ *in two dimensions.*

6.3. **Calculus Functions.**

Cauchy constructed reals via convergent sequences (i.e. sequences equivalent in the limit). Instead of defining calculus operations (derivative & integral) in terms of these real numbers which are already defined by limits, redundant limits are added.

6.3.1. *Difference Quotient.*

Cauchy constructed reals via convergent sequences (i.e. sequences equivalent in the limit). Instead of defining derivatives in terms of these real numbers which are already defined by limits, redundant limits are added, and the derivative is defined as the limit of Difference Quotients.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

We can do without the redundant limit.

Find the derivative of $F(X) = X^2$ at $X = [1]$

Example 6.17 (Difference Quotient)**.**

> *Consider the rational function* $g(h) = \frac{(1+h)^2-1^2}{h} = 2+h$, *defined for* $h \neq 0$.
> *Construct* ${}^c g(H)$.
> *Take* $h = \{h_1, h_2, h_3, \ldots\} \in H$ *with all* $h_i \neq 0$,
> *so* $g[h] = \{2+h_1, 2+h_2, 2+h_3, \ldots\} \in [2]+H$.
> *So,* ${}^c g(H) = [2]+H$.
> *For* $H = [0]$, *use* $h = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\}$,
> *yielding* $\{2+1, 2+\frac{1}{2}, 2+\frac{1}{3}, \ldots\} \in [2]$.

Thus, ${}^c g([0]) = [2]$, the derivative $F'([1])$, directly obtained, without redundant limits.

6.3.2. *Riemann Sum.*

Cauchy constructed reals via convergent sequences (i.e. sequences equivalent in the limit). Instead of defining integrals in terms of these real numbers which are already defined by limits, redundant limits are added, and the Riemann Integral is defined as the limit of Riemann Sums.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

We can do without the redundant limit.

Find the integral of $F(X) = X$ from $X = [0]$ to $[1]$.

Example 6.18 (Riemann Sum)**.**

> *Consider the rational Riemann Sum function* $g(h) = h \cdot \sum_{i=1}^{|1/h|} ih$.
> *Construct* ${}^c g(H)$ *at* $H = [0]$.
> *We want Upper-Sum-Index* $1/h$ *to be an integer.*
> *So, we choose* $h = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \in [0]$.
> *This simplifies* $g(h) = h^2 \cdot \frac{\frac{1}{h}(\frac{1}{h}+1)}{2} = \frac{1+h}{2}$.
> *Then,* $g[h] = \{\frac{1+1}{2}, \frac{1+\frac{1}{2}}{2}, \frac{1+\frac{1}{3}}{2}, \ldots\} = \{1, \frac{3}{4}, \frac{2}{3}, \ldots\} \in [\frac{1}{2}]$.

Thus, ${}^c g([0]) = [\frac{1}{2}]$, the integral $\int_0^1 F(x)\,dx$, directly obtained, again without redundant limits.

7. Discussion

Our approach evaluates functions at exact real points, enhancing intuitiveness, making neighborhood analysis redundant.

This approach simplifies real analysis by:

- Eliminating Redundant Limits: Functions are defined directly via equivalence classes.
- Removing Pathologies: Discontinuities like $f(x) = \frac{x}{x}$ at $x = 0$ vanish.
- Consistency: All functions align with the Cauchy construction.

Unlike extended number systems, which introduce complexity (e.g., hyperreals' ultrafilters, surreals' vastness), our method requires no additional structures or operators like std($\cdot$). It alters standard results (e.g., unary $X/X = [1]$ everywhere §6.2.1), adding a feature typically added manually after the fact.

8. Conclusion

By mandating that all functions respect Cauchy equivalence classes, we redefine real analysis to be more coherent and intuitive, avoiding redundant limits and pathologies inherent in the classical approach. Examples confirm practicality for diverse functions. Future work could extend to other quotient constructions, potentially reinterpreting mathematical analysis.

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GPT was utilized to generate sections of this work, including text, tables, graphs, code, data, citations, etc.

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Extended semi-parallel tensor product surfaces in $\mathbb{E}^4_2$

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Abstract. Tensor product immersion of two immersions of a given Riemannian manifold has begun to be studied by B.Y. Chen. In the light of Chen's definition, many researchers studied the tensor product of two immersions. Then, tensor product surfaces of Euclidean plane curves were investigated by Mihai and Rouxel. Moreover, tensor product surfaces of Lorentzian plane curves were investigated by Mihai et al. Recently Bulca and Arslan studied some special semi-parallel surfaces in Euclidean spaces. Further, Yildirim and Ilarslan considered tensor product surfaces in 4-dimensional semi-Euclidean space with index 2, $\mathbb{E}^4_2$, satisfying the semi-parallelity condition. In present study, we consider and characterize tensor product surfaces in $\mathbb{E}^4_2$ satisfying the extended semi-parallelity condition.

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1. Introduction

B.Y. Chen introduced tensor product immersion of two immersions of a given Riemannian manifold, (*cf.* [9]). Motivated by Chen's definition, F. Decruyenaere et al. studied the tensor product of two immersions of different manifolds, (*cf.* [11]). This gives an immersion of the product manifold, under some conditions. Let M and N be differentiable manifolds and assume that $f : M \to \mathbb{E}^m$ and $h : N \to \mathbb{E}^n$ are two immersions. Then the tensor product map is defined by

$$f \otimes h : M \times N \to E^{mn}$$

$$(p, q) \mapsto f(p) \otimes h(q) = (f^1(p)h^1(q), \dots, f^1(p)h^n(q), \dots, f^m(p)h^n(q)).$$

Necessary and sufficient conditions for $f \otimes h$ to be an immersion were obtained in (*cf.* [11]). Further, tensor products of spherical and equivariant immersions were studied in (*cf.* [10]). Tensor product surfaces of Euclidean plane curves were studied by Mihai and Rouxel (*cf.* [17]) and tensor product surfaces of Lorentzian plane curves by Mihai et al., (*cf.* [18]) (See also (*cf.* [1])).

Let M be a submanifold of a $(n + d)$-dimensional Euclidean space $\mathbb{E}^{n+d}$. Denote by $\overline{R}$ the curvature tensor of the Vander Waerden-Bortoletti connection $\overline{\nabla}$ of M and h is the second fundamental form of M in $\mathbb{E}^{n+d}$. The submanifold M is called semi-parallel if $\overline{R} \cdot h = 0$, (*cf.* [12]). This notion is an extrinsic analogue for semi-symmetric spaces, i.e. Riemannian manifolds for which $R \cdot R = 0$ and a direct generalization of parallel submanifolds, i.e. submanifolds for which $\overline{\nabla} h = 0$, (*cf.* [14]). For more details see also (*cf.* [12, 13, 16, 19]). In (*cf.* [12]), J. Deprez showed the fact that the submanifold $M \subset \mathbb{E}^{n+d}$ is semi-parallel implies that (M, g) is semi-symmetric. For references on semi-symmetric spaces, see (*cf.* [2, 3, 4, 12, 20]); for references on parallel immersions, see (*cf.* [14]). In (*cf.* [12]), J. Deprez gave a local classification of semi-parallel hypersurfaces in $\mathbb{E}^n$. Recently Bulca and Arslan considered some special surfaces in $\mathbb{E}^n$ satisfying the semi-parallelity condition $\overline{R} \cdot h = 0$ (*cf.* [5, 6, 7]). If the curvature tensor $\overline{R}$ and the tensor Q are linearly dependent, that is; $\overline{R} \cdot h = L_h Q(g, h)$, then M is called extended semi-parallel. Extended semi-parallel surfaces in $\mathbb{E}^n$ are classified by Ozgur et al., (*cf.* [19]).

Yildirim and Ilarslan studied tensor product surfaces of an Euclidean plane curve and a Lorentzian plane curve in $\mathbb{E}^4_2$, satisfying the semi-parallelity condition, (*cf.* [22]). In the present study we consider tensor product surfaces in $\mathbb{E}^4_2$, satisfying the extended semi-parallelity condition.

1.1. **Preliminaries.** In the present section, we begin by giving some basic concepts about Riemannian submanifolds from (*cf.* [8]). Let $\iota : M \to \mathbb{E}^n$ be an immersion from an m-dimensional connected Riemannian manifold M into $\mathbb{E}^n$. We denote the metric tensor of $\mathbb{E}^n$ by $<\cdot,\cdot>$ or $\mathfrak{g}$ as well as that induced on M. Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\mathbb{E}^n$ and ∇ the induced connection on M. Then the Gaussian and Weingarten formulas are given respectively by

$$\nabla_X Y = \nabla_X Y + h(X,Y) \text{ and } \widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N,$$

where X, Y are vector fields tangent to M and N is normal to M. Moreover, h is the second fundamental form, $\nabla^{\perp}$ is linear connection induced in the normal bundle $T^{\perp}M$, called normal connection and A_N is the shape operator in the direction of N that is related with h by

$$< h(X,Y), N > = < A_N X, Y > .$$

If the set $\{X_1, ..., X_m\}$ is a local basis for $\chi(M)$ and $\{N_1, ..., N_{n-m}\}$ is an orthonormal local basis for $\chi^{\perp}(M)$, then h can be written as

$$h = \sum_{k=1}^{n-m} \sum_{i,j=1}^{m} h^k_{ij} N_k,$$

where $h^k_{ij} =< h(X_i, X_j), N_k >$. The covariant differentiation $\overline{\nabla} h$ of the second fundamental form h on the direct sum of the tangent bundle and the normal bundle $TM \oplus T^{\perp}M$ of M is defined by

$$(\overline{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

for any vector fields X, Y and Z tangent to M. Then we have the Codazzi equation as

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z),$$

where $\overline{\nabla}$ is called the Vander Waerden-Bortoletti connection of M. We denote by R the curvature tensor associated with ∇;

$$R(X,Y)Z \; = \; -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z,$$

and denote by $R^{\perp}$ the curvature tensor associated with $\nabla^{\perp}$;

$$R^{\perp}(X,Y)\eta \; = \; \nabla^{\perp}_Y \nabla^{\perp}_X \eta - \nabla^{\perp}_X \nabla^{\perp}_Y \eta - \nabla^{\perp}_{[X,Y]} \eta,$$

where X, Y, Z are tangent vector fields to M and η is normal vector field to M. The Gaussian curvature of M is defined by (see (*cf.* [15]))

$$K =< h(X_1, X_1), h(X_2, X_2) > - < h(X_1, X_2), h(X_1, X_2) >,$$

where the set $\{X_1, X_2\}$ is a linearly independent subset of $\chi(M)$. The normal curvature K_N of M is defined by (see (*cf.* [11]))

$$K_N = \left\{ \sum_{1=k<l}^{n-m} < R^{\perp}(X_1, X_2)N_k, N_l >^2 \right\}^{(1/2)}, \tag{1}$$

where $\{N_k, N_l\}$ forms an orthonormal basis of $\chi^{\perp}(M)$. From (1), we conclude that $K_N = 0$ if and only if $\nabla^{\perp}$ is a flat normal connection of M. Let us consider the product tensor $\overline{R}.h$ of the curvature tensor $\overline{R}$ with the second fundamental form h. This product tensor is defined by

$$(\overline{R}(X,Y) \cdot h)(Z,T) = \overline{\nabla}_X (\overline{\nabla}_Y h(Z,T)) - \overline{\nabla}_Y (\overline{\nabla}_X h(Z,T)) - \overline{\nabla}_{[X,Y]} h(Z,T),$$

for all X, Y, Z, T tangent to M. Let $M \subset \mathbb{E}^n$ be a smooth surface, then M is said to be semi-parallel if $\overline{R} \cdot h = 0$, i.e., $\overline{R}(X,Y) \cdot h = 0$, (*cf.* [12, 21]). One can easily get

$$(\overline{R}(X,Y) \cdot h)(Z,T) = R^{\perp}(X,Y)h(Z,T) - h(R(X,Y)Z,T) - h(Z, R(X,Y)T).$$

Lemma 1.1 (*cf.* [12]). *Let $M \subset \mathbb{E}^n$ be a smooth surface, then the followings hold;*

$$(\overline{R}(X_1,X_2)\cdot h)(X_1,X_1) = \left(\sum_{k=1}^{n-2} h_{11}^k(h_{22}^k - h_{11}^k + 2K)\right) h(X_1,X_2)$$

$$+\sum_{k=1}^{n-2} h_{11}^k h_{12}^k (h(X_1,X_1) - h(X_2,X_2)),$$

$$(\overline{R}(X_1,X_2)\cdot h)(X_1,X_2) = \left(\sum_{k=1}^{n-2} h_{12}^k(h_{22}^k - h_{11}^k)\right) h(X_1,X_2)$$

$$+\left(\sum_{k=1}^{n-2} h_{12}^k h_{12}^k - K\right)(h(X_1,X_1) - h(X_2,X_2)),$$

$$(\overline{R}(X_1,X_2)\cdot h)(X_2,X_2) = \left(\sum_{k=1}^{n-2} h_{22}^k(h_{22}^k - h_{11}^k - 2K)\right) h(X_1,X_2)$$

$$+\sum_{k=1}^{n-2} h_{22}^k h_{12}^k (h(X_1,X_1) - h(X_2,X_2)).$$

Definition 1.2 (*cf.* [13]). *Let $M \subset \mathbb{E}^n$ be a smooth surface. A tensor Q is defined by*

$$Q(\mathfrak{g},h)(X_i,X_j;X_k,X_l) = -h((X_k\wedge X_l)X_i,X_j) - h(X_i,(X_k\wedge X_l)X_j),$$

for vector fields X_i, X_j, X_k, X_l tangent to M. Here, $\wedge$ denotes the endomorphism with respect to the metric tensor $\mathfrak{g}$.

Definition 1.3 (*cf.* [19]). *Let $M \subset \mathbb{E}^n$ be a smooth surface, then M is said to be extended semi-parallel if the tensors $\overline{R}\cdot h$ and $Q(\mathfrak{g},h)$ are linearly dependent, that is, the equality*

$$\overline{R}\cdot h = L_h Q(\mathfrak{g},h),$$

holds on the set $U_h = \{p \in M : Q(\mathfrak{g},h) \neq 0\}$, where L_h is some function on U_h.

Lemma 1.4 (*cf.* [19]). *Let $M \subset \mathbb{E}^n$ be a smooth surface and X_1, X_2 be an orthonormal basis of the tangent space T_pM. Then, the followings hold;*

$$Q(\mathfrak{g},h)(X_1,X_1;X_1,X_2) = 2h(X_1,X_2),$$

$$Q(\mathfrak{g},h)(X_1,X_2;X_1,X_2) = h(X_2,X_2) - h(X_1,X_1),$$

$$Q(\mathfrak{g},h)(X_2,X_2;X_1,X_2) = -2h(X_1,X_2).$$

Yildirim and Ilarslan studied tensor product surfaces of an Euclidean plane curve and a Lorentzian plane curve in $\mathbb{E}_2^4$. Let $c_1 : \mathbb{R} \to \mathbb{E}^2$ be a differentiable Euclidean plane curve and $c_2 : \mathbb{R} \to \mathbb{E}_1^2$ be a differentiable non-null Lorentzian plane curve. Put $c_1(t) = (\gamma(t), \delta(t))$ and $c_2(s) = (\alpha(s), \beta(s))$. Then their tensor product surface patch is given by

$$f = c_1 \otimes c_2 : \mathbb{R}^2 \to \mathbb{E}_2^4$$

$$f(t,s) = (\gamma(t)\alpha(s), \gamma(t)\beta(s), \delta(t)\alpha(s), \delta(t)\beta(s)).$$

The metric tensor on $\mathbb{E}_1^2$ and $\mathbb{E}_2^4$ is given by

$$g = -dx_1^2 + dx_2^2$$

and

$$\mathbf{g} = -dx_1^2 + dx_2^2 - dx_3^2 + dx_4^2,$$

respectively. Let $c_1(t) = (\cos t, \sin t)$ and $c_2(s) = (\alpha(s), \beta(s))$ is a spacelike or a timelike curve with unit speed. Then, the surface patch becomes

$$M : f(t,s) = (\alpha(s)\cos t, \beta(s)\cos t, \alpha(s)\sin t, \beta(s)\sin t). \tag{2}$$

An orthonormal frame tangent to M is given by

$$\begin{aligned} e_1 &= \left(\frac{1}{\|c_2(s)\|}\right)\frac{\partial f}{\partial t} \\ &= \left(\frac{1}{\|c_2(s)\|}\right)(-\alpha(s)\sin t, -\beta(s)\sin t, \alpha(s)\cos t, \beta(s)\cos t), \\ e_2 &= \frac{\partial f}{\partial s} \\ &= (\alpha^{'}(s)\cos t, \beta^{'}(s)\cos t, \alpha^{'}(s)\sin t, \beta^{'}(s)\sin t). \end{aligned}$$

The normal space of M is spanned by

$$\begin{aligned} n_1 &= (\beta^{'}(s)\cos t, \alpha^{'}(s)\cos t, \beta^{'}(s)\sin t, \alpha^{'}(s)\sin t), \\ n_2 &= \left(\frac{1}{\|c_2(s)\|}\right)(-\beta(s)\sin t, -\alpha(s)\sin t, \beta(s)\cos t, \alpha(s)\cos t) \end{aligned}$$

where

$$\mathbf{g}(e_1, e_1) = -\mathbf{g}(n_2, n_2) = \frac{g(c_2(s), c_2(s))}{\|c_2(s)\|^2} = \varepsilon_1,$$

$$\mathbf{g}(e_2, e_2) = -\mathbf{g}(n_1, n_1) = g(c_2^{'}(s), c_2^{'}(s)) = \varepsilon_2$$

and $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1$. By covariant differentiation with respect to e_1 and e_2 a straightforward calculation gives

$$\begin{aligned} \widetilde{\nabla}_{e_1} e_1 &= a\varepsilon_2 e_2 - b\varepsilon_2 n_1, \\ \widetilde{\nabla}_{e_1} e_2 &= -a\varepsilon_1 e_1 - b\varepsilon_1 n_2, \\ \widetilde{\nabla}_{e_1} n_1 &= -b\varepsilon_1 e_1 - a\varepsilon_1 n_2, \\ \widetilde{\nabla}_{e_1} n_2 &= -b\varepsilon_2 e_2 + a\varepsilon_2 n_1, \\ \widetilde{\nabla}_{e_2} e_1 &= -b\varepsilon_1 n_2, \\ \widetilde{\nabla}_{e_2} e_2 &= -c\varepsilon_2 n_1, \\ \widetilde{\nabla}_{e_2} n_1 &= -c\varepsilon_2 e_2, \\ \widetilde{\nabla}_{e_2} n_2 &= -b\varepsilon_1 e_1, \end{aligned}$$

where a, b and c are Christoffel symbols and as in follows

$$\begin{aligned} a = a(s) &= \frac{\alpha\alpha^{'} - \beta\beta^{'}}{\|c_2(s)\|^2}, \\ b = b(s) &= \frac{\alpha\beta^{'} - \alpha^{'}\beta}{\|c_2(s)\|^2}, \\ c = c(s) &= \alpha^{'}\beta^{''} - \alpha^{''}\beta^{'}. \end{aligned}$$

Second fundamental form of this surface is written as

$$h = \sum_{i,j,k=1}^{2} \varepsilon_k h_{ij}^k n_k,$$

where

$$h_{11}^1 = b, \qquad h_{11}^2 = 0,$$

$$h_{12}^1 = h_{21}^1 = 0, \qquad h_{12}^2 = h_{21}^2 = b,$$

$$h_{22}^1 = c, \qquad h_{22}^2 = 0.$$

In the next section, we will use the results obtained in (*cf.* [22]).

2. Main results

In this section, we present the main results that we obtained.

Theorem 2.1. *Let M be a tensor product surface given with the surface patch* (2)*. Then, M is an extended semi-parallel surface if and only if there exists a function L_h such that the following equalities hold;*

$$
\begin{aligned}
b^2\varepsilon_1(c-b+2b\varepsilon_1-2c\varepsilon_2) &= L_h 2b,\\
b\varepsilon_2(b-b\varepsilon_1+c\varepsilon_2)(c-b) &= L_h(c-b),\\
b\varepsilon_1(2b\varepsilon_1+bc-c-2bc\varepsilon_2) &= -L_h 2b.
\end{aligned}
\tag{3}
$$

Proof. Since M is an extended semi-parallel surface, i.e., $\overline{R}\cdot h = L_h Q(g,h)$, where the induced metric on M is denoted by g. Then, using Lemmas (1.1) and (1.4), we obtain

$$
\begin{aligned}
b^2\varepsilon_1(c-b+2b\varepsilon_1-2c\varepsilon_2)n_2 &= L_h 2bn_2,\\
b\varepsilon_2(b-b\varepsilon_1+c\varepsilon_2)(c-b)n_1 &= L_h(c-b)n_1,\\
b\varepsilon_1(2b\varepsilon_1+bc-c-2bc\varepsilon_2)n_2 &= -L_h 2bn_2.
\end{aligned}
$$

So the result is clear. □

Theorem 2.2. *Let M be a tensor product surface given with the surface patch* (2) *and $\varepsilon_1=\varepsilon_2=1$. Then, M is an extended semi-parallel surface if and only if M is a semi-parallel surface.*

Proof. Suppose that M is an extended semi-parallel surface, so M satisfies the extended semi-parallelity condition (3). There are two possible cases:

1) Let $L_h = K$. Then by (3), we obtain

$$
\begin{aligned}
bK &= 0,\\
(b-c)(bc-K) &= 0,\\
b(3K+b^2-c^2) &= 0.
\end{aligned}
$$

If $b=0$, then $c=0$. Since $K=b(b-c)$, also $K=0$. In this case, M is a semi-parallel surface given with the surface patch (*cf.* [22])

$$
f(t,s) = (\lambda\beta(s)cost, \beta(s)cost, \lambda\beta(s)sint, \beta(s)sint),\ \ \lambda\neq 1,\ \ \lambda\in\mathbb{R}. \tag{4}
$$

If $b\neq 0$, then $K=0$, consequently $b=c$. In this case, M is again a semi-parallel surface.

2) Let $L_h\neq K$. Then by (3), we obtain

$$
\begin{aligned}
b(K-2L_h) &= 0,\\
(b-c)(bc-L_h) &= 0,\\
b(K+b^2-c^2+2L_h) &= 0.
\end{aligned}
$$

We suppose $b,c\neq 0$ and $b\neq c$. So, we get

$$
\begin{aligned}
K &= 2L_h,\\
bc &= L_h,\\
K+2L_h &= c^2-b^2.
\end{aligned}
$$

Since $K=b(b-c)$, these three equations give us $b=0$, so this is a contradiction. Consequently, the case $L_h\neq K$ is not possible. □

Theorem 2.3. *Let M be a tensor product surface given with the surface patch* (2) *and $\varepsilon_1=-1, \varepsilon_2=1$. Then, M is an extended semi-parallel surface if and only if M is a semi-parallel surface.*

Proof. Suppose that M is an extended semi-parallel surface, so M satisfies the extended semi-parallelity condition (3). There are two possible cases:

1) Let $L_h = K$. Then by (3), we obtain

$$\begin{aligned} b(2b^2 - 3K) &= 0, \\ (c-b)(b^2 - 2K) &= 0, \\ b(K + b^2 + c^2) &= 0. \end{aligned}$$

If $b = 0$, then $c = 0$. Since $K = -b(b + c)$, also $K = 0$. In this case, M is a semi-parallel surface given with the surface patch (4). If $b \neq 0$, we get $3K = 2b^2 = -3b^2 - 3c^2$, this equation gives us a contradiction. Consequently, in this case $b \neq 0$ is not possible.

2) Let $L_h \neq K$. Then by (3), we obtain

$$\begin{aligned} b(-K + 2b^2 - 2L_h) &= 0, \\ (c-b)(-K + b^2 - L_h) &= 0, \\ b(-K + b^2 + c^2 + 2L_h) &= 0. \end{aligned}$$

We suppose $b \neq 0$. If $b = c$, we obtain $L_h = 0$ and $K = 2b^2$. But, since $K = -b(b + c)$, this gives us $b = 0$, so this is a contradiction. If $b \neq c$, we obtain $K = 0$, this gives us the same contradiction. Consequently, the case $L_h \neq K$ is not possible. □

Theorem 2.4. *Let M be a tensor product surface given with the surface patch (2) and $\varepsilon_1 = 1, \varepsilon_2 = -1$. Then, M is an extended semi-parallel surface if and only if M is a semi-parallel surface.*

Proof. Suppose that M is an extended semi-parallel surface, so M satisfies the extended semi-parallelity condition (3). There are two possible cases:

1) Let $L_h = K$. Then by (3), we obtain

$$\begin{aligned} b(2bc - K) &= 0, \\ (c-b)(bc - K) &= 0, \\ b(4K + bc - c^2) &= 0. \end{aligned}$$

If $b = 0$, then $c = 0$. Since $K = b(b + c)$, also $K = 0$. In this case, M is a semi-parallel surface given with the surface patch (4). If $b \neq 0$, we get $K = 2bc = b^2 + bc$, so we find $b(b - c) = 0$, this implies $b = c$. Consequently, we find $K = 0$, so $b = 0$. This equation gives us a contradiction. Consequently, in this case $b \neq 0$ is not possible.

2) Let $L_h \neq K$. Then by (3), we obtain

$$\begin{aligned} b(-K + 2b^2 - 2L_h) &= 0, \\ (c-b)(-K + b^2 - L_h) &= 0, \\ b(-K + b^2 + c^2 + 2L_h) &= 0. \end{aligned}$$

We suppose $b \neq 0$. If $b = c$, we obtain $L_h = 0$ and $K = 2b^2$. But, since $K = b(b + c)$, this gives us $b = 0$, so this is a contradiction. If $b \neq c$, we obtain $K = 0$, then we find $b = c$ and this gives us a contradiction. Consequently, the case $L_h \neq K$ is not possible. □

Theorem 2.5. *Let M be a tensor product surface given with the surface patch (2) and $\varepsilon_1 = \varepsilon_2 = -1$. Then, M is an extended semi-parallel surface if and only if M is a semi-parallel surface.*

Proof. Suppose that M is an extended semi-parallel surface, so M satisfies the extended semi-parallelity condition (3). There are two possible cases:

1) Let $L_h = K$. Then by (3), we obtain

$$\begin{aligned} 5bK &= 0, \\ 0 &= 0, \\ cK &= 0. \end{aligned}$$

If $b = 0$, then $c = 0$. Since $K = b(c - b)$, also $K = 0$. In this case, M is a semi-parallel surface given with the surface patch (4). If $b \neq 0$, then $K = 0$, consequently $b = c$. In this case, M is again a semi-parallel surface.

2) Let $L_h \neq K$. Then by (3), we obtain

$$\begin{aligned} b(3K + 2L_h) &= 0, \\ (c-b)(K - L_h) &= 0, \\ b(-2K + c^2 - bc + 2L_h) &= 0. \end{aligned}$$

We suppose $b, c \neq 0$ and $b \neq c$. So, we get $K = L_h$. This is a contradiction. Consequently, the case $L_h \neq K$ is not possible.

□

3. Conclusion

We examined tensor product surfaces given with the surface patch (2) in $\mathbb{E}_2^4$ and investigated such kind of surfaces satisfying extended semi-parallelity condition (3). We conclude that extended semi-parallel tensor product surfaces coincide with semi-parallel tensor product surfaces in $\mathbb{E}_2^4$.

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Power series statistical relative uniform convergence of a double sequence of functions at a point and approximation results

Sevda Yıldız

Abstract. In this paper, we introduce a novel concept in statistical convergence for double sequences, namely power series statistical uniform convergence at a point, defined with respect to the power series method. We establish an approximation theorem for sequences of functions under this convergence. Furthermore, we provide an illustrative example that satisfies our new theorem but fails to hold under previously studied convergence methods. Additionally, we investigate the rate of convergence, offering a quantitative analysis of the proposed method's efficiency.

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1. Introduction

Convergence methods have been a central focus in approximation theory, functional analysis, and summability theory for a considerable period. Convergence for double sequences has attracted considerable attention due to its aplications in numerical analysis, functional equations, and the theory of positive linear operators. The classical notion of convergence, as proposed by Pringsheim [25], provided the foundation for this line of research. The introduction of the notion of statistical convergence by Fast [17] and Steinhaus [29] resulted in significant theoretical contributions, including further generalisations and extensions to double sequences by Moricz [24]. These developments have opened a wide avenue for new approximation results and summability methods (see e.g. [1, 3, 4, 6, 7, 13, 15, 18, 19, 22, 27]).

A more recent and powerful framework involves the use of power series methods, non-matrix methods (including Abel, Borel and logarithmic methods), which were investigated for double sequences by Baron and Stadtmüller [2]. In particular, Ünver and Orhan [31] introduced the notion of statistical convergence with respect to power series methods for single sequences, while Yıldız et al. [34] extended this idea to double sequences, showing that such convergence is incompatible with classical statistical convergence. These methods not only generalize classical convergence techniques but also provide a flexible structure for studying statistical-type convergence (see e.g. [9, 12, 14, 28, 30, 33, 35]).

Another interesting new forms of uniform convergence have been proposed in order to better capture the local behavior of single-double sequences of functions. Klippert and Williams [20] introduced the concept of uniform convergence at a point for single sequences, and later Dirik et al. [16] extended it to the double sequences. More recently, Demirci et al. [10] and Boccuto et al. [5] introduced the notion of relative uniform convergence ([8, 23]), which serves as a refined tool for developing Korovkin-type approximation theorems [21] (see also [11, 32]).

Motivated by these observations, the present paper introduces a new notion titled "power series statistical relative uniform convergence of a double function sequence at a point" . This concept combines two fundamental ideas: the power of statistical convergence with respect to power series methods and the flexibility of relative uniform convergence. This combination fills a gap in the existing literature. We establish a Korovkin-type approximation theorem within this new framework, together with example that demonstrates its effectiveness in cases where classical methods fail. Furthermore, we analyze the rate of convergence with the modulus of continuity.

The structure of the paper is as follows. In Section 2, preliminary definitions are revisited and the notion of power series statistical relative uniform convergence at a point is introduced. Section 3 is dedicated to the presentation of Korovkin-type approximation results for double sequences of positive linear operators, accompanied by a detailed application. The subsequent section, Section 4, discusses the rate of convergence. Finally, the concluding remarks provide a summary of the significance and potential extensions of the results obtained.

2. Preliminaries

In this section, the fundamental concepts of definitions are revisited and we introduce the notion of power series statistical relative uniform convergence at a point.

2.1. The Notions of Convergences for Double Sequences.

Definition 2.1. [25] *A double sequence $y = \{y_{k,l}\}$ is said to converge in Pringsheim's sense if, for every $\varepsilon > 0$, there exists an integer $L = L(\varepsilon)$ such that*

$$|y_{k,l} - \kappa| < \varepsilon \text{ whenever } k, l > L,$$

where κ is called the Pringsheim limit of y and is denoted by $P - \lim_{k,l} y_{k,l} = \kappa$. For brevity, we refer to such sequences as P-convergent.

A double sequence $\{y_{k,l}\}$ is bounded provided there exists a constant $K > 0$ with $|y_{k,l}| \leq K$ for every $(k,l) \in \mathbb{N}^2$. One should observe that, unlike the case of ordinary (single) sequences, P-convergence of a double sequence does not in general guarantee boundedness.

Let $\{p_{k,l}\}$ be a double sequence of nonnegative real numbers with $p_{00} > 0$, and consider the associated power series

$$p(t,u) := \sum_{k,l=0}^{\infty} p_{k,l} t^k u^l$$

which is assumed to have a radius of convergence R (with $R \in (0,\infty]$) so that the series converges for all $t, u \in (0,R)$.

Definition 2.2. [2] *A double sequence $y = \{y_{k,l}\}$ is said to converge according to the power series method, denoted by $P_p^2 - \lim y_{k,l} = \kappa$, if, for every $t, u \in (0,R)$, the limit*

$$\lim_{t,u \to R^-} \frac{1}{p(t,u)} \sum_{k,l=0}^{\infty} p_{k,l} t^k u^l y_{k,l} = \kappa$$

holds.

Furthermore, this method is called regular if, for every fixed index pair (μ, κ), the conditions

$$\lim_{t,u \to R^-} \frac{\sum_{k=0}^{\infty} p_{k,\kappa} t^k}{p(t,u)} = 0 \text{ and } \lim_{t,u \to R^-} \frac{\sum_{l=0}^{\infty} p_{\mu,l} u^l}{p(t,u)} = 0, \tag{1}$$

hold true (cf. [2]).

For the remainder of this paper, we shall assume that the power series method under consideration is regular, unless explicitly stated otherwise.

In a recent contribution, Ünver and Orhan [31] proposed the concept of P_p-density of $E \subset \mathbb{N}_0$ together with the associated idea of P_p-statistical convergence (st_{P_p}-limit) for single sequences. Their analysis revealed that classical statistical convergence and P_p-statistical convergence are fundamentally incompatible. Inspired by this observation, Yıldız et al. [34] extended this idea to double sequences by defining the P_p^2-density of $F \subset \mathbb{N}_0^2 = \mathbb{N}_0 \times \mathbb{N}_0$ and P_p^2-statistical convergence.

Definition 2.3. [34] *Let $F \subset \mathbb{N}_0^2$. If the limit*

$$\delta_{P_p}^2(F) := \lim_{t,u \to R^-} \frac{1}{p(t,u)} \sum_{(k,l) \in F} p_{k,l} t^k u^l$$

exists, then $\delta_{P_p}^2(F)$ is called the P_p^2-density of F.

Definition 2.4. [34] *Let $y = \{y_{k,l}\}$ be a double sequence. Then y is said to be power series statistically convergent (P_p^2-statistically convergent) to κ if for any $\varepsilon > 0$*

$$\lim_{t,u \to R^-} \frac{1}{p(t,u)} \sum_{(k,l) \in F_\varepsilon} p_{k,l} t^k u^l = 0$$

where $F_\varepsilon = \left\{(k,l) \in \mathbb{N}_0^2 : |y_{k,l} - \kappa| \geq \varepsilon\right\}$, that is $\delta_{P_p}^2(F_\varepsilon) = 0$ for any $\varepsilon > 0$. In this case we write $st_{P_p}^2 - \lim y_{k,l} = \kappa$.

The notion of uniform convergence of a sequence of functions at a point, which is strictly stronger than the classical concept of uniform convergence, was originally formulated by J. Klippert and G. Williams [20]. Subsequently, Demirci et al. [10] defined relative uniform convergence of sequences of functions at a point and established several Korovkin-type approximation theorems in this setting. More recently, Dirik et al. [16] extended this line of research to the setting of double sequences by formulating the notion of uniform convergence at a point for such sequences.

Definition 2.5. [16] *Let $\{f_{k,l}\}$ be a double sequence of real functions defined on $M^2 \subset \mathbb{R}^2$. Let $(y_0, z_0) \in M^2$. We say that $\{f_{k,l}\}$ converges uniformly at the point (y_0, z_0) to $f : M^2 \to \mathbb{R}$ iff for every $\varepsilon > 0$, there are $\eta > 0$ and $L \in \mathbb{N}$ such that for every $k, l \geq L$, if $\sqrt{(y - y_0)^2 + (z - z_0)^2} \leq \eta$ (or $|y - y_0| \leq \eta$ and $|z - z_0| \leq \eta$), then*

$$|f_{k,l}(y,z) - f(y,z)| < \varepsilon.$$

2.2. **Power Series Statistical-Type Convergence at a Point.**

Definition 2.6. *Let $M^2 \subset \mathbb{R}^2$ be an interval and suppose that $\{f_{k,l}\}$ is a double sequence of real functions defined on M^2. Let $(y_0, z_0) \in M^2$. We say that $\{f_{k,l}\}$ converges power series statistically relatively uniformly at the point (y_0, z_0) to $f : M^2 \to \mathbb{R}$ iff for every $\varepsilon > 0$ there exist a function σ, $|\sigma(y,z)| \neq 0$, $\eta > 0$ and $F \subseteq \mathbb{N}^2$ with $\delta_{P_p}^2(F) = 0$ such that for every $(k,l) \in \mathbb{N}_0^2 \backslash F$, if $\sqrt{(y - y_0)^2 + (z - z_0)^2} \leq \eta$ (or $|y - y_0| \leq \eta$ and $|z - z_0| \leq \eta$), then*

$$|f_{k,l}(y,z) - f(y,z)| < \varepsilon |\sigma(y,z)|.$$

Remark 1. (*i*) *It is immediate that if $\{f_{k,l}\}$ is P_p^2-statistically relatively uniformly convergent to a function f on M^2, then $\{f_{k,l}\}$ is also converges P_p^2-statistically relatively uniformly at each point in M^2.*

(*ii*)*If the scale function σ is bounded, P_p^2-statistical relative uniform convergence at a point implies P_p^2-statistical uniform convergence at a point. In contrast, when σ is unbounded, P_p^2-statistical relative uniform convergence at a point does not necessarily imply P_p^2-statistical uniform convergence at a point.*

(*iii*) *The notion of P_p^2-statistical uniform convergence of a double sequence of functions at a point is the special case of P_p^2-statistical relative uniform convergence of a double sequence of functions at a point in when the scale function is taken to be a nonzero constant.*

Now we give the following example to show the effectiveness of newly proposed method:

Example 2.7. *Define $g_{k,l} : [0,1]^2 \to \mathbb{R}$ by*

$$g_{k,l}(y,z) = \begin{cases} k^2 l y^2 z, & k = 2i \text{ or } l = 2j \\ \frac{k^2 l y^2 z}{4 + k^2 l y^2 z}, & \text{otherwise} \end{cases}, \tag{2}$$

$i, j = 1, 2, \ldots$. In addition, let the power series method be defined by

$$p_{k,l} = \begin{cases} 0, & k = 2i \text{ or } l = 2j \\ 1, & \text{otherwise} \end{cases} \tag{3}$$

$i, j = 1, 2, \ldots$. We claim that $\{g_{k,l}\}$ converges power series statistically relatively uniformly at $(y_0, z_0) = (0,0)$ to $g = 0$ to the scale function

$$\sigma(y,z) = \begin{cases} 1, & y = 0 \text{ or } z = 0 \\ \frac{1}{yz}, & (y,z) \in \ ]0,1]^2 \end{cases}. \tag{4}$$

Indeed, let $\varepsilon > 0$ *be given and choose* $\eta = \sqrt{\varepsilon}$ *and* $F = \{(k,l) : k = 2i \text{ or } l = 2j,\ i,j = 1,2,...\}$. *Then* $\delta^2_{P_p}(F) = 0$. *Let* $(k,l) \in \mathbb{N}_0^2 \backslash F$ *and* $(y,z) \in [0,1]^2$ *with* $|y| \leq \eta$ *and* $|z| \leq \eta$. *Then,*

$$\left|\frac{g_{k,l}(y,z)}{\sigma(y,z)}\right| \leq \left|\frac{k^2 l y^3 z^2}{4 + k^2 l y^2 z}\right| \leq |y|\,|z| < \eta^2 = \varepsilon.$$

However, $\{g_{k,l}\}$ *does not convergence power series statistically uniformly at* $(0,0)$. *Indeed, for* $\varepsilon = \frac{1}{6}$, $(y,z) = \left(\frac{1}{k}, \frac{1}{l}\right) \in]0,1]^2$ *with* $\frac{1}{k} < \eta$, $\frac{1}{l} < \eta$ *and* $(k,l) \in \mathbb{N}_0^2 \backslash F$, *we get*

$$\frac{k^2 l y^2 z}{4 + k^2 l y^2 z} = \frac{1}{5} > \frac{1}{6}.$$

Furthermore, the sequence fails to converge to $g = 0$, *both in the sense of uniform convergence and in the sense of relative uniform convergence.*

3. Korovkin-Type Approximation

In this section we apply the concept of power series statistical relative uniform convergence of double sequences of functions at a point in order to establish a Korovkin type approximation theorem. Let $C(M^2)$ denote the space of all functions f continuous on M^2. It is well known that $C(M^2)$ is a Banach space equipped with the supremum norm $\|f\| = \sup_{(y,z)\in M^2} |f(y,z)|$.

For a function $f \in C(M^2)$, we denote the value of the operator $T_{k,l}(f)$ at a point $(y,z) \in M^2$ by $T_{k,l}(f(s,v); y,z)$ or briefly, $T_{k,l}(f; y,z)$.

Throughout, we employ the following standard test functions:

$$e_0(y,z) = 1,\ e_1(y,z) = y,\ e_2(y,z) = z \text{ and } e_3(y,z) = y^2 + z^2.$$

At this stage, it is now convenient to recall the following P_p^2-statistical Korovkin-type theorem, which will serve as a useful reference point.

Theorem 3.1. [34] *Suppose that* $\{T_{k,l}\}$ *is a double sequence of positive linear operators acting from* $C(M^2)$ *into itself. Then, for all* $f \in C(M^2)$,

$$st^2_{P_p} - \lim \|T_{k,l}(f) - f\| = 0$$

iff

$$st^2_{P_p} - \lim \|T_{k,l}(e_r) - e_r\| = 0, \quad (r = 0,1,2,3).$$

We are now in a position to present the main theorem.

Theorem 3.2. *Let* $\{T_{k,l}\}$ *be a double sequence of positive linear operators acting from* $C(M^2)$ *into itself. Then* $\{T_{k,l}(e_r)\}$ $(r = 0,1,2,3)$ *converges power series statistically relatively uniformly at* (y_0, z_0) *to* e_r *with respect to the scale function* σ_r *iff for each* $f \in C(M^2)$, $\{T_{k,l}(f)\}$ *converges power series statistically relatively uniformly at* (y_0, z_0) *to* f *with respect to the scale function* σ *where* $|\sigma_r(y,z)| > 0$, $|\sigma(y,z)| > 0$ *and* σ_r, σ *possibly unbounded,* $r = 0,1,2,3$.

Proof. First, we begin the "if" part. Since $e_r \in C(M^2)$, $r = 0,1,2,3$, the necessity of the condition is straightforward. We now proceed to establish the sufficiency by considering the "only if" direction. Let $f \in C(M^2)$ and $(y,z) \in M^2$ be fixed. Let $B_1 = \max\left\{|y|, |y|^2\right\}$, $B_2 = \max\left\{|z|, |z|^2\right\}$ and $B_3 = \max\{B_1, B_2\}$. Also, by the continuity of f on M^2, we can write $|f(y,z)| \leq B_4$. Hence,

$$|f(s,v) - f(y,z)| \leq |f(s,v)| + |f(y,z)| \leq 2B_4.$$

Moreover, since f is uniformly continuous on M^2, we write that for every $\varepsilon > 0$, there exists a number $\eta > 0$ such that $|f(s,v) - f(y,z)| < \varepsilon$ holds for all $(s,v) \in M^2$ satisfying $\sqrt{(s-y)^2 + (v-z)^2} < \eta$. Hence, we get

$$|f(s,v) - f(y,z)| < \frac{\varepsilon}{4} + \frac{2B_4}{\eta^2}\left\{(s-y)^2 + (v-z)^2\right\}. \tag{5}$$

This means

$$-\frac{\varepsilon}{4}-\frac{2B_4}{\eta^2}\left\{(s-y)^2+(v-z)^2\right\}<f(s,v)-f(y,z)$$
$$<\frac{\varepsilon}{4}+\frac{2B_4}{\eta^2}\left\{(s-y)^2+(v-z)^2\right\}.$$

It is sufficient, without loss of generality, to restrict attention to values of ε satisfying $0<\varepsilon\leq 1$. By the given hypothesis, in correspondence with $B_5:=\min\left\{\frac{\varepsilon}{4},\frac{\varepsilon}{4B_4},\frac{\varepsilon\eta^2}{56B_4},\frac{\varepsilon\eta^2}{56B_4B_3}\right\}$ and $r=0,1,2,3$, there are σ_r, scale functions, $\eta_r>0$ and $F_r\subseteq\mathbb{N}_0^2$ with $\delta_{P_p}^2(F_r)=0$ such that

$$|T_{k,l}(e_r;y,z)-e_r(y,z)|\leq B_5|\sigma_r(y,z)|$$

whenever $(k,l)\in\mathbb{N}_0^2\backslash F_r$, $|y-y_0|\leq\eta_r$ and $|z-z_0|\leq\eta_r$. Now, choose

$$\sigma(y,z)=\max\{|\sigma_r(y,z)|,r=0,1,2,3\}.$$

Then, we get

$$|T_{k,l}(e_r;y,z)-e_r(y,z)|\leq B_5\sigma(y,z)$$

whenever $(k,l)\in\mathbb{N}_0^2\backslash F$ and $|y-y_0|\leq\eta$ and $|z-z_0|\leq\eta$, where

$$F=\bigcup_{r=0}^{3}F_r \text{ with } \delta_{P_p}^2(F)=0 \text{ and } \eta=\min\{\eta_r:r=0,1,2,3\}.$$

We write

$$\begin{aligned}&T_{k,l}\left((.-y)^2+(.-z)^2;y,z\right)\\&\leq|T_{k,l}(e_3;y,z)-e_3(y,z)|+2|y||T_{k,l}(e_1;y,z)-e_1(y,z)|\\&+2|z||T_{k,l}(e_2;y,z)-e_2(y,z)|+\left|y^2+z^2\right||T_{k,l}(e_0;y,z)-e_0(y,z)|\\&\leq\frac{\varepsilon\eta^2}{8B_4}\sigma(y,z)\end{aligned}$$

for $(k,l)\in\mathbb{N}_0^2\backslash F$ and $(y,z)\in M^2$ with $|y-y_0|\leq\eta$ and $|z-z_0|\leq\eta$. Using the linearity and the positivity of the operators $T_{k,l}$ together with (5), it follows that

$$\begin{aligned}&|T_{k,l}(f;y,z)-f(y,z)|\\\leq\;&|T_{k,l}(f;y,z)-f(y,z)T_{k,l}(e_0;y,z)|+|f(y,z)||T_{k,l}(e_0;y,z)-e_0(y,z)|\\\leq\;&T_{k,l}\left(\frac{\varepsilon}{4}+\frac{2B_4}{\eta^2}\left\{(.-y)^2+(.-z)^2\right\};y,z\right)+B_4|T_{k,l}(e_0;y,z)-e_0(y,z)|\\=\;&\frac{\varepsilon}{4}T_{k,l}(e_0;y,z)+\frac{2B_4}{\eta^2}T_{k,l}\left((.-y)^2+(.-z)^2;y,z\right)\\&+B_4|T_{k,l}(e_0;y,z)-e_0(y,z)|\\\leq\;&\left\{\frac{\varepsilon}{4}+B_4\right\}|T_{k,l}(e_0;y,z)-e_0(y,z)|+\frac{\varepsilon}{4}+\frac{2B_4}{\eta^2}T_{k,l}\left((.-y)^2+(.-z)^2;y,z\right)\\\leq\;&\varepsilon\sigma(y,z)\end{aligned}$$

whenever $(k,l)\in\mathbb{N}_0^2\backslash F$ and $(y,z)\in M^2$ with $|y-y_0|\leq\eta$ and $|z-z_0|\leq\eta$, hence the result. □

When the scale function is substituted by a nonzero constant, the following statement can be derived directly as an immediate consequence of the main Korovkin-type approximation theorem established herein.

Corollary 3.3. *Let* $\{T_{k,l}\}$ *be a double sequence of positive linear operators acting on* $C\left(M^2\right)$ *into itself. Then* $\{T_{k,l}(e_r)\}$ $(r=0,1,2,3)$ *converges power series statistically uniformly at* (y_0,z_0) *to* e_r *iff for each* $f\in C\left(M^2\right)$, $\{T_{k,l}(f)\}$ *converges power series statistically uniformly at* (y_0,z_0) *to* f.

An illustrative example is provided below, highlighting the extent to which our new theorem exhibits a higher degree of precision compared to earlier results.

Example 3.4. *(i) Let $M^2 = [0,1]^2$. Consider the following Bernstein operators (see [26]) given by*

$$B_{k,l}(f;y,z) = \sum_{i=0}^{k}\sum_{j=0}^{l} f\left(\frac{i}{k},\frac{j}{l}\right)\binom{k}{i}\binom{l}{j} y^i (1-y)^{k-i} z^j (1-z)^{l-j} \tag{6}$$

where $(y,z) \in M^2$; $f \in C(M^2)$.From these polynomials, we introduce the corresponding positive linear operators on $C(M^2)$:

$$T_{k,l}(f;y,z) = (1+g_{k,l}(y,z))\, B_{k,l}(f;y,z) \tag{7}$$

where $g_{k,l}(y,z)$ is given by (2) and the power series method is given with (3). Now, observe that

$$\begin{aligned}
T_{k,l}(e_0;y,z) &= (1+g_{k,l}(y,z))\, e_0(y,z), \\
T_{k,l}(e_1;y,z) &= (1+g_{k,l}(y,z))\, e_1(y,z), \\
T_{k,l}(e_2;y,z) &= (1+g_{k,l}(y,z))\, e_2(y,z), \\
T_{k,l}(e_3;y,z) &= (1+g_{k,l}(y,z))\left[e_3(y,z) + \frac{y-y^2}{k} + \frac{z-z^2}{l}\right].
\end{aligned}$$

Now we claim that $\{T_{k,l}(e_r)\}$ converges power series statistically uniformly at $(y_0,z_0) = (0,0)$ to e_r to the scale function $\sigma_r := \sigma$, is given by (4) $(r = 0,1,2,3)$.

Let $\varepsilon > 0$ be given and $F_r := F = \{(k,l) : k = 2i \text{ or } l = 2j,\ i,j = 1,2,...\}$, then $\delta^2_{P_p}(F_r) = 0$ $(r = 0,1,2,3)$.

Now, let $(k,l) \in \mathbb{N}_0^2 \backslash F_0$ and $(y,z) \in [0,1]^2$ with $|y| \le \eta_0$ and $|z| \le \eta_0$ where $\eta_0 = \sqrt{\varepsilon}$. Then,

$$\left|\frac{T_{k,l}(e_0;y,z) - e_0(y,z)}{\sigma_0(y,z)}\right| = \left|\frac{g_{k,l}(y,z)}{\sigma(y,z)}\right| \le |y|\,|z| < 2\eta_0^2 = \varepsilon,$$

our claim is true for $r = 0$. Also,

$$\begin{aligned}
\left|\frac{T_{k,l}(e_1;y,z) - e_1(y,z)}{\sigma_1(y,z)}\right| &= \left|\frac{e_1(y,z)\, g_{k,l}(y,z)}{\sigma(y,z)}\right| \\
&\le |y|^2\,|z| < \eta_1^3 = \varepsilon
\end{aligned}$$

whenever $(k,l) \in \mathbb{N}_0^2 \backslash F_1$ and $|y| \le \eta_1$, $|z| \le \eta_1$ where $\eta_1 = \sqrt[3]{\varepsilon}$. Similarly,

$$\left|\frac{T_{k,l}(e_2;y,z) - e_2(y,z)}{\sigma_2(y,z)}\right| \le \left|\frac{e_2(y,z)\, g_{k,l}(y,z)}{\sigma(y,z)}\right| \le |y|\,|z|^2 < \eta_2^3 = \varepsilon$$

whenever $(k,l) \in \mathbb{N}_0^2 \backslash F_2$ and $|y| \le \eta_2$, $|z| \le \eta_2$ where $\eta_2 = \sqrt[3]{\varepsilon}$. Hence, we get that our claim is true for $r = 1,2$. Finally,

$$\begin{aligned}
&\left|\frac{T_{k,l}(e_3;y,z) - e_3(y,z)}{\sigma_3(y,z)}\right| \\
= &\left|\frac{1}{\sigma(y,z)}\left[(1+g_{k,l}(y,z))\left[e_3(y,z) + \frac{y-y^2}{k} + \frac{z-z^2}{l}\right] - e_3((y,z))\right]\right| \\
\le &\left|\frac{g_{k,l}(y,z)}{\sigma(y,z)}\left[e_3(y,z) + \frac{y-y^2}{k} + \frac{z-z^2}{l}\right]\right| + \left|\frac{y-y^2}{k\sigma(y,z)}\right| + \left|\frac{z-z^2}{l\sigma(y,z)}\right| \\
\le &\ 4\left|\frac{g_{k,l}(y,z)}{\sigma(y,z)}\right| + 2\,|y|\,|z| \le 6\,|y|\,|z| \le 6\eta_3^2 = \varepsilon
\end{aligned}$$

whenever $(k,l) \in \mathbb{N}_0^2 \backslash F_3$ and $|y| \le \eta_3$, $|z| \le \eta_3$ where $\eta_3 = \sqrt{\frac{\varepsilon}{6}}$ and our claim is true for $r = 3$. Hence from our main Theorem 3.2, we get

$$\left|\frac{T_{k,l}(f;y,z) - f(y,z)}{\sigma(y,z)}\right| \le \varepsilon$$

whenever $(k,l) \in \mathbb{N}_0^2 \backslash F$, $|y| \le \eta$ and $|z| \le \eta$ where $\eta = \min\left\{\sqrt{\varepsilon}, \sqrt[3]{\varepsilon}, \sqrt{\frac{\varepsilon}{6}}\right\}$. Since

$|T_{k,l}(e_0;y,z) - e_0(y,z)| = g_{k,l}(y,z)$, it follows that the sequence $\{T_{k,l}(e_0)\}$ is not power series statistically uniformly convergent at $(0,0)$ to e_0. Therefore, neither Theorem 3.1 (power series statistical Korovkin type theorem) nor Corollary 3.3 are valid for the operators defined in (7). Furthermore, even ordinary uniform convergence of $\{T_{k,l}(e_0)\}$ at $(0,0)$ to e_0 fails.

(ii) Let for any $\alpha_k > 0$ and $\beta_l > 0$, $M_{\alpha_k\beta_l} := [0,\alpha_k] \times [0,\beta_l]$ with $\lim_k \alpha_k = \lim_l \beta_l = +\infty$ and

$$st_{P_p} - \lim \frac{\alpha_k}{k} = 0,\ st_{P_p} - \lim \frac{\beta_l}{l} = 0.$$

By these weaker conditions, consider the double Bernstein-Chlodowsky operators:

$$C_{k,l}(f; y, z) = \sum_{i=0}^{k}\sum_{j=0}^{l} f\left(\frac{i\alpha_k}{k}, \frac{j\beta_l}{l}\right)\binom{k}{i}\left(\frac{y}{\alpha_k}\right)^i\left(1-\frac{y}{\alpha_k}\right)^{k-i}\binom{l}{j}\left(\frac{z}{\beta_l}\right)^j\left(1-\frac{z}{\beta_l}\right)^{l-j}.$$

It is well known that

$$\begin{aligned} C_{k,l}(e_0; y, z) &= e_0(y,z), \\ C_{k,l}(e_1; y, z) &= e_1(y,z), \\ C_{k,l}(e_2; y, z) &= e_2(y,z), \\ C_{k,l}(e_3; y, z) &= y^2 + z^2 - \frac{y^2}{k} - \frac{z^2}{l} + \frac{y\alpha_k}{k} + \frac{z\beta_l}{l}. \end{aligned}$$

Let b, d are any suficiently large fixed positive real numbers such that $b \le \alpha_k$ and $d \le \beta_l$, $f \in C(M_{bd})$, then, from Theorem 3.1, we have

$$st^2_{P_p} - \lim \|C_{k,l}(f) - f\|_{C(M_{bd})} = 0.$$

Using now $C_{k,l}$, we define the following

$$C^*_{k,l}(f; y, z) = (1 + g_{k,l}(y,z))\, C_{k,l}(f; y, z) \tag{8}$$

where $g_{k,l}(y,z)$ is given by (2) and the power series method is given with (3). It is easy to see that $\left\{C^*_{k,l}(e_r)\right\}$ converges power series statistically uniformly at $(y_0, z_0) = (0,0)$ to e_r to the scale function $\sigma_r := \sigma$, is given by (4) $(r = 0,1,2,3)$. Hence from our main Theorem 3.2, we can say that, for all $f \in C(M_{bd})$, $\left\{C^*_{k,l}(f)\right\}$ converges power series statistically uniformly at $(y_0, z_0) = (0,0)$ to f to the scale function σ. However, neither Theorem 3.1 (power series statistical Korovkin type theorem) nor Corollary 3.3 are valid for the operators defined in (8).

4. Rate of Convergence

The focus of this section is on studying the rate of convergence through the modulus of continuity, defined below:

$$\omega_2(f;\eta) := \sup\left\{|f(s,v) - f(y,z)| : (s,v), (y,z) \in M^2, \sqrt{(s-y)^2 + (v-z)^2} \le \eta\right\}$$

where $f \in C(M^2)$ and $\eta > 0$. For the purpose of establishing our result, we make use of the elementary inequality, for all $f \in C(M^2)$ and for $\lambda, \eta > 0$,

$$\omega_2(f;\lambda\eta) \le (1 + [\lambda])\,\omega_2(f;\eta)$$

where $[\lambda]$ is defined to be the greatest integer less than or equal to λ.

Theorem 4.1. *Let $\{T_{k,l}\}$ denote a double sequence of positive linear operators acting from $C(M^2)$ into itself. Suppose that the following conditions are satisfied:*

(i) $\{T_{k,l}(e_0)\}$ converges power series statistically uniformly at (y_0, z_0) to e_0 with respect to the scale function σ_0,

(ii) $st^2_{P_p}-\lim \frac{\omega_2(f;\eta_{k,l})}{|\sigma_1(y,z)|} = 0$ for each $(y,z) \in M^2$ where $\eta_{k,l} := \sqrt{T_{k,l}\left((. - y)^2 + (. - z)^2; y, z\right)}$.

Then we have, for all $f \in C(M^2)$, $\{T_{k,l}(f)\}$ converges power series statistically uniformly at (y_0, z_0) to f with respect to the scale function σ, where $\sigma(y,z) = \max\{|\sigma_r(y,z)| : r = 0,1\}$.

Proof. Let $(y,z)\in M^2$and $f\in C\left(M^2\right)$ be fixed. Using the linearity and the positivity of the operators $T_{k,l}$, for all $(k,l)\in\mathbb{N}_0^2$ and any $\eta>0$, we have

$$\begin{aligned}
&\left|T_{k,l}\left(f;y,z\right)-f\left(y,z\right)\right|\\
\leq\ &\left|T_{k,l}\left(f;y,z\right)-f\left(y,z\right)T_{k,l}\left(e_0;y,z\right)\right|+\left|f\left(y,z\right)\right|\left|T_{k,l}\left(e_0;y,z\right)-e_0\left(y,z\right)\right|\\
\leq\ &T_{k,l}\left(\left(1+\frac{\left(.-y\right)^2+\left(.-z\right)^2}{\delta^2}\right)\omega_2\left(f;\eta\right);y,z\right)\\
+\ &\left|f\left(y,z\right)\right|\left|T_{k,l}\left(e_0;y,z\right)-e_0\left(y,z\right)\right|\\
=\ &\omega_2\left(f;\eta\right)T_{k,l}\left(e_0;y,z\right)+\frac{\omega_2\left(f;\eta\right)}{\eta^2}T_{k,l}\left(\left(.-y\right)^2+\left(.-z\right)^2;y,z\right)\\
+\ &\left|f\left(y,z\right)\right|\left|T_{k,l}\left(e_0;y,z\right)-e_0\left(y,z\right)\right|.
\end{aligned}$$

Put $\eta:=\eta_{k,l}=\sqrt{T_{k,l}\left(\left(.-y\right)^2+\left(.-z\right)^2;y,z\right)}$. Hence, we get

$$\begin{aligned}
&\frac{\left|T_{k,l}\left(f;y,z\right)-f\left(y,z\right)\right|}{\sigma\left(y,z\right)}\\
\leq\ &\left[\omega_2\left(f;\eta_{k,l}\right)+\left|f\left(y,z\right)\right|\right]\left|\frac{T_{k,l}\left(e_0;y,z\right)-e_0\left(y,z\right)}{\sigma_0\left(y,z\right)}\right|+2\frac{\omega_2\left(f;\eta_{k,l}\right)}{\left|\sigma_1\left(y,z\right)\right|}.
\end{aligned}$$

By using (i) and (ii) the proof is completed. □

5. Conclusion

In this work, we introduced and investigated the concept of statistical relative uniform convergence with respect to power series methods of double sequences of functions at a point. Within this framework, we proved a Korovkin-type approximation theorem and presented an example to illustrate this. We also derived results on the rate of convergence using the modulus of continuity. The proposed approach extends classical convergence notions and offers new insights in approximation theory. It is suggested that future work should extend nonlinear operators and establish connections with probabilistic methods.

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RESUME

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